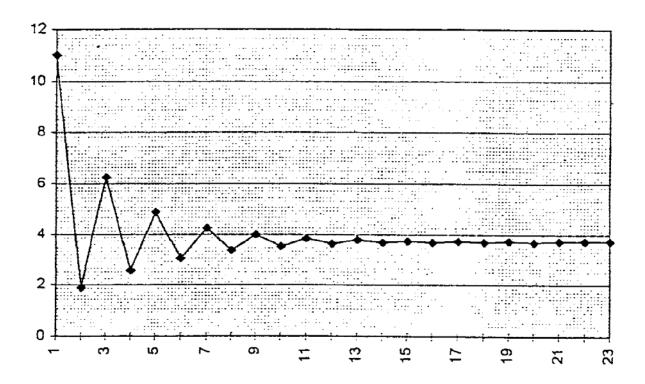
# SMARANDACHE NOTIONS JOURNAL



Number Theory Association of the UNIVERSITY OF CRAIOVA

The scheme on the first cover represents a general continued fraction as a function of n [see p. 45].

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The title of the paper should be writting with capital letters. The author's name has to apply in the middle of the line, near the title. References should be mentioned in the text by a number in square brackets and should be listed alphabetically. Current address followed by e-mail address should apply at the end of the paper, after the references.

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# Smarandache Factors and Reverse Factors

### Micha Fleuren

### November 1998

#### Abstract

This document will describe the current status on the search for factors of Smarandache consecutive numbers and their reverse. A complete list up to index 200 will be given, with all known factors. Smarandache numbers are the concatenation of the natural numbers from one up to the given index, and reverse Smarandache numbers are the concatenation of the natural numbers from the given index down to 1.

## 1 Introduction

As a followup to Ralf Stephan's article in this journal [St], I decided to extend his factorizations to index 200. The Smarandache consecutive sequence, as well as their reverse is described in [Sm]. In this article Sm11 denotes 1234567891011 for example, and Rsm11 denotes 1110987654321.

Most of the factors that have been found by me and others, have been found by using the elliptic curve method (ECM) [Le], some have been found using the Multiple Polynomial Quadratic Sieve (MPQS) [Si].

All factors and remaining cofactors have been proven prime or composite by means of Elliptic Curve Primality Proving (ECPP) [At], or the Adleman-Pomerance-Rumely test [Ad], which has been simplified in 1984 by Cohen and Lenstra [Co].

### 2 Software used

The main factoring program used has been GMP-ECM by Paul Zimmermann [Zi, Le, Gr]. The first small factors were filtered out quickly by ECMX, a program of the UBASIC package [Ki, Le].

The factors which were probably prime were then tested with François Morain's ECPP [Mo, At]. Some factors have been proven prime by use of APRT-CLE [Ad] from the UBASIC package [Ki].

All these fine pieces of software are freely available from the internet. The appropriate addresses are enlisted in the references.

# 3 Progress of calculations

All numbers have been factored using GMP-ECM up to 20 digits. First 25 runs with B1 = 2000 were run, and if the factorization wasn't complete, 90 runs with B1 = 11000 were run.

Work is in progress to extend this to 25 digits. Some factors have already been tried to 25 digits (300 curves with B1 = 50000). For more detail on the progress check the following URL:

http://www.sci.kun.nl/sigma/Persoonlijk/michaf/ecm/ecmtries.html Currently the lowest not-completely factored numbers are Sm63 and Rsm59.

## 4 Factorization results

The lists presented here are an up to date representation of the factors known so far. When more factors are found they will be added to the list, which can be found on the internet at the following URL:

http://www.sci.kun.nl/sigma/Persoonlijk/michaf/ecm/

Most of the factors up to Sm80 and Rsm80 should be creditted to Ralf Stephan. (unless otherwise stated). All contributors, together with their email-addresses can be found in tables 1 and 3.

A '\*' denotes an uncomplete factorization, pxx denotes a prime of xx digits and cxx denotes a composite number of xx digits.

#### 4.1 Smarandache Factors

Contributors of Smarandache factors		
RB	Robert Backstrom	bobb@atinet.com.au
TC	Tim Charron	tcharron@interlog.com
BD	Bruce Dodson	bad0@lehigh.edu
MF	Micha Fleuren	michaf@sci.kun.nl

AM	Allan MacLeod	MACL-MS0@wpmail.paisley.ac.uk
RS	Ralf Stephan	stephan@tmt.de
EW	Egon Willighagen	egonw@sci.kun.nl
PZ	Paul Zimmermann	zimmerma@loria.fr
		(LORIA, Nancy, France)

Table 1: Contributors of Smarandache factors

	T
n	Factors of $Sm(n)$
2	$2^2 \cdot 3$
3	3 · 41
4	$2 \cdot 617$
5	$3 \cdot 5 \cdot 823$
6	$2^6 \cdot 3 \cdot 643$
7	$127 \cdot 9721$
8	$2\cdot 3^2\cdot 47\cdot 14593$
9	$3^2 \cdot 3607 \cdot 3803$
10	2 · 5 · 1234567891
11	3 · 7 · 13 · 67 · 107 · 630803
12	$2^3 \cdot 3 \cdot 2437 \cdot 2110805449$
13	113 · 125693 · 869211457
14	2 · 3
	p18: 205761315168520219
15	3.5
	p19:8230452606740808761
16	$2^2$
	p10: 2507191691
	p13:1231026625769
17	$3^2 \cdot 47 \cdot 4993$
	p18: 584538396786764503
18	$2 \cdot 3^2 \cdot 97 \cdot 88241$
	p18:801309546900123763
19	13 · 43 · 79 · 281 · 1193
	p18: 833929457045867563
20	$2^5 \cdot 3 \cdot 5 \cdot 323339 \cdot 3347983$
	p16:2375923237887317
21	$3 \cdot 17 \cdot 37 \cdot 43 \cdot 103 \cdot 131 \cdot 140453$
	p18:802851238177109689
22	2 · 7 · 1427 · 3169 · 85829
	continued

