Problems

Edited by

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Welcome to the latest installment of the problems section! Our goal as always is to present interesting and challenging problems in all areas and at all levels of difficulty with the only limits being good taste. Readers are encouraged to submit new problems and solutions to the editor at one of the addresses given above. All solvers will be acknowledged in a future issue. Please submit a solution along with your proposals if you have one. If there is no solution and the editor deems it appropriate, that problem may appear in the companion column of unsolved problems. Feel free to submit computer related problems and use computers in your work. Programs can also be submitted as part of the solution. While the editor is fluent in several programming languages, be cautious when submitting programs as solutions. Wading through several pages of an obtuse program to determine if the submitter has done it right is not the editors idea of a good time. Make sure you explain things in detail.

If no solution is currently available, the program will be flagged with an asterisk*. The deadline for submission of solutions will generally be six months after the date appearing on that issue. Regardless of deadline, no problem is ever officially closed in the sense that new insights or approaches are always welcome. If you submit problems or solutions and wish to guarantee a reply, please include a self-addressed stamped envelope or postcard with appropriate postage attached. Suggestions for improvement or modification are also welcome at any time. All proposals in this offering are by the editor.

Definition: Given any positive integer n, the value of the Smarandache function S(n) is the smallest integer m such that n divides m!.

Definition: Given any positive integer $n \geq 1$, the value of the Pseudo-Smarandache function Z(n) is the smallest integer m such that n divides $\sum_{k=1}^{m} k$. Note that this is equivalent to n divides $\frac{m(m+1)}{2}$.

New Problems

16. Prove that there are an infinite number of integers n such that S(n) = Z(n).

17. Prove that if n is an even perfect number, then S(n) and Z(n) are equal and prime.

18. The Smarandache Square-Partial-Digital Subsequence(SPDS) is the set of square numbers that can be partitioned into a set of square integers. For example, 101 = 1 | 0 | 1 and 1449169 = 144 | 9 | 169 are in SPDS. Widmer[1] closes his paper with the comment, "It is relatively easy to find two consecutive squares in SPDS. One example is $12^2 = 144$ and $13^2 = 169$. Does SPDS also contain a sequence of three or more consecutive integers?"

Find a sequence of three consecutive squares in SPDS.

19. Prove that if $k > 0$, then

$$Z(2^k * 3) = \begin{cases} 2^{k+1} - 1 & \text{if k is odd} \\ 2^{k+1} & \text{if k is even} \end{cases}$$

20. Prove that if $k > 0$, then $Z(2^k * 5) =$

a) 2^{k+2} if k is congruent to 0 modulo 4
b) 2^{k+1} if k is congruent to 1 modulo 4
c) $2^{k+2} - 1$ if k is congruent to 2 modulo 4
d) $2^{k+1} - 1$ if k is congruent to 3 modulo 4.

21. a) Prove that

S(Z(n)) - Z(S(n))

is positive infinitely often.

b) Prove that

S(Z(n)) - Z(S(n))

is negative infinitely often.

22. It is clear that if p is an odd prime,

Z(S(n)) = Z(n)

since S(p) = p. Prove that there are an infinite number of composite numbers that also satisfy the equation.

Reference

1. Lamarr Widmer, 'Construction of Elements of the Smarandache Square-Partial-Digital Subsequence', **Smarandache Notions Journal,** Vol. 8, No. 1-2-3, Fall, 1997.

Problem 23 (by Sabin Tabirca, England)

Prove the following equation $(\forall n > 1)$ $\sum_{i=1,(i,n)=1}^{n} i = \frac{n \cdot \varphi(n)}{2}$.

Proof

This proof is made based on the *Inclusion & Exclusion* principle.

Let $D_p = \{i = 1,2,\dots,n \mid p|n\}$ be the set which contains the multiples of p.

This set satisfies

$$D_p = p \cdot \left\{1,2,\dots,\frac{n}{p}\right\} \text{ and } \sum_{i\in D_p} i = p \cdot \sum_{i=1}^{\frac{n}{p}} i = p \cdot \frac{\frac{n}{p}\cdot\left(\frac{n}{p}+1\right)}{2} = \frac{n}{2}\cdot\left(\frac{n}{p}-1\right).$$

Let $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_s^{k_s}$ be the prime number decomposition of n.

The following intersection of sets

$$D_{p_{j_1}} \cap D_{p_{j_2}} \cap \ldots \cap D_{p_{j_m}} = \{i = 1, 2, \dots, n \mid p_{j_1} | n \wedge p_{j_2} | n \wedge \ldots \wedge p_{j_m} | n\}$$

is evaluated as follows

$$D_{p_{j_1}} \cap D_{p_{j_2}} \cap \ldots \cap D_{p_{j_m}} = \{i = 1, 2, \dots, n \mid p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m} | n\} = D_{p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m}}$$

Therefore, the equation

$$\sum_{i\in D_{p_{j_1}} \cap D_{p_{j_2}} \cap \ldots \cap D_{p_{j_m}}} i = \sum_{i\in D_{p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m}}} i = \frac{n}{2}\cdot\left(\frac{n}{p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m}} + 1\right) \quad (1)$$

holds.

