PROPERTIES OF THE TRIPLETS p^*

I. Balacenoiu, V. Seleacu

Univ. of Craiova

For every natural number p we define p^* as the following triplet $(p^* - 1, p^*, p^* + 1)$, where $p^* = 2.3.5....p$

Let us consider the following requence of prime numbers :

$2 = p_1 < 3 = p_2 < 5 = p_3 < ... p_k < ....$

We call the triplets $(p_k^* - 1, p_k^*, p_k^* + 1)$, where $p_k^* = p_1 \cdot p_2 \cdot ... \cdot p_k$, $k = 1,2...$ as p^* triplets.

It is casy to observe that :

i) $(p_k^* - 1, p_k^* + 1) = 1$, because $p_k^* - 1, p_k^* + 1$ are both of them are add numbers, and $(p_k^* + 1) - (p_k^* - 1) = 2$

ii) if $n = s_1^{\alpha_1} . s_2^{\alpha_2} ..... s_t^{\alpha_t}$ divides $p_k^* - 1$ *or* $p_k^* + 1$, because $(p_k^* - 1, p_k^*) = 1$, this implies $s_i > p_k$, for every $i \in \overline{1,t}$.

iii) if n divides $p_k^* - 1$ *or* $p_k^* + 1$, then $(n, p_h) = 1$, for $h \le k$

Proposition. The triplets p^* are separated .

Proof. Let us consider the consecutive triplets :

$p_{k-1}^* - 1, p_{k-1}^*, p_{k-1}^* + 1$

$p_k^* - 1, p_k^*; p_k^* + 1$

Because $p_k^* - 1 - (p_{k-1}^* + 1) = p_{k-1}^*(p_k - 1) - 2 > 0$ it results that every two consecutive triplets are separated , so we have :

$p_1^* - 1 < p_1^* < p_1^* + 1 < p_2^* - 1, p_2^* < p_2^* + 1 < ... < p_k^* - 1 < p_k^* < p_k^* + 1 < ...$

Remark. Let us consider the triplets :

$p_k^* - 1, p_k^*, p_k^* + 1$

$p_h^* - 1, p_h^*, p_h^* + 1,$ *where* $k < h,$ *and*

$M_{kh} = \{n \in N / p_k^* + 1 < n < p_h^* - 1\}$

Then we have :

a) if $h - k$ is constant , then card M increases simultaneously with k .

b) card M_{kh} increans when $h - k$ increases.

Definition . We say that the triplets p_k^*, p_h^* , where $k < h$, are F - prime triplets iff there is no $n \in N, n > 1$ so that $n / p_k^* \pm 1$ *and* n / p_h^k *or* $n / p_h^* \pm 1$

Examples. The triplets :

$5^* - 1 = 29, 5^* = 30, 5^* + 1 = 31$

$7^* - 1 = 209, 7^* = 210, 7^* + 1 = 211$ *are*

F - prime triplets.

The triplets :

$7^* - 1 = 209, 7^* = 210, 7^* + 1 = 211$

$11^* - 1 = 2309, 11^* = 2310, 11^* + 1 = 2311$

are not F - prime triplets, because $(7^* - 1, 11^*) = 11$

Definition . The triplets : $(p^* - 1, p^*, p^* + 1)$ and $(q^* - 1, q^*, q^* + 1)$ where $p^* - 1 = q$ or $p^* + 1 = q$ are called tinked triplets.

Remark. i) If q and p are two consecutive prime numbers, then we call p^* and q^* as consecutive linked triplets. For example 3^* and 5^* are consecutive linked triplets.

ii) Two linked triplets are not F- prime triplets.

Proposition . There is no consecutive linked triplets with $p < q$, for every $p \geq 5$.

Proof. Because p and q , $p < q$, are two consecutive prime numbers , we have : $p < q < 2p$.