n	Factors of $Sm(n)$
	p22 : 2271991367799686681549
23	$3 \cdot 41 \cdot 769$
	p32 : 13052194181136110820214375991629
24	$2^2 \cdot 3 \cdot 7$
	p18: 978770977394515241
	p19: 1501601205715706321
25	$5^2 \cdot 15461$
	p11:31309647077
	p25 : 1020138683879280489689401
26	$2 \cdot 3^4 \cdot 21347 \cdot 2345807$
	p12: 982658598563
	p18: 154870313069150249
27	$3^3 \cdot 19^2 \cdot 4547 \cdot 68891$
_ `	p32 : 40434918154163992944412000742833
28	$2^3 \cdot 47 \cdot 409$
	p15: 416603295903037
	p27 : 192699737522238137890605091
29	3 · 859
	p20 : 24526282862310130729
	p26: 19532994432886141889218213
30	$2 \cdot 3 \cdot 5 \cdot 13 \cdot 49269439$
	p18 : 370677592383442753
	p23:17333107067824345178861
31	29
	p10: 2597152967
	<i>p</i> 42 : 163915283880121143989433769727058554332117
32	$2^2 \cdot 3 \cdot 7$
	p23: 45068391478912519182079
	<i>p</i> 30 : 326109637274901966196516045637
33	$3 \cdot 23 \cdot 269 \cdot 7547$
	<i>p</i> 18 : 116620853190351161
	p31:7557237004029029700530634132859
34	2
	<i>p</i> 50 : 6172839455055606570758085909601061116212631364146515661667
35	$3^2 \cdot 5 \cdot 139 \cdot 151 \cdot 64279903$
	p10 : 4462548227
	p37: 4556722495899317991381926119681186927
36	$2^4 \cdot 3^2 \cdot 103 \cdot 211$
	<i>p</i> 56
	continued

	To story of Com(m)
n	Factors of Sm(n)
37	71 · 12379 · 4616929
-	<i>p</i> 52
38	2 · 3
	p23:86893956354189878775643
	p43: 2367958875411463048104007458352976869124861
39	3 · 67 · 311 · 1039
	p25:6216157781332031799688469
	p36: 305788363093026251381516836994235539
40	$2^2 \cdot 5 \cdot 3169 \cdot 60757 \cdot 579779$
	p10:4362289433
	p20:79501124416220680469
	p26: 15944694111943672435829023
41	3 · 487 · 493127 · 32002651
4.0	p56
42	2 · 3 · 127 · 421
	p11: 22555732187
	p25: 4562371492227327125110177
	p34: 3739644646350764691998599898592229
43	7 · 17 · 449
	p72
44	23 · 32
	p26: 12797571009458074720816277
4=	p52
45	$3^2 \cdot 5 \cdot 7 \cdot 41 \cdot 727 \cdot 1291$
	p13: 2634831682519 p18: 379655178169650473
	p41: 10181639342830457495311038751840866580037
46	2 · 31 · 103 · 270408101
40	p18: 374332796208406291
ĺ	p25:3890951821355123413169209
	p28: 4908543378923330485082351119
47	3 · 4813 · 679751
41	p22: 4626659581180187993501
	p53
48	$\begin{array}{c} p_{53} \\ 2^2 \cdot 3 \cdot 179 \cdot 1493 \cdot 1894439 \end{array}$
40	p29: 15771940624188426710323588657
	p46: 1288413105003100659990273192963354903752853409
49	23 · 109 · 3251653
49	p10: 2191196713
-	continued
1	continued

n	Factors of $Sm(n)$
	p23:53481597817014258108937
	p47: 12923219128084505550382930974691083231834648599
50	$2 \cdot 3 \cdot 5^2 \cdot 13 \cdot 211 \cdot 20479$
	p18: 160189818494829241
	p20 : 46218039785302111919
	p44: 19789860528346995527543912534464764790909391
51	3
	p20:17708093685609923339
	p73
52	$\frac{1}{2}$ 7
	p17: 43090793230759613
	p76
53	$3^3 \cdot 7^3$
	p18: 127534541853151177
	p76
54	$2 \cdot 3^6 \cdot 79 \cdot 389 \cdot 3167 \cdot 13309$
	p11:69526661707
	p22:8786705495566261913717
	p51
55	5 · 768643901
	p15:641559846437453
	p22:1187847380143694126117
	p55
56	2 <sup>2</sup> ·3
	p25: 4324751743617631024407823 (BD)
	p77
57	3 · 17 · 36769067 p13 : 2205251248721
	p37 : 2128126623795388466914401931224151279 (RB)
	p47: 14028351843196901173601082244449305344230057319
58	2 · 13
00	2-13   p31 : 1448595612076564044790098185437 (BD)
	p75
59	3
	p18: 340038104073949513
	p36: 324621819487091567830636828971096713 (RB)
	p55
60	$2^3 \cdot 3 \cdot 5 \cdot 97 \cdot 157$
	p103
	continued

