PROBLEM OF NUMBER THEORY (5)

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Prove that the equation

$S(x) = p$, where p is a given prime number,

has just $D((p-1)!)$ solutions, all of them in between p and p!
[$S(n)$ is the Smarandache Function: the smallest integer such that $S(n)!$ is divisible by n, and $D(n)$ is the number of positive divisors of n].

PROOF (inspired by a remark of D. W. Sharpe) :

Of course the smallest solution is $x = p$, and the largest one is $x = p!$
Any other solution should be an integer number divided by p, but not by p^2 (because $S(kp^2) >= S(p^2) = 2p$, where k is a positive integer).

Therefore $x = pq$, where q is a a divisor of $(p-1)!$

Reference: "The Smarandache Function", by J. Rodriguez (Mexico) & T. Yau (USA), in <Mathematical Spectrum>, Sheffield, UK, 1993/4, Vol. 26, No. 3, pp. 84-5; Editor: D. W. Sharpe.

Examples (of D. W. Sharpe) :

$S(x) = 5$, then $x \in \{ 5, 10, 15, 20, 30, 40, 60, 120 \}$ (eight solutions).

$S(x) = 7$ has just 30 solutions, because $6! = 2^4 x 3^2 x 5^1$ and 6! has just 5x3x2 = 30 positive divisors.