For every $p \geq 5$ we have :

$$\left[\frac{p^* + 1}{q}\right] = \left[\frac{p^*}{q} + \frac{1}{q}\right] \geq \left[\frac{p^*}{2p} + \frac{1}{q}\right] = \left[\frac{s^*}{2} + \frac{1}{q}\right] = \frac{s^*}{2} \geq 3,$$

where s is such that $s < p$ and s and p are two consecutive prime number, so we have : $p^* + 1 \neq q$.

Because $\left[\frac{p^* - 1}{q}\right] \geq \left[\frac{s^*}{2} - \frac{1}{q}\right] = \frac{s^*}{2} - 1 \geq 2$, then we have $p^* - 1 \neq q$

Remark i) There are p^* triplets such that $p^* - 1 \; and \; p^* + 1$ are friend prime numbers (for example for p = 5)

There are friend prime numbers which do not belong to a p^* triplet . For example the friend prime number 11 and 13 do not belong to any triplet p^* , because 12 is not a p^* .

ii) The friend prime numbers which belong to a triplet p^* are called friend prime numbers with the triplet p^* .

There are the pairs of friend prime numbers (5,7) and (29,31) with the triplet p^* which correspond to p^* linked consecutive triplets.

Unsolved problem

i) There are an infinite set of friend prime numbers which the triplet p^* .

ii) There are an infinite set of friend prime numbers which the triplet is not p^* .

Proposition. For every $k \in N^*$ there is a natural number $h, h > k$ such that for every $s \geq h$, the triplets $(p_k^* - 1, p_k^*, p_k^* + 1)$ and $(p_s^* - 1, p_s^*, p_s^* + 1)$ are not F - prime.

Proof. If n divides p_k^* or $p_k^* + 1$, then $n = t_1^{\alpha_1} \ldots t_i^{\alpha_i}$, where $t_j > p_k$ for every $j \in \overline{1, i}$.

Let $\overline{n}$ be $\overline{n} = t_1 . t_2 . \ldots . t_i$.

Then $\overline{n}$ divides $p_k^* - 1 \; or \; p_k^* + 1$. If $p_h = \max\{t_j\}$, then $h > k, \overline{n}$ divides p_k^* and , of course , $\overline{n}$ divides p_s^* , for every $s \geq h$. Then the triplets p_k^*, p_s^* are not F - prime.

Definition. If $n = p_{i_1}^{\alpha_1} \ldots p_{i_r}^{\alpha_r}$, then $\tilde{n}$ is denoted by $\tilde{n} = \{p_{i_1}^{\alpha_1}, p_{i_2}^{\alpha_2}, \ldots p_{i_r}^{\alpha_r}\}$.

Definition. Let us consider $M = \{\tilde{n}\}_{n \in M}$ and let $\subsetneqq$ be the partial ordering relation on M , defined by

$\{p_{i_1}^{\alpha_1},\ldots,p_{i_s}^{\alpha_s}\} \subsetneqq \{q_{j_1}^{\beta_1},\ldots,q_{j_t}^{\beta_t}\} \Leftrightarrow \{p_{i_1},p_{i_2},\ldots,p_{i_s}\} \subset \{q_{j_1},\ldots,q_{j_t}\}$ and $p_{i_k}=q_{j_l}$ implies $\alpha_k \le \beta_l$.

Definition. Let us consider $M_k=\tilde{p}_k^* \cup \tilde{p}_k^*-1 \cup \tilde{p}_k^*+1, k \in N^*$.

Then we define $\hat{p}_k^*=\{n \in N^* / \tilde{n} \subset M_k\}$ and $\not\subsetneqq$, for $h<k$.

Remark. For $n=t_1^{\gamma_1}\ldots t_i^{\gamma_i}$, if $n \in \tilde{p}_k^*$, then there are the following cases :

i) n / p_k^* and

ii) $n / p_k^*-1 \; or \; n / p_k^*+1$

In the first case, $n \in \{p_k, p_1 p_k, \ldots, p_{k-1} p_k, \ldots, p_k^*\}$.