n	Factors of $Sm(n)$
61	10386763
	p14:35280457769357
ļ	p92
62	$2 \cdot 3^2 \cdot 1709 \cdot 329167 \cdot 1830733$
	p34:9703956232921821226401223348541281(TC)
	p64
63*	3 <sup>2</sup>
	p11:17028095263
	c105
64	$2^2 \cdot 7 \cdot 17 \cdot 19 \cdot 197 \cdot 522673$
	<i>p</i> 19 : 1072389445090071307
	p29: 20203723083803464811983788589 (PW)
	p60
65*	$3 \cdot 5 \cdot 31 \cdot 83719$
	c113
66*	$2 \cdot 3 \cdot 7 \cdot 20143 \cdot 971077$
	clll
67	397
	p18: 183783139772372071
	p105
68*	$2^4 \cdot 3 \cdot 23 \cdot 764558869$
	p10:1811890921
	c105
69	$3 \cdot 13 \cdot 23$
	p22:8684576204660284317187
	p24 : 281259608597535749175083
	p32:15490495288652004091050327089107 (RB)
	<i>p</i> 49 : 3637485176043309178386946614318767365372143115591
70	2 · 5 · 2411111
	p24 : 109315518091391293936799
	p41:11555516101313335177332236222295571524323
	<i>p</i> 60
71	$3^2 \cdot 83 \cdot 2281$
-	p31 : 7484379467407391660418419352839 (AM)
	<i>p</i> 95
72	$2^2 \cdot 3^2 \cdot 5119$
	p27: 596176870295201674946617769 (BD)
	p103
73*	37907
	continued

$\lceil n \rceil$	Factors of Sm(n)
<u> </u>	c132
74	2 · 3 · 7 · 1788313 · 21565573
' -	p20: 99014155049267797799
	p25: 1634187291640507800518363 (PW)
	p31: 1981231397449722872290863561307
	p49: 2377534541508613492655260491688014802698908815817
75*	$\begin{array}{c} 745 \cdot 2377034041008013432033200491688014802698908815817 \\ 3 \cdot 5^2 \cdot 193283 \end{array}$
.0	c133
76	23
''	p18:828699354354766183
	p27: 213643895352490047310058981
	p27 . 213043693332490047310038981 p97
77	3
• • •	
	p24 : 383481022289718079599637 (PW)   p24 : 874911832937988998935021
	p39:164811751226239402858361187055939797929 (RB)
78*	2 · 3 · 31 · 185897
•0	c139
79*	73 · 137
	p20 : 22683534613064519783
	p24 : 132316335833889742191773
	c102
80	$2^2 \cdot 3^3 \cdot 5 \cdot 101 \cdot 10263751$
	p25 : 1295331340195453366408489
	p115
81	3 <sup>3</sup> · 509
-	p30 : 152873624211113444108313548197 (AM)
	p119
82*	$2 \cdot 29 \cdot 4703 \cdot 10091$
	p35 : 12295349967251726424104854676730107 (AM)
	cll1
83*	3 · 53 · 503
	p18: 177918442980303859 (MF)
1	c134
84	25 · 3
	p157
85*	$5 \cdot 7^2$
	c158
	continued

n	Factors of Sm(n)
86*	2 · 3 · 23 · 1056149
	c155
87*	3 · 7 · 231330259
	p10: 4275444601 (MF)
1	c145
88*	$2^2$
	p14:12414068351873 (MF)
ļ	c153
89*	3 · 3 · 13 · 31 · 97 · 163060459
	p18: 789841356493369879 (MF)
	c137
90*	$2 \cdot 3 \cdot 3 \cdot 5 \cdot 1987 \cdot 179827 \cdot 2166457$
	c154
91*	37 · 607
	p16:5713601747802353 (MF)
	p24:100397446615566314002487 (MF)
	c130
92*	$2^3 \cdot 3 \cdot 75503$
00*	c168
93*	3 · 73 · 1051
	p10: 3298142203 (MF) c162
94*	2 · 12871181
34	p11: 98250285823 (MF)
	c160
95*	3 · 5 · 7 · 401
	c176
96	$2 \cdot 2 \cdot 3 \cdot 23 \cdot 60331$
	p175
97	13
	p183
98*	$2\cdot 3^2\cdot 23\cdot 37\cdot 199$
	p16: 1495444452918817 (MF)
	c165
99*	$3^2 \cdot 31601$
	p12:786576340181 (MF)
100+	c171
100*	$2^2 \cdot 5^2 \cdot 7^3 \cdot 8171 \cdot 1065829$
	p10 : 2824782749 (AM)
	continued