The *Inclusion & Exclusion* principle is applied based on

$$D = \{i = 1, 2, \dots, n \mid (i,n) = 1\} = \{1, 2, \dots, n\} - \bigcup_{j=1}^{s} D_{p_k}$$

and it gives

$$\sum_{i<n,(i,n)=1} i = \sum_{i=1}^{n} i - \sum_{m=1}^{n} (-1)^{m-1} \cdot \sum_{1\le j_1<j_2<\ldots<j_m\le n} \sum_{i\in D_{p_{j_1}}\cap D_{p_{j_2}}\cap\ldots\cap D_{p_{j_m}}} i \qquad (2)$$

Applying (1), the equation (2) becomes

$$\sum_{i<n,(i,n)=1} i = \sum_{i=1}^{n} i + \sum_{m=1}^{n} (-1)^{m} \cdot \sum_{1\le j_1<j_2<\ldots<j_m\le n} \frac{n}{2}\cdot\left(\frac{n}{p_{j_1}\cdot p_{j_2}\cdots p_{j_m}}+1\right) \cdot \qquad (3)$$

The right side of the equation (3) is simplified by reordering the terms as follows

$$\sum_{i<n,(i,n)=1} i = \frac{n^2}{2}\cdot\left(1+\sum_{m=1}^{n}(-1)^m \cdot \sum_{1\le j_1<j_2<\ldots<j_m\le n} \frac{1}{p_{j_1}\cdot p_{j_2}\cdots p_{j_m}}\right) + \frac{n}{2}\cdot\left(1+\sum_{m=1}^{n}(-1)^m\cdot\sum_{1\le j_1<j_2<\ldots<j_m\le n} 1\right)$$

$$\sum_{i<n,(i,n)=1} i = \frac{n^2}{2}\cdot\prod_{m=1}^{s}\left(1-\frac{1}{p_{j_m}}\right)+\frac{n}{2}\cdot\left(1+\sum_{m=1}^{n}(-1)^m\cdot\binom{n}{m}\right) = \frac{n^2}{2}\cdot\prod_{m=1}^{s}\left(1-\frac{1}{p_{j_m}}\right) = \frac{n}{2}\cdot\varphi(n)\cdot$$

Therefore, the equation (14) holds.♣

Remark

Obviously, the equation does not hold for $n=1$ because $\sum_{i=1,(i,1)=1}^{1} i = 1$ and $\frac{n\cdot\varphi(n)}{2}=\frac{1}{2}$.

Problem 24 (by Sabin Tabirca, England)

Prove that there is no a magic square made with the numbers $S(1), S(2),\ldots, S(n^2)$ where $n\in\{2, 3, 4, 5, 7, 8, 10\}$.

Proof

Let n be a number in the set $\{2, 3, 4, 5, 7, 8, 10\}$.

Let us suppose that there is a magic $x=\left(x_{i,j}\right)_{i,j=\overline{1,n}}$ square made with the number $S(1), S(2),\ldots, S(n^2)$.

In this case, the following equations are true:

$$\left(\forall\ i=\overline{1,n}\right)\sum_{j=1}^{n} x_{i,j} = C \qquad (1)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{i,j} = \sum_{i=1}^{n^2} S(i) = n\cdot C \qquad (2)$$

Therefore, the sum of the numbers $S(1), S(2),\ldots, S(n^2)$ is divisible by n.

Let us denote $SS(n)=\sum_{i=1}^{n^2} S(i)$. In the cases $n\in\{2, 3, 4, 5, 7, 8, 10\}$, we have:

n=2 $\Rightarrow$ SS(2)=9 is not divisible by 2.

n=3 $\Rightarrow$ SS(3)=34 is not divisible by 3.

n=4 $\Rightarrow$ SS(4)=85 is not divisible by 4.

n=5 $\Rightarrow$ SS(5)=187 is not divisible by 5.

n=7 $\Rightarrow$ SS(7)=602 is not divisible by 7.

n=10 $\Rightarrow$ SS(10)=2012 is not divisible by 10.

A contradiction has been found for each case. Therefore, there is no a magic square with the elements $S(1)$, $S(2)$,..., $S(n^2)$.

Problem 25 (by Jose Castillo, Arizona)

The following number, which has 155 digits,

82818079787776...1110987654321

has been proved (Stephan [1]) with a computer to be a prime number called Smarandache Reverse Prime and it belongs to the sequence:

1,21,321,4321,54321,... .