In the second case, becasuse $t_j > p_k$ for every $j \in \overline{1,i}$ it implies that there is $s \in \overline{1,i}$ for every $h, 1 \le h \le k$, such that $t_s^{\alpha_s} \notin P_h^*-1$, respectively $t_s^{\gamma_s} \notin p_h^*+1$.

In the paper [1] it is defined the Primorial Smarandache function, denoted by SP_r , where $SP_r : A \subset N^* \to N^*$ and $SP_r(n)=p$, where p is the smallest prime number such that n divides one of the numbers which belong to the triplet $p^* : p^*-1, p^*, p^*+1$, where $p^*=2.3.5.\ldots.p$ (the product of the prime numbers which are $\le p$)

In the paper [1] it is proved that the free of quadrates numbers belongs to the domain of definition of the function SP_r . The problem is : There are numbers which are not free of quadrates numbers which belongs to the domain of definition of the function SP_r ?

We study if there is $x^2 \in N^*$, where x is a prime number, such that x^2 divides one of the numbers of the triplet $p^* : p^*-1, p^*, p^*+1$, where p is a prime number .

It is casy to see that $x^2 \nmid p^*$, for every prime number p .

We proof that every prime number $x \in N^*$ has the property $x^2 \nmid p^* \pm 1$. If $x<p$, then $x^2 \nmid p^* \pm 1$.

Proposition . $x^2 \nmid p^* \pm 1$

Proof. In the case $x^2=p^*+1$, then $x^2-1=p^*$. It is casy to see that x = 2 do not verify this property.

Because $x^2-1=M4$ and $p^*=M4+2$, then $x^2-1 \ne p^*$

If $x=p^*-1, x^2+1 \ne M3$ and $p^*=M3$, then $x^2+1 \ne p^*$

Remark. Every free of quadrates number could be of onc of the following kinds : $4kx^2, (4k+1)x^2, (4k+2)x^2 \; or \; (4k+3)x^2$, where $k \in N$ and x is a prime number.

Proposition. For every prime number $x, x \in N$, we have :

a) $4kx^2 \ne p^* \pm 1$

b) $(4k+2)x^2 \ne p^* \pm 1$

c) $(4k+1)x^2 \ne p^*+1$

d) $(4k+3)x^2 \ne p^*-1$

Proof. a) Because $4kx^2$ is an even number and $p^* \pm 1$ are odd numbers, then it results that $4kx^2 \ne p^* \pm 1$

b) In an analogue way $(4k+2)x^2 \ne p^* \pm 1$, because $(4k+2)x^2$ is an even number.

c) Because $(4k+1)x^2-1=M4, x>2 \; and \; p^*=M4+2$ then it results that $(4k+1)x^2 \ne p^*+1$. For $x=2$ it can be directly proved.

d) Because $(4k+3)x^2+1=M4$, then it implies $(4k+3)x^2 \neq p^* - 1$. For $x=2$ it is directly proved.
In order to proved the proposed problem it is necessary to study the following cases, too:
$\exists x$ *and* p which are prime numbers, so that :
a) $(4k+1)x^2 = p^* - 1$, where $4k+1, 4k+3$ are prime number greater than x.
b) $(4k+3)x^2 = p^* + 1$ or products of primes greater than x.
It is easy to see that in the case when $4k+1$ *and* $4k+3$ have a prime factor q smallest than p $(q \leq p)$ the assertions a) and b) are not proved.

References.

1. **Charles Ashbacher** A Note an the Smarandache Near-To-Primorial Function Smarandache Notions Journal Vol. 7 No -1-2-3 August 1996 p. 56-49.
2. **I Balacenoiu** The Factorial Sygnature of Natural Numbers.
3. **W Sierpinski** Elementary Theory of Numbers. Warszawa 1964.