n	Factors of $Sm(n)$
<b>—</b>	p20: 20317177407273276661 (MF)
	c149
101*	3 · 8377
101	p21:799917088062980754649 (AM)
ŀ	p21
102	$2 \cdot 3 \cdot 19 \cdot 89 \cdot 3607 \cdot 15887 \cdot 32993$
102	
	p10: 2865523753 (MF)
103*	p172
103"	131 · 1231
	p16: 1713675826579469 (MF)
10.0	c180
104*	$2^6 \cdot 3 \cdot 59 \cdot 773$
	p20: 19601852982312892289 (AM)
	c177
105*	3 · 5 · 193
	p13: 6942508281251 (MF)
	c190
106*	$2 \cdot 11 \cdot 127 \cdot 827$
	c203
107	$3^3$
	p12 : 536288185369 (MF)
	$p_1^{199}$
108*	$2^2 \cdot 3^3$
	p18: 128451681010379681 (AM)
	c196
109*	7 - 1559 - 78176687
	p20:73024355266099724939 (AM)
	c187
110	$2\cdot 3\cdot 5\cdot 4517$
	p20:18443752916913621413  (AM)
	p197
111	$3 \cdot 293 \cdot 431 \cdot 230273 \cdot 209071 \cdot 241423723$
	p10:3182306131 (MF)
	p12:171974155987 (MF)
	p13:1532064083461 (MF)
	p17:59183601887848987 (MF)
	p19:8526805649394145853  (AM)
	p23:27151072184008709784271  (AM)
	p109
	continued

112 2	<sup>3</sup> · 16619 · 449797 · 894009023
112   2	· 10019 · 449191 · 694009023
1 !	17:74225338554790133 (MF)
1 -	23:10021106769497255963093 (MF)
	169
113* 3	· 11 · 13 · 5653 · 1016453 · 16784357
p	18 : 116507891014281007 (AM)
p	37:6844495453726387858061775603297883751 (AM)
	157
114* 2	$\cdot$ 3 $\cdot$ 7 $\cdot$ 178333
c:	227
115* 5	$\cdot$ 17 $\cdot$ 19 $\cdot$ 41 $\cdot$ 36606 $\cdot$ 71518987
p	18 : 283858194594979819 (AM)
	202
1	$^{2}\cdot3^{2}\cdot2239$
	235
1	<sup>2</sup> ·31883
	12: 333699561211 (MF)
	20 : 28437086452217952631 (MF)
1	206
	. 83
	1:33352084523 (MF)
	20 : 20481677004050305811 (MF)
	70 101 100 0001
l i	· 59 · 101 · 139 · 2801
	39
1	· 3 · 5 · 13 · 16693063 41
,	8240783
$\begin{vmatrix} 121 & 27 \\ c2 & \end{vmatrix}$	
1	3 · 23 · 618029123
	4:31949422933783 (MF)
$p_1$	
	7 · 37 · 413923
! -	0:1565875469 (MF)
	6: 5500432543504219 (MF)
c2	` '
1	· 739393
1	6: 1958521545734977 (MF)
c2	
	continued

n	Factors of $Sm(n)$
125*	$3^2 \cdot 5^3 \cdot 4019$
1	p13:7715697265127 (MF)
	c247
126	$2 \cdot 3^2 \cdot 29 \cdot 103 \cdot 70271$
	p20:11513388742821485203 (MF)
	p241
127*	53 · 269 · 4547
	p20:56560310643009044407  (AM)
	c245
128*	$2^3 \cdot 3 \cdot 7 \cdot 11 \cdot 59 \cdot 215329$
	p22:8154316249498591416487 (MF)
	c243
129*	3 · 19
}	c277
130*	2.5
	p12:166817332889 (MF)
	c269
131*	$3 \cdot 19 \cdot 83 \cdot 1693$
	p11:23210501651 (MF)
	p12: 575587270441 (MF)
	c256
132*	$2^2 \cdot 3 \cdot 79$
	p13: 2312656324607 (MF)
100+	c272
133*	p19:8223519074965787731 (AM)
124*	c272
134*	2 · 3 <sup>3</sup> · 73 · 6173
	p16: 5527048386371021 (AM)
	p28 : 1417349652747970442615118133 (AM)   c243
135*	$3^3 \cdot 5 \cdot 11 \cdot 37 \cdot 647$
133	p10:1480867981 (MF)
	p10 : 1480807981 (MF) p12 : 174496625453 (MF)
	-
	p15 : 151994480112757 (MF) c255
136*	$2^{5} \cdot 1259 \cdot 4111$
130	p13: 9485286634381 (MF)
	p13 : 9485280034381 (MF) p26 : 10151962417972135624157641 (AM)
	c253
	continued
L	continued

n	Factors of $Sm(n)$
137*	$3 \cdot 7^2$
	p13:7459866979837 (MF)
	c288
138*	2 · 3 · 181 · 78311 · 914569
	p15:413202386279227 (MF)
	c277
139*	13
	p11 : 62814588973 (MF)
	p12:115754581759 (MF)
	p12:964458587927 (MF)
	p22:9196988352200440482601 (MF)
	c252
140*	$2^2 \cdot 3 \cdot 5 \cdot 23 \cdot 761 \cdot 1873 \cdot 12841$
	p11:34690415939 (MF)
	p18: 226556543956403897 (AM)
	p23: 10856300652094466205709 (AM)
	c248
141	3 · 107171
1.0*	p309
142*	2 · 7 · 4523 · 14303 · 76079
	p22: 2244048237264532856611 (AM)
143*	$c282$ $3^2 \cdot 859$
143	c317
144	$2^3 \cdot 3^2 \cdot 6361$
1.4.4	p13 : 6585181700551 (MF)
	p14: 81557411089043 (MF)
	p21: 165684233831183308123 (MF)
	p271
145*	5 · 96151639
	c326
146*	$2 \cdot 3 \cdot 13 \cdot 83$
	p12:720716898227 (MF)
	p19:1122016187632880261 (MF)
	c296
147*	3 · 59113
	p22: 1833894252004152212837 (AM)
	p31:1519080701040059055565669511153 (MF)
	c276
	continued