What is the sum of the digits of this number?

Solution:

Write the number per groups:

```
                 digit sum
      828180 ---------------> 8*3+2+1+0              = 27
7978...727170 ---------------> 7*10+(9+8+...+2+1+0) = 70+45
6968...626160 ---------------> 6*10+(9+8+...+2+1+0) = 60+45
5958...525150 ---------------> 5*10+(9+8+...+2+1+0) = 50+45
----------------------------------------------
1918...121110 ---------------> 1*10+(9+8+...+2+1+0) = 10+45
   98...   21 ---------------> 0*10+(9+8+...+2+1+0) = 0+45
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Total = 27+(70+60+50+...+10)+45*8 = 27+280+360 = 667

References:

[1] Stephan, Ralf W., "Factors and Primes in two Smarandache Sequences", URL: http://rws.home.pages.de, E-mail address: stephan@tmt.de .

[2] Sloane, N.J.A., "Enciclopedia of Integer Sequences", online, 1995-1998.

Solutions to Vol. 7, 1-2-3 Problems

1. The Euler phi function $\phi(n)$ is defined as the number of positive integers not exceeding n that are relatively prime to n.

a) Prove that there are no solutions to the equation

$$\phi(S(n)) = n$$

Proof: It is well-known that $S(n) \leq n$ and $\phi(n) < n$ for all $n > 0$.

b) Prove that there are no solutions to the equation

$$S(\phi(n)) = n$$

Proof: Use the same reasoning as in part (a).

c) Prove that there are an infinite number of solutions to the equation

$$n - \phi(S(n)) = 1$$

Proof: It is well-known that if p is an odd prime, $S(p) = p$ and $\phi(p) = p - 1$. Since there are an infinite number of odd primes, the result follows.

d) Prove that for every odd prime p, there is a number n such that

$$n - \phi(S(n)) = p + 1$$

Proof: It is well-known that if p is an odd prime, then $S(2p) = p$ and if p is an odd prime, $\phi(p) = p - 1$. Therefore,

$$\phi(S(2p)) = p - 1.$$

The result follows.

2) This problem was proposed in **Canadian Mathematical Bulletin** by P. Erdös and was listed as unsolved in the book **Index to Mathematical Problems 1980-1984,** edited by Stanley Rabinowitz and published by MathPro Press.

Prove that for infinitely many n

$$\phi(n) < \phi(n - \phi(n)).$$

Proof: It is easily verified that

$$\phi(30) = \phi(2)*\phi(3)*\phi(5) = 1*2*4 = 8 \text{ and}$$

$$\phi(30 - 8) = \phi(22) = \phi(2)*\phi(11) = 1*10 = 10$$

Now multiply 30 by any power of 2, 2^k. It is easy to verify using the well-known formula for the computation of the phi function

If $n = p_1^{a1} \ldots p_k^{ak}$ is the prime factorization of n, then

$$\phi(n) = n(1 - \frac{1}{p_1}) \ldots (1 - \frac{1}{p_k})$$

that

$$\phi(30 * 2^k) = 8 * 2^k \text{ and } \phi(30 * 2^k - 8 * 2^k) = 10 * 2^k.$$

which creates the infinite set.

3) The following appeared as unsolved problem(21) in **Unsolved Problems Related to Smarandache Function**, edited by R. Muller and published by Number Theory Publishing Company.

Are there m, n, k non-null positive integers, m, n $\neq$ 1 for which

$$S(mn) = m^k * S(n)?$$

Find a solution.

Solution: m = n = 2 and k = 1 is a solution.

4) The following appeared as unsolved problem(22) in **Unsolved Problems Related to Smarandache Function**, edited by R. Muller and published by Number Theory Publishing Company.

Is it possible to find two distinct numbers k and n such that

$$\log_{(k^n)} S(n^k)$$

is an integer?

Find two integers n and k that satisfy these conditions.

Solution: For k = n = 2.

$$\log_{(2^2)} S(2^2) = \log_4 S(4) = \log_4 4 = 1$$

5) Solve the following doubly true Russian alphametic

ДВА	2
ДВА	2
ТРИ	3
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СЕМЬ	7

Solution:

There are many solutions, one is

```
 572
 572
 690
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1834
```

Solution:

There are many solutions, one is

