PROOF OF FUNCTIONAL SMARANDACHE ITERATIONS

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ABSTRACT: The paper makes use of method of Mathematics Analytic to prove Functional Smarandache Iterations of three kinds.

1. Proving Functional Smarandache Iterations of First Kind.

Kind 1.

Let $f: A \to A$ be a function, such that $f(x) \le x$ for all x, and $\min\{f(x), x \in A\} \ge m_0$, different from negative infinity.

Let f have $p \ge 1$ fix points: $m_0 \le x_1 < x_2 < \cdots < x_p$. [The point x is called fix, if $f(x) = x$.].

Then:

$\underset{f}{SI1}(x)$ = the smallest number of iterations k such that

$\underbrace{f(f(\cdots f(x)\cdots))}_{\text{iterted } k \text{ times}} =$ constant.

Proof: I. When A⊆Q or A⊆R, conclusion is false.

Counterexample:

Let A=[0, 1] with $f(x) = x^2$, then $f(x) \le x$, and $x_1 = 0$, $x_2 = 1$ are fix points.

Denote: $A_n(x) = \underbrace{f(f(\ldots f(x)\ldots))}_{n \text{ times}}$, $A_1(x) = f(x)$, (n=1, 2, ⋯).

then $A_n(x) = x^{2^n}$ (n=1, 2, ⋯).

For any fixed $x \ne 0$, $x \ne 1$, assumed that the smallest positive integer k exist, such that $A_n(x) = a$ (constant), hence, $A_{k+1}(x) = f(A_k(x)) = f(a) = a$, that is to say a be fix point. So $x^{2^{k+1}}$=0 or 1, $\Rightarrow$ x=0 or 1, this appear contradiction. If A⊆Z, let A be set of all rational number on [0, 1] with $f(x) = x^2$, using the same methods we can also deduce contradictory result.

This shows the conclusion is false where A⊆Q or A⊆R.

II. when A⊆Z, the conclusion is true.

(1). If $x = x_i$ (x_i is fix point, i=1, ⋯p). Then $f(x) = f(x_i) = x_i = A_1(x)$. So for any positive integer n, $A_n(x) = x_i$ (i=1, ⋯p), $\Rightarrow$ $\underset{f}{SI1}(x)$=1.

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(2). Let $x \neq x_i$ (x is fixed, i=1, ⋯ p), if $f(x) = x_i$ (i=1, ⋯p), then $\underset{f}{SI1}(x)$=1, if $f(x) \neq x_i$ but $f(f(x)) = A_2(x) = x_i$ (i=1, ⋯p), then $\underset{f}{SI1}(x)$=2. In general, for fixed positive integer k, if $A_1(x) \neq x_i$, $A_2(x) \neq x_i$ ⋯ $A_{k-1}(x) \neq x_i$, but $A_k(x) = x_i$ then $\underset{f}{SI1}(x)$=k.

(3). Let $x \neq x_i$ (x is fixed), and for $\forall n \in N$ $A_n(x) \neq x_i$ (i=1, ⋯p), this case is no exist.

Because x is fix point, $m_0 < \cdots < A_n(x) < \cdots < A_2(x) < A_1(x) < x$. So sequence $\{A_n(x)\}$ is descending and exist boundary, this makes know that $\{A_n(x)\}$ is convergent. But, each item of $\{A_n(x)\}$ is integer, it is not convergent, this appear contradiction. This shows that the case is no exist.

(4). Let $x \neq x_i$ (x is fixed, i=1, ⋯ p), if exist the smallest positive integer k such that $A_k(x) = a$ ($a \neq x_i$), it is yet unable. Because $A_{k+1}(x) = A_k(x) = a$, $A_{k+1}(x) = f(A_k(x)) = f(a) = a$, this shows that a is fix point, namely, $a = x_i$, this also appear contradiction.

Combining (1), (2), (3) and (4) we have

$\underset{f}{SI1}(x)$ = the smallest number of iterations k such that

$$\underbrace{f(f(\cdots f(x)\cdots))}_{\text{iterted } k \text{ times}} = x_i \quad (x_i \text{ is fix point, } i=1, \cdots p).$$

This proves Kind 1.

We easily give a simple deduction.

Let $f : A \to A$ be a function, such that $f(x) \le x$ for all x, and $\min\{f(x), x \in A\} \ge m_0$, different from negative infinity.

Let $f(m_0) = m_0$, namely, m_0 is fix point, and only one.