n	Factors of $Sm(n)$
148*	$2^2 \cdot 197 \cdot 11927 \cdot 17377 \cdot 273131 \cdot 623321$
140	p13 : 3417425341307 (AM)
	p13 : 3417423341307 (AM) p13 : 4614988413949 (MF)
	c288
149*	3 · 103 · 131 · 1399
149	c331
150*	$2 \cdot 3 \cdot 5^2 \cdot 11 \cdot 23$
100	p16: 2315007810082921 (MF)
	p26: 92477662071402284092009799 (MF)
	c296
151*	7 · 53 · 1801 · 3323
101	c335
152*	$2^4 \cdot 3^2 \cdot 131 \cdot 10613$
102	p20: 29354379044409991753 (AM)
	p22 : 2587833772662908004979 (MF)
	c298
153*	$3^2 \cdot 29 \cdot 7237 \cdot 6987053 \cdot 8237263 \cdot 389365981$
100	c322
154*	$2 \cdot 17 \cdot 19 \cdot 43$
	p18: 444802312089588077 (MF)
	p21 : 855286987917657769927 (EW)
	c311
155	3 · 5 · 66500999
	p24:223237752082537677918401 (EW)
	p323
156*	$2^2 \cdot 3 \cdot 7 \cdot 3307$
	c354
157*	11 · 53 · 492601 · 43169527
	p12:645865664923 (MF)
	p18: 125176035875938771 (MF)
	c318
158*	$2 \cdot 3 \cdot 17 \cdot 29 \cdot 53854663$
	p21 : 164031369541076815133 (EW)
	c334
159*	3 · 71 · 647
	p10:3175105177 (AM)
	p25: 1957802969152764074566129 (EW)
	<i>c</i> 330
160*	$2^3 \cdot 5 \cdot 37 \cdot 130547 \cdot 859933 \cdot 21274133$
	continued

n	Factors of $Sm(n)$
	p27:122800249349203273846720291 (EW)
	c324
161	$3^4 \cdot 59 \cdot 491 \cdot 81705851$
	p360
162*	$2 \cdot 3^5 \cdot 2999$
	p21:393803780657062026421 (AM)
	c351
163*	2381
	p11:72549525869 (AM)
	p12:666733067809 (AM)
	p25: 1550529016982764630292633 (AM)
	c330
164*	$2^2 \cdot 3$
	c383
165*	$3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 247007767$
	p15: 490242053931613 (MF)
	c359
166	2 · 89
	p23:55566524959746113370037 (AM)
	p365
167*	3 · 3313
	c389
168	$2^7 \cdot 3 \cdot 532709$
	p387
169*	2671 - 5233
	c392
170*	$2\cdot 3^2\cdot 5\cdot 7\cdot 701$
	p14:73406007054077 (MF)
	c382
171*	$3^2 \cdot 1237$
	p19:6017588157881558471 (AM)
	c382
172*	$2^2 \cdot 11 \cdot 13 \cdot 37$
	c403
173*	3 · 17 · 53 · 101 · 153 · 11633 · 228673
	c394
174*	
	p22:2928995151034569627547 (AM)
	c381
	continued

n	Factors of $Sm(n)$
175*	52
110	p13 : 2606426254567 (MF)
	c403
176*	$2^3 \cdot 3 \cdot 19 \cdot 1051$
170	p19: 1031835687651103571 (AM)
	c396
177*	3 · 109 · 153277 · 6690569
111	p11: 32545700623 (MF)
	p16: 2984807754776597 (MF)
	c382
178	2
110	
	p13:3144036216187 (MF)
	p17 : 11409535046513339 (MF) p397
179*	$\frac{p_{397}}{3^2 \cdot 7 \cdot 11 \cdot 359}$
119	c423
180*	$2^2 \cdot 3^2 \cdot 5 \cdot 43 \cdot 89 \cdot 7121$
100	c422
181*	31 · 197 · 70999
101	p20: 46096011552749697739 (AM)
	c406
182*	2 · 3 · 123529391
102	c429
183*	$3 \cdot 29 \cdot 661 \cdot 1723$
100	p16: 3346484052265661 (AM)
	c417
184*	$2^4 \cdot 7 \cdot 59 \cdot 191 \cdot 1093 \cdot 1223$
	p11: 22521973429 (MF)
	p17:15219125459582087 (MF)
	p18: 158906425126963139 (MF)
	p19: 2513521443592870099 (MF)
	c369
185*	3 · 5 · 94050577
	p13: 4716042857821 (MF)
	p16: 3479131875325867 (MF)
	c409
186*	$2 \cdot 3 \cdot 1201$
	p21:574850252802945786301 (MF)
	c425
	continued