```
 572
 572
 690
-----
1834
```

Unsolved Problems

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Welcome to another installment of the unsolved problems column! In this section, problems are presented where the solution is either unknown or incomplete. This is meant to be an interactive endeavor, so input from readers is strongly encouraged. Always feel free to contact the editor at any of the addresses given above. It is hoped that we can work together to advance the flow of mathematics in some small way. There will be no deadlines here, and even if a problem is completely solved, new insights or more elegant proofs are always welcome. All correspondents who are the first to resolve any issue appearing here will have their efforts acknowledged in a future issue.

Definition of the Smarandache function, S(n).

S(n) = m where m is the smallest integer such that n divides m!.

Definition of the Pseudo-Smarandache function, Z(n).

Z(n) = m, where m is the smallest number such that n divides $\sum_{k=1}^{m} k$.

It is easy to verify that the expression

S(Z(n)) - Z(S(n))

is positive and negative an infinite number of times. It is also occasionally zero. A computer program was created to check the percentages. When run for $1 \leq n \leq 10{,}000$, the numbers were

Positive	4,744
Negative	5,227
Zero	29

This precentage was fairly constant for runs with smaller upper limits. Which leads to the question

Unsolved Question: What are the percentages of numbers for which the expression

$$S(Z(n)) - Z(S(n))$$

is positive, negative and zero?

It is possible to create polynomials with the variables the values of the Smarandache function. For example, the polynomial

$$S(n)^2 + S(n) = n$$

is such an expression. A computer search for all n $\geq$ 10,000 yielded 23 values of n for which the expression is true.

A computer search for all values of n $\leq$ 10,000 for which the expression

$$S(n)^2 + S(n) = 2n$$

is true yielded 33 solutions.

A computer search for all values of n $\leq$ 10,000 for which the expression

$$S(n)^2 + S(n) = 3n$$

is true yielded 20 solutions.

A computer search for all values of n $\leq$ 10,000 for which the expression

$$S(n)^2 + S(n) = 4n$$

is true yielded 24 solutions.

A computer search for all values of n $\leq$ 10,000 for which the expression

$$S(n)^2 + S(n) = 5n$$

is true yielded 11 solutions.

A computer search for all values of n $\leq$ 10,000 for which the expression

$$S(n)^2 + S(n) = 6n$$

is true yielded 26 solutions.

Unsolved Question: Is the number of solutions to each of the expressions above finite or infinite?

Unsolved Question: Is there a number k such that there is no number n for which

$$S(n)^2 + S(n) = kn?$$

Unsolved Question: Is there a largest number k for which there is some number n that satisfies the expression

$$S(n)^2 + S(n) = kn?$$

Unsolved Question: In examining the number of solutions for the runs for k = 1, 2, 3, 4, 5 and 6, it appears that there are more solutions when k is even than when k is odd. Is this true in general?

A computer search was performed for the expression

$$S(n)^3 + S(n)^2 + S(n) = n$$

for all n $\leq$ 10,000 and no solutions were found.

Unsolved Question: What is the largest value of k such that there is a solution to the expression

$$S(n)^k + S(n)^{k-1} + \ldots + S(n) = n?$$

A computer search for solutions for all n $\leq$ 10,000 was performed for the expression

$$S(n)^3 + S(n)^2 + S(n) = kn$$

for k=2, 3, 4, 5, and 6 and no solutions were found. However, two solutions were found for k=7.

Another computer search for all n $\leq$ 10,000 for the expression

$$S(n)^4 + S(n)^3 + S(n)^2 + S(n) = kn$$

for k = 1, 2, 3, 4, 5, 6 and 7. One solution was found for k = 5.

Unsolved Question: Is there a largest value of m for which there are no values of n and k for which

$$S(n)^m + S(n)^{m-1} + \ldots + S(n) = kn?$$

There are several classic functions of number theory, and it is in some sense natural to examine problems with the Smarandache and Pseudo-Smarandache functions combined with the classic functions.

Definition: For $n \geq 1$, the divisors function d(n) is the number of integers m, where $1 \leq m \leq n$, such that m evenly divides n.

Unsolved Question: How many solutions are there to the equation

$$Z(n) = d(n)?$$

A computer search up through n = 10,000 yielded only the solutions n = 1, 3 and 10.

Unsolved Question: How many solutions are there to the equation

$$Z(n) + d(n) = n?$$

A computer search up through n = 10,000 yielded only the solution n = 56, as d(56) = 8 and Z(56) = 48.

Unsolved Question: How many solutions are there to the equation

$$S(n) = d(n)?$$

A computer search up through n = 10,000 yielded 12 solutions, 10 of which were less than 5,000 and the last two were n = 5,000 and n = 8750. Given the obvious thinning of the solutions as n gets larger, it may be that there are very few solutions.

Definition: For $n \geq 1$, the Euler phi function $\phi(n)$ is the number of integers k, $1 \leq k \leq n$ that are relatively prime to n.

Using the Euler phi function, we can create an additional problem.

Unsolved Problem: How many solutions are there to the expression

$$S(n) + d(n) + \phi(n) = n?$$

A compute search for all n up through 10,000 yielded only the trivial solutions n =1.