Then: $\underset{f}{SI1}(x)$ = the smallest number of iterations k such that

$$\underbrace{f(f(\cdots f(x)\cdots))}_{\text{iterted } k \text{ times}} = m_0.$$

2. Proving Functional Smarandache Iterations of Second Kind.

Kind 2.

Let $g : A \to A$ be a function, such that $g(x) > x$ for all x, and let $b > x$.

Then:

$\underset{g}{SI2}(x,b)$ = the smallest number of iterations k such that

$$\underbrace{g(g(\cdots g(x)\cdots))}_{\text{iterted } k \text{ times}} \ge b.$$

Proof: Firstly, denote: $B_n(x) = \underbrace{g(g(\cdots g(x)\cdots))}_{n\ times}$, (n=1, 2, ⋯).

I. Let A⊆Z, for $\forall\ x<b$, $x \in Z$, assumed that there are not the smallest positive integer k such that $B_k(x) \geq b$, then for $\forall\ n \in N$ have $B_n(x) < b$, so

$$x < B_1(x) < B_2(x) < \cdots < B_n(x) < \cdots < b.$$

This makes know that { $B_n(x)$ } is convergent, but it is not convergent. This appear contradiction, then, there are the smallest k such that $B_n(x) \geq b$.

II. Let A⊆Q or A⊆R.

(1). For fixed $x<b$. If $g(x) \geq g(b) > b$, then $B_n(x) \geq g(x) > b$ ($n \in N$), $\underset{g}{SI2}(x,b)$=1, if $g(x) < g(b)$ but $B_2(x) \geq g(b) > b$, then $B_n(x) \geq g(b) > b$ ($n \geq 2$), $\underset{g}{SI2}(x,b)$ =2. In general, if $B_1(x) < g(b)$, $B_2(x) < g(b)$, ⋯ $B_{k\ 1}(x) < g(b)$, but $B_k(x) \geq g(b) > b$, then $\underset{g}{SI2}(x,b)$=k.

(2). For fixed $x<b$, $B_n(x) < g(b)$, ($n \in N$) then

$$x < B_1(x) < B_2(x) < \cdots < B_n(x) < \cdots < g(b),$$

so $\{B_n(x)\}$ is convergent. Let $\lim_{\to\infty} B_n(x) = b^*$ $\because$ $B_n(x) < g(b)$ ($n \in N$), $\therefore$ $b^* \leq g(b)$.

1). $b^* = g(b)$. $\because$ $\lim_{n\to\infty} B_n(x) = b^*$ $\therefore$ for $\varepsilon = g(b) - b > 0$, $\exists$ positive integer k, when n>k such that $|B_n(x)\ \ g(b)| < \varepsilon$. So $B_n(x) > g(b)\ \ \varepsilon = g(b)\ \ (g(b)\ \ b) = b$. That is to say there are the smallest k such that $B_n(x) > b$. 2). $b^* < g(b)$. $\because$ $g(b^*) > b^*$, $\therefore$ $\{B_n(x)\}$ does not converge at $g(b^*)$. So $\exists\,\varepsilon_0 > 0$, for $\forall$N, $\exists\, n_1$, when $n_1 > N$, such that $|B_{n_1}(x) - g(b^*)| \geq \varepsilon_0$, then, $B_{n_1}(x) \geq g(b^*) + \varepsilon_0$ $\therefore$ $B_{n_1}(x) > b^* + \varepsilon_0$. On the other hand, $B_n(x) \leq b^*$ ($n \in N$), $\therefore$ $B_{n_1}(x) \leq b^*$ then $b^* + \varepsilon_0 < B_{n_1}(x) \leq b^*$, but this is unable. This makes know that there is not the case.

By (1) and (2) we can deduce the conclusion is true in the case of A belong to Q or R.

Combining I. and II., we have: for any fixed $x>b$ there is

$\underset{g}{SI2}(x,b)$ = the smallest number of iterations k such that

$$\underbrace{g(g(\cdots g(x)\cdots))}_{iterted\ k\ times} \geq b.$$

This proves Kind 2.

3. Proving Functional Smarandache Iterations of Second Kind.

Kind 3.

Let h: $A \to A$ be a function, such that $h(x) < x$ for all x, and let $b < x$. Then:

$SI3_h(x,b)$ = the smallest number of iterations k such that

$$\underbrace{h(h(\cdots h(x)\cdots))}_{\text{iterted } k \text{ times}} \le b.$$

Using similar methods of proving Kind 2 we also can prove Kind 3, we well not prove again in the place.

We complete the proofs of Functional Smarandache Iterations of all kinds in the place.

REFERNECES

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