REMARKABLE INEQUALITIES

by

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In this paper are presented inequalities between factors of canonical decomposition. Let

$$n! = p_1^{e_{p_1}(n)} \cdot p_2^{e_{p_2}(n)} \cdot \ldots \cdot p_{\pi(n)}^{e_{p_{\pi(n)}}(n)}$$

be the decomposition of n! into primes with $2 = p_1 < 3 = p_2 < \ldots < p_{\pi(n)}$, and $\pi(n)$ is the number of prime numbers smaller or equal to n. Of course, $e_{p_i}(n)$, $i = \overline{1, \pi(n)}$ are Legendre's exponents. It is said that:

$$e_p(n) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ . \ . \ .$$

1. Proposition.

For every $n \geq 4$, holds: $2^{e_2(n)} > 3^{e_3(n)}$

Proof. Because

$$\frac{\left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right]}{\left[\frac{n}{3}\right] + \left[\frac{n}{3^2}\right]} > \frac{\frac{n}{2} - 1 + \frac{n}{2^2} - 1}{\frac{n}{3} + \frac{n}{3^2}} = \frac{9(3n-8)}{16n} \text{ and } \frac{9(3n-8)}{16n} \geq \frac{5}{3} \text{ for } n \geq 216$$ it resultats that:

$$\frac{\left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right]}{\left[\frac{n}{3}\right] + \left[\frac{n}{3^2}\right]} \geq \frac{5}{3} \quad \text{for} \quad n \geq 216 \qquad (1)$$

If $n = 2k$, then $\dfrac{\left[\frac{n}{2}\right]}{\left[\frac{n}{3}\right]} \geq \dfrac{\frac{n}{2}}{\frac{n}{3}} = \dfrac{3}{2}$

If $n = 2k+1$, it results that we have the following possibilities: $2k+1 = 3m$ or $2k+1 = 3m+1$ or $2k+1 = 3m+2$ and consequently $\frac{k}{m} = \frac{3}{2} - \frac{1}{2m}$ or $\frac{k}{m} = \frac{3}{2}$ or $\frac{k}{m} = \frac{3}{2} + \frac{1}{2m}$.

It results

$$\frac{\left[\frac{n}{2}\right]}{\left[\frac{n}{3}\right]} = \frac{k}{m} \geq \frac{3}{2} - \frac{1}{14} = \frac{10}{7}, \quad \text{that is } \left[\frac{n}{2}\right] \geq \frac{10}{7}\left[\frac{n}{3}\right] \quad \text{for} \quad n \geq 21.$$

While

$$\frac{\left[\frac{n}{2^2}\right]}{\left[\frac{n}{3^2}\right]} = \frac{\left[\frac{1}{2}\left[\frac{n}{2}\right]\right]}{\left[\frac{1}{3}\left[\frac{n}{3}\right]\right]} \geq \frac{\left[\frac{3}{2} \cdot \frac{10}{7} \cdot \frac{1}{3}\left[\frac{n}{3}\right]\right]}{\left[\frac{1}{3}\left[\frac{n}{3}\right]\right]} \geq \frac{\left[\frac{15}{7}\right]\left[\frac{1}{3}\left[\frac{n}{3}\right]\right]}{\left[\frac{1}{3}\left[\frac{n}{3}\right]\right]} = \left[\frac{15}{7}\right] = 2$$

that is $\left[\frac{n}{2^2}\right] \geq 2\left[\frac{n}{3^2}\right]$.

And
$$\frac{\left[\frac{n}{2^3}\right]}{\left[\frac{n}{3^3}\right]}=\frac{\left[\frac{1}{2}\left[\frac{n}{2^2}\right]\right]}{\left[\frac{1}{3}\left[\frac{n}{3^2}\right]\right]}\geq\frac{\left[\frac{3}{2}\cdot 2\cdot\frac{1}{3}\left[\frac{n}{3^2}\right]\right]}{\left[\frac{1}{3}\left[\frac{n}{3^2}\right]\right]}\geq 3$$
Generally, is true that:
$$\frac{\left[\frac{n}{2^i}\right]}{\left[\frac{n}{3^i}\right]}\geq\frac{5}{3}\quad\text{for } i\geq 3 \qquad (2)$$
From (1) and (2) it results $\frac{e_2(n)}{e_3(n)}\geq\frac{5}{3}$ for $n\geq 216$ and so $2^{\frac{e_2(n)}{e_3(n)}}\geq 2^{\frac{5}{3}}>3$ or $2^{e_2(n)}>2^{e_3(n)}$ for $n\geq 216$.

It may be verified directly that $2^{e_2(n)}>3^{e_3(n)}$ for $4\leq n<216$.

2. Proposition.

For $p\geq 5$ and $n\geq 2$ it is true that $2^{e_2(n)}>p^{e_p(n)}$.