	· · · · · · · · · · · · · · · · · · ·
n	Factors of $Sm(n)$
187*	349 · 506442073
	c442
188*	$2^2 \cdot 3^3$
	c454
189*	$3^3 \cdot 47 \cdot 1515169$
	p10:1550882611 (MF)
	p10:1687056803 (MF)
	p21: 348528133548561476953 (AM)
	c410
190	2 · 5 · 379
	p23:46645758388308293907739 (AM)
	p435
191*	3 · 13 · 5233
	p12:164130096629 (MF)
	p20: 13806214882775315521 (MF)
4004	c429
192*	$2^3 \cdot 3 \cdot 29 \cdot 41$
100*	c463
193*	7 · 419
104*	c467
194*	2 · 3 · 11 · 31 · 491 · 34188439
	p14: 28739332991401 (MF)
	p16 : 8203347603076921 (MF) p19 : 1507421050431503839 (MF)
	p20: 22805873052490568609 (MF)
	p21:168560953170124281211 (MF)
	p21 : 100500955170124261211 (MF)
195*	$3 \cdot 5 \cdot 397 \cdot 21728563 \cdot 300856949 \cdot 554551531$
150	p10 : 8174619091 (MF)
	c438
196	$2^2 \cdot 17 \cdot 73 \cdot 79$
130	p10:3834513037 (MF)
	p465
197*	$3^2 \cdot 37 \cdot 6277$
10.	p16: 1368971104990459 (MF)
	c461
198*	$2 \cdot 3^2 \cdot 7^2 \cdot 13$
100	c482
199*	151
	continued

n	Factors of $Sm(n)$
200*	c487 2 <sup>5</sup> · 3 · 5 <sup>2</sup> c488

Table 2: Factorizations of Sm(n),  $1 < n \le 200$ 

# 4.2 Reverse Smarandache Factors

	Contributors of Reverse Smarandache factors		
RB	Robert Backstrom	bobb@atinet.com.au	
BD	Bruce Dodson	bad0@lehigh.edu	
MF	Micha Fleuren	michaf@sci.kun.nl	
AM	Allan MacLeod	MACL-MS0@wpmail.paisley.ac.uk	
RS	Ralf Stephan	stephan@tmt.destephan@tmt.de	
PZ	Paul Zimmermann	Paul.Zimmermann@loria.fr	
continued			

Table 3: Contributors of Reverse Smarandache factors

n	Factors of $Rsm(n)$
2	3.7
3	3.107
4	29.149
5	3.19.953
6	3.218107
7	19.402859
8	3 <sup>2</sup> .1997.4877
9	$3^2.17^2.379721$
10	7.28843.54421
11	3
	p12:370329218107
12	3.7
	p13:5767189888301
13	17.3243967.237927839
14	3.11.24769177
	p10:1728836281
15	$3.13.19^2.79$
	continued

n	Factors $Rsm(n)$
Ì	p15: 136133374970881
16	23.233.2531
	p16:1190788477118549
17	$3^2.13.17929.25411.47543.677181889$
18	3 <sup>2</sup> .11 <sup>2</sup> .19.23.281.397.8577529.399048049
19	17.19
	p13: 1462095938449
	p14:40617114482123
20	3.89.317.37889
	p21:629639170774346584751
21	3.37
	p12:732962679433
	p19:2605975408790409767
22	13.137.178489
	p13:1068857874509
	p14:65372140114441
23	3.7.191
	p32:578960862423763687712072079528211
24	3.107.457.57527
	p28:28714434377387227047074286559
25	11.31.59.158820811.410201377
	p20:19258319708850480997
26	3 <sup>3</sup> .929.1753.2503.4049.11171
	p24: 527360168663641090261567
27	3 <sup>5</sup> .83
	p10: 3216341629
	p13:7350476679347
20	p18:571747168838911343
28	23.193.3061
	p19: 2150553615963932561
	p21:967536566438740710859
29	3.11.709.105971.2901761
	p10:1004030749
	p24:405373772791370720522747
30	3.73.79.18041.24019.32749
	<i>p</i> 10 : 5882899163
.	p24 : 209731482181889469325577
31	7.30331061
	<i>p</i> 45 : 147434568678270777660714676905519165947320523
	continued

	Footoes Dom(n)
n 20	Factors Rsm(n)
32	3.17.1231.28409
	p12: 103168496413
	p35: 17560884933793586444909640307424273
33	3.7.7349
	p10:9087576403
	p42: 237602044832357211422193379947758321446883
34	89.488401.2480227.63292783.254189857
İ	p10: 3397595519
	p19: 5826028611726606163
35	3 <sup>2</sup> .881.1559.755173.7558043
	p10:1341824123
	p16: 4898857788363449
_	p16: 7620732563980787
36	$3^2.11^2.971$
	p13: 1114060688051
	p22:1110675649582997517457
	p24:277844768201513190628337
37	29.2549993
	p20:39692035358805460481
00	p38:12729390074866695790994160335919964253
38	3.9833
00	<i>p</i> 63
39	3.19.73.709.66877
40	<i>p</i> 58
40	11.41.199
	p27: 537093776870934671843838337
41	<i>p</i> 39 : 837983319570695890931247363677891299117
41	3.29.41.89.3506939
	p14:18697991901857
	p20: 59610008384758528597
42	p28 : 3336615596121104783654504257
42	3.13249.14159.25073
	p10 : 6372186599
43	<i>p</i> 52 52433
43	
	p20 : 73638227044684393717
44	p53
44	3 <sup>2</sup> .7.3067.114883.245653
	p23:65711907088437660760939
	continued

