PRIMES IN THE SEQUENCES $\{n^n + 1\}_{n=1}^{\infty}$ and $\{n^n + 1\}_{n=1}^{\infty}$

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let n be a positive integer. In this paper we prove that (i) if $n > 2$, then $n^n - 1$ is not a prime; (ii) if $n > 2$ and $n^n + 1$ is a prime, then $n = 2^{2^r}$, where r is a positive integer.

Let n be a positive integer. In [1, Problem 17], Smarandache posed the following questions

Question A. How many primes belong to the sequence $\{n^n - 1\}_{n=1}^{\infty}$?

Question B. How many primes belong to the sequence $\{n^n + 1\}_{n=1}^{\infty}$?

In this paper we prove the following results:

Theorem 1. 3 is the only prime belonging to $\{n^n - 1\}_{n=1}^{\infty}$.

Theorem 2. If $n > 2$ and $n^n + 1$ is a prime, then we we have $n = 2^{2^r}$, where r is a positive integer.

Proof of Theorem 1. If $n = 2$, then $2^2 - 1 = 3$ is a prime. If $n > 2$, then we have

(1) $n^n - 1 = (n-1)(n^{n-1} + n^{n-2} + \ldots + n + 1)$

Since $n - 1 > 1$ and $(n^{n-1} + n^{n-2} + \ldots + n + 1)$ if $n > 2$, we see from (1) that $n^n - 1$ is not a prime. The theorem is proved.

Proof of Theorem 2. Let $n^n + 1$ be a prime with $n > 2$. Since $n^n + 1$ is an even integer greater than 2 if $2 \nmid n$, we get $2 \mid n$. Let $n = 2^s n_1$, where s, n_1 are positive integers with $2 \nmid n_1$. If $n_1 > 1$, then we have

(2) $n^n + 1 = (n^{2^s})^{n_1} + 1 = (n^{2^s} + 1)(n^{2^s(n_1-1)} - n^{2^s(n_1-2)} + \ldots - n^{2^s} + 1)$.

It is not a prime. So we have $n_1 = 1$ and $n = 2^s$. It implies that

(3) $n^n + 1 = 2^{s \cdot 2^s} + 1$.

By the same method, we see from (3) that if $n^n + 1$ is a prime, then s must be a power of 2. Thus, we get $n = 2^{2^r}$. The Theorem is proved.

Reference

1. F.Smarandache, Only Problems, not Solutions!, Xiquan Pub. House, Phoenix, Chicago, 1990.