Proof.

i) If $2\leq n<p$ because $e_p(n)=0$, it results $2^{e_2(n)}>p^{e_p(n)}=1$

ii) If $5\leq p\leq n<p^2$, then it may be showed that:
$$\frac{p}{2}\leq\frac{\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]}{\left[\frac{n}{p}\right]}\leq\frac{e_2(n)}{e_p(n)} \qquad (3)$$
iii) For $n\geq p^2$ we have:
$$\frac{\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]}{\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]}>\frac{\frac{n}{2}-1+\frac{n}{2^2}-1}{\frac{n}{p}+\frac{n}{p^2}}=\frac{p^2(3n-8)}{4n(p+1)}\quad\text{and of course:}$$
$$\frac{p^2(3n-8)}{4n(p+1)}>\frac{p}{2}\iff n>\frac{8p}{p-2}.$$
Therefore $n\geq p^2\geq n>\frac{8p}{p-2}$ for $p\geq 5$, it results
$$\frac{\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]}{\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]}>\frac{p}{2} \qquad (4)$$
If $n=2k$, then:
$$\frac{\left[\frac{n}{2}\right]}{\left[\frac{n}{p}\right]}>\frac{\frac{n}{2}}{\frac{n}{p}}=\frac{p}{2}>\frac{p-1}{2} \qquad (5)$$
If $n=2k+1\geq p$, then $2k+1$ is of the form: $2k+1=pm+i$, $i\in\overline{0,p-1}$ .For $i=0$, it results $\frac{k}{m}=\frac{p}{2}-\frac{1}{2m}\geq\frac{p-1}{2}$ and for $i\in\overline{1,p-1}$, $\frac{k}{m}=\frac{p}{2}+\frac{i}{2m}$.

Finally we get

$$\frac{\left[\frac{n}{2}\right]}{\left[\frac{n}{p}\right]} = \frac{k}{m} \geq \frac{p-1}{2} \qquad (5')$$

We have also

$$\frac{\left[\frac{n}{2^2}\right]}{\left[\frac{n}{p^2}\right]} = \frac{\left[\frac{1}{2}\left[\frac{n}{2}\right]\right]}{\left[\frac{1}{p}\left[\frac{n}{p}\right]\right]} \geq \frac{\left[\frac{p}{2}\frac{p-1}{2}\frac{1}{p}\left[\frac{n}{p}\right]\right]}{\left[\frac{1}{p}\left[\frac{n}{p}\right]\right]} \geq \frac{\left[\frac{p(p-1)}{4}\right]\left[\frac{1}{p}\left[\frac{n}{p}\right]\right]}{\left[\frac{1}{p}\left[\frac{n}{p}\right]\right]} = \left[\frac{p(p-1)}{4}\right]$$

While

$$\frac{\left[\frac{n}{2^3}\right]}{\left[\frac{n}{p^3}\right]} \geq \frac{\left[\frac{p}{2}\left[\frac{p(p-1)}{4}\right]\frac{1}{p}\left[\frac{n}{p^2}\right]\right]}{\left[\frac{1}{p}\left[\frac{n}{p^2}\right]\right]} \geq \left[\frac{p}{2}\right]\left[\frac{p(p-1)}{4}\right]$$

Generally

$$\frac{\left[\frac{n}{2^i}\right]}{\left[\frac{n}{p^i}\right]} \geq \left[\frac{p}{2}\right]^{i-2}\left[\frac{p(p-1)}{4}\right] \geq \frac{p}{2} \quad \text{for} \quad p \geq 5, \quad i \geq 2 \qquad (6)$$

From (4) and (6) it results that

$$\frac{e_2(n)}{e_p(n)} \geq \frac{p}{2} \quad \text{for} \quad n \geq p^2 \qquad (7)$$

From (3) and (7) it results $2^{\frac{e_2(n)}{e_p(n)}} \geq 2^{\frac{p}{2}} > p$, that is: $2^{e_2(n)} > 2^{e_p(n)}$

3. Proposition.

Let p, q be prime numbers, $n = p \cdot q \cdot x$ with $x \in N^*$. If $3 \leq p < q$ and $\left[\frac{q^2}{p^2}\right] > \frac{q}{p}$, it results $p^{e_p(n)} > q^{e_q(n)}$.

Proof. Obviously, if $n = p \cdot q \cdot x$, then:

$$\frac{\left[\frac{n}{p}\right]}{\left[\frac{n}{q}\right]} = \frac{qx}{px} = \frac{q}{p} > 1 \qquad (8)$$

We shall prove that

$$\frac{e_p(n)}{e_q(n)} = \frac{\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots}{\left[\frac{n}{q}\right] + \left[\frac{n}{q^2}\right] + \left[\frac{n}{q^3}\right] + \ldots} \geq \frac{q}{p} \qquad (9)$$

For $n = p \cdot q \cdot x$, $x \in N^*$ is true the following inequality

$$\frac{\left[\frac{n}{p^i}\right]}{\left[\frac{n}{q^i}\right]} \geq \left[\frac{q}{p}\right]^{i-2}\left[\frac{q^2}{p^2}\right] \geq \left[\frac{q^2}{p^2}\right], \quad \text{if} \quad p < q, \quad i \geq 2 \qquad (10)$$