	To-the D ( )
$\frac{n}{n}$	Factors $Rsm(n)$
ا	p41:12400566709419342558189822382901899879241
45	$3^2.23.167.15859.25578743$
	<i>p</i> 65
46	23.35801
	p12:543124946137
ļ	p23:45223810713458070167393
	p43:2296875006922250004364885782761014060363847
47	3.11.31.59
	p16:1102254985918193
	p28: 4808421217563961987019820401
	p38:14837375734178761287247720129329493021
48	3.151.457.990013
	p15: 246201595862687
	p24:636339569791857481119613
	p39:15096613901856713607801144951616772467
49	71
	p10:9777943361
	p77
50	3.157.3307
	p13:3267926640703
	p30:771765128032466758284258631297
	p43:1285388803256371775298530192200584446319323
51	3.11
	p92
52	7.29.670001
	p12:403520574901
	p14:70216544961751
	p16:1033003489172581
	p47: 13191839603253798296021585972083396625125257997
53	$3^4.499.673.6287.57653.199236731$
	p16: 1200017544380023
	p28:1101541941540576883505692003
	p31:2061265130010645250941617446327
54	$3^3.7^4.13.1427.632778317$
	p11:57307460723
	p13:7103977527461
	p15:617151073326209
	p43: 2852320009960390860973654975784742937560247
55	357274517.460033621
	continued

n	Factors $Rsm(n)$
	184
56	$\begin{vmatrix} p_{04} \\ 3.13^2 \end{vmatrix}$
30	p14:85221254605693
	<b>1</b> -
57	<i>p</i> 87   3.41
31	3.41   p11 : 25251380689
	, -
58	p93
90	11.2425477
	p15: 178510299010259
	p18: 377938364291219561
	p28: 5465728965823437480371566249
F0*	p40:5953809889369952598561290100301076499293
59*	3
60	c109
00	3
	p10:8522287597
61	p101
01	13.373
	p22:6399032721246153065183
	p42: 214955646066967157613788969151925052620751 (RB)
62	p46:9236498149999681623847165427334133265556780913
02	p13:3921939670009
	p13: 3921939070009 p14: 11729917979119
	p14 : 11729917979119 p28 : 9383645385096969812494171823
	<i>p</i> 50 : 43792191037915584824808714186111429193335785529359
63	3 <sup>2</sup> .97.26347
00	p24 : 338856918508353449187667
	p86
64	397.653
01	p12 : 459162927787
	p14 : 27937903937681
	p24 : 386877715040952336040363
İ	p65
65*	3.7.23.13219.24371
	c110
66	3.53.83.2857.1154129.9123787
	p103
67*	43
	continued
	continued

$\overline{n}$	Factors $Rsm(n)$
	p11:38505359279
	c113
68	3.29.277213.68019179.152806439
	p18: 295650514394629363
	p20: 14246700953701310411
	p67
69	3.11.71.167.1481
	p10: 2326583863
	p23:19962002424322006111361
	p89
70	1157237.41847137
	p22:8904924382857569546497
	p96
71	3 <sup>2</sup> .17.131.16871
	p10:1504047269
	p11:82122861127
	p19:1187275015543580261
	p87
72	$3^2.449.1279$
	p129
73	7.11.21352291
	p10:1051174717
	p17: 92584510595404843
	p20: 33601392386546341921
	p83
74	3.177337
	p10:6647068667
	p11:31386093419
	p15: 669035576309897
	p16: 4313244765554839
	p32: 67415094145569534144512937880453 (PW)
	<i>p</i> 51
75	3.7.230849.7341571.24260351
	p10:1618133873
	p14: 19753258488427
	p17: 46752975870227777
	p28: 7784620088430169828319398031 (PW)
76*	<i>p</i> 53
10	53
l	continued

n	Factors Rsm(n)
	c142
77	3.919
	p15: 571664356244249
	p22 : 6547011663195178496329 (PW)
	p27:591901089382359628031506373 (BD)
	p33: 335808390273971395786635145251293 (PW)
	p46: 3791725400705852972336277620397793613760330637
78*	3.17.47
	p14:17795025122047
	c131
79	160591
	p15: 274591434968167
	p19:1050894390053076193
	p112
80*	3 <sup>3</sup> .11.443291.1575307
	p17:19851071220406859
	c121
81	$3^3.23^2.62273.22193.352409$
	p15:914359181934271 (MF)
	p120
82	PRIME! (RS)
83*	3
	c157
84*	3.11.47.83.447841.18360053
	p14:53294058577163  (MF)
	c130
85	p12:465619934881 (MF)
	p22:5013354844603778080337 (AM)
0.0*	p128
86*	3.7.3761.205111.16080557.16505767
077	c139
87	3.2423
	p25 : 4433139632126658657934801 (AM)
	p30:951802198132419645688492825211 (MF)
88*	p107 73.8747
00	c162
89*	3 <sup>2</sup> .19.7052207
09	319.7052207 c161
	continued
	continued

n	Factors Rsm(n)
90*	3 <sup>2</sup> .157.257.691
	c140
91*	11.29.163.3559.2297.22899893
	p15: 350542343218231 (MF)
-	p25:8365221234379371317434883 (MF)
	c115
92*	3.17.113.376589.3269443.6872137
	c153
93*	3.13.69317.14992267
	c164
94*	7.593.18307
	p11:51079607083 (MF)
	c161
95*	3.11.13.53.157.623541439
1	c166
96*	3.7.211.14563.2297
	c172
97*	1553
	c182
98	$3^2.101.401.5741.375373$
	p173
99*	$3^2.109.41829209$
	p12:174489586693 (MF)
	c168
100*	13.6779
	p11:48856332919 (MF)
	p26: 41858129936073024200781901 (MF)
	c150
101*	3
	<i>p</i> 11 : 16320902651 (MF)
	p19:3845388775716560041 (MF)
	p33: 693173763848292948494434792706137 (AM)
	c132
102*	3.101.103.36749
	p11:10189033219 (MF)
	p20: 23663501701518727831 (AM)
	p26: 52648894306108287380398039 (AM)
	c133
103*	19.29.103.3119.154009291
	continued