We prove this inequality using the mathematical induction. Obviously

$$\frac{\left[\frac{n}{p^2}\right]}{\left[\frac{n}{q^2}\right]}=\frac{\left[\frac{1}{p}\left[\frac{n}{p}\right]\right]}{\left[\frac{1}{q}\left[\frac{n}{q}\right]\right]}\geq\frac{\left[\frac{q^2}{p^2}\frac{1}{q}\left[\frac{n}{q}\right]\right]}{\left[\frac{1}{q}\left[\frac{n}{q}\right]\right]}\geq\frac{\left[\frac{q^2}{p^2}\right]\left[\frac{1}{q}\left[\frac{n}{q}\right]\right]}{\left[\frac{1}{q}\left[\frac{n}{q}\right]\right]}=\left[\frac{q^2}{p^2}\right]$$

We suppose that is true (10) for i=k-1, that is:

$$\frac{\left[\frac{n}{p^{k-1}}\right]}{\left[\frac{n}{q^{k-1}}\right]}\geq\left[\frac{q}{p}\right]^{k-3}\left[\frac{q^2}{p^2}\right]\geq\left[\frac{q^2}{p^2}\right] \tag{10'}$$

And we demonstrate that (10) is true for $i=k$:

$$\frac{\left[\frac{n}{p^k}\right]}{\left[\frac{n}{q^k}\right]}=\frac{\left[\frac{1}{p}\left[\frac{n}{p^{k-1}}\right]\right]}{\left[\frac{1}{q}\left[\frac{n}{q^{k-1}}\right]\right]}\geq\frac{\left[\frac{q}{p}\left[\frac{q}{p}\right]^{k-3}\left[\frac{q^2}{p^2}\right]\frac{1}{q}\left[\frac{n}{q^{k-1}}\right]\right]}{\left[\frac{1}{q}\left[\frac{n}{q^{k-1}}\right]\right]}\geq\left[\frac{q}{p}\right]^{k-2}\left[\frac{q^2}{p^2}\right]\geq\left[\frac{q^2}{p^2}\right]$$

Finally the formula (10) is true for $i\geq 2$.

If $\left[\frac{q^2}{p^2}\right]>\frac{q}{p}$ then from (8) and (10) it results (9). Using (9) it results:

$$p^{\frac{e_p(n)}{e_q(n)}}\geq p^{\frac{q}{p}} \tag{11}$$

Because $p^q>q^p$ is $q>p\geq 3$ it results $p^{\frac{q}{p}}>q$ and therefore $p^{\frac{e_p(n)}{e_q(n)}}\geq p^{\frac{q}{p}}>q$ that is:

$$p^{e_p(n)}>p^{e_q(n)} \tag{11'}$$

4. Remark.

The restriction $3\leq p<q$ it suppressed in following cases:

i) $p=2$ and $q\geq 5$, because in Proposition 2 it is showed that:

$$2^{e_2(n)}>q^{e_q(n)},\quad \text{for } n\geq 2.$$

ii) $p=2$, $q=3$ for $n=2\cdot 3\cdot x$, $x\in N^*$, $6|x$ and $x\geq 18$. Is true that:

$$\frac{\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]}{\left[\frac{n}{3}\right]+\left[\frac{n}{3^2}\right]}\geq\frac{5}{3},\quad \text{that is}\quad \frac{3x+\left[\frac{3x}{2}\right]}{2x+\left[\frac{2x}{3}\right]}\geq\frac{5}{3}.$$

Obviously $$\frac{3x+\left[\frac{3x}{2}\right]}{2x+\left[\frac{2x}{3}\right]}\geq\frac{5}{3}\Leftrightarrow 0<3\left[\frac{3x}{2}\right]-5\left[\frac{2x}{3}\right]-x$$

Because $3\left(\frac{3x}{2}-1\right)-5\frac{2x}{3}-x \le 3\left[\frac{3x}{2}\right]-5\left[\frac{2x}{3}\right]-x$, and $0 \le \frac{x-18}{6}$ for $x \ge 18$, it results:

$$\frac{\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]}{\left[\frac{n}{3}\right]+\left[\frac{n}{3^2}\right]} \ge \frac{5}{3} \tag{12}$$

Using (2) and (12) it, results:

$$2^{\frac{e_2(n)}{e_3(n)}} \ge 2^{\frac{5}{3}} > 3 \quad \text{and therefore} \quad 2^{e_2(n)} > 3^{e_3(n)}$$

iii) $p = 2$, $q = 3$ and $n = 2^2 \cdot 3^2 \cdot x$, where $x \in N^*$.

Indeed

$$\frac{\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]}{\left[\frac{n}{3}\right]+\left[\frac{n}{3^2}\right]} = \frac{27}{16} > \frac{5}{3} \tag{13}$$

Using (2) and (13) it results:

$$2^{\frac{e_2(n)}{e_3(n)}} \ge 2^{\frac{5}{3}} > 3 \quad \text{and therefore} \quad 2^{e_2(n)} > 3^{e_3(n)}.$$

References

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