n	Factors Rsm(n)
90*	3 <sup>2</sup> .157.257.691
	c140
91*	11.29.163.3559.2297.22899893
	p15: 350542343218231 (MF)
-	p25:8365221234379371317434883 (MF)
	c115
92*	3.17.113.376589.3269443.6872137
	c153
93*	3.13.69317.14992267
	c164
94*	7.593.18307
	p11:51079607083 (MF)
	c161
95*	3.11.13.53.157.623541439
1	c166
96*	3.7.211.14563.2297
	c172
97*	1553
	c182
98	$3^2.101.401.5741.375373$
	p173
99*	$3^2.109.41829209$
	p12:174489586693 (MF)
	c168
100*	13.6779
	p11:48856332919 (MF)
	p26: 41858129936073024200781901 (MF)
	c150
101*	3
	<i>p</i> 11 : 16320902651 (MF)
	p19:3845388775716560041 (MF)
	p33: 693173763848292948494434792706137 (AM)
	c132
102*	3.101.103.36749
	p11:10189033219 (MF)
	p20: 23663501701518727831 (AM)
	p26: 52648894306108287380398039 (AM)
	c133
103*	19.29.103.3119.154009291
	continued

n	Factors $Rsm(n)$
<u> </u>	p12: 329279243129 (MF)
	p13: 1240336674347 (MF)
	i <del>-</del>
104*	c161
104*	3.7.60953.1890719
	p11: 10446899741 (MF)
	p15: 216816630080837 (MF)
	p19:1614245774588631629 (MF)
	c149
105*	3.7.859.6047.63601
	c194
106*	p22:1912037972972539041647 (AM)
	p22: 3052818746214722908609 (AM)
	c167
107*	3 <sup>3</sup> .13.4519.114967
	p10:1425213859 (MF)
	p14:17641437858251 (MF)
	c179
108	3 <sup>3</sup> .23.457.1373
	p12:605434593221 (MF)
	p12 : 703136513561 (MF)
	p183
109	11.29.31 <sup>2</sup> .1709.30345569
	p14: 42304411918757 (MF)
	p189
110*	3.11.19.53.229.24672421
110	p24 : 611592384837948878235019 (AM)
	c183
111*	3.61.269.470077.143063.544035253
111	c200
112*	137
***	p12 : 262756224547 (MF)
	c214
113*	3.19.45061.111211
113	c219
114*	3.19.53.59
114	c228
115*	137.509.1720003
110	c226
116	3==3
116	32.83.103.156307.176089.21769127
	continued

n	Factors Rsm(n)
	p217
117	$\frac{1}{3^2}$
	p242
118	7.4603
	p241
119*	3.7
	c247
120*	3.73
	c249
121*	31.371177
	c248
122*	3.17
	p11 : 91673873887 (MF)
	c245
123	3.1197997
	p11: 15744706711 (MF)
1015	p244
124*	37.1223
105	c259
125	32.59.83
	p10:5961006911 (MF)
	p13:1096598255677 (MF) p240
126*	$\begin{vmatrix} p^{240} \\ 3^{2}.13.68879.135342173 \end{vmatrix}$
120	c255
127*	97
12.	p16: 1385409249340483 (AM)
	c255
128*	3.34613.29497667
	c263
129*	3.23.1213.82507
	p12: 420130412231 (MF)
	c257
130*	31.263.86969.642520369
	c264
131*	3.11.4111.852143
	p12:606617222863 (MF)
	p23: 33247682213571703426139 (AM)
	c239
	continued

n	Factors Rsm(n)
132	3.7.11.41.43.31259.69317.180307.199313
	p17: 16995472858509251 (MF)
Ì	p20:56602777258539682957 (AM)
	p226
133	7.13
	p20: 22533511116338912411 (AM)
	p269
134*	33.37.29004967
	p17: 60164048964096599 (AM)
40=#	c266
135*	33.211.5393.98563
	p12: 207481965329 (MF)
	p22:6789282931372049267693 (AM)
196*	c251
136* 137*	3.179
137	p22: 6796599525965619205571 (AM)
İ	c278
138*	3.119611.314087617
100	c292
139*	
140*	3.317.772477
	p15:153629260660723 (AM)
	c289
141*	3.631.65831
	c307
142*	859.2377.2909.6521
	p14:41190901651547 (MF)
	c291
143	32.93971
	p12: 9053448211979 (MF)
144*	<i>p</i> 302 3 <sup>2</sup>
144"	<u> </u>
	p19:5028055908018884749 (MF) c304
145*	57719.2691841
140	p20: 45690580335973653419 (MF)
	c296
146*	3.7 <sup>2</sup> .277.19319.55807
	continued

	·
n	Factors $Rsm(n)$
	p13: 2454423915989 (MF)
	c304
147*	3.7 <sup>2</sup> .19.31.15467623
	c321
148*	p20: 33825333713396366003 (AM)
	p23: 25082957895838310384953 (AM)
	c294
149*	3.109.34442413
	c329
150*	3.59.257
	c337
151*	p10:7134941903 (MF)
	c335
152	$3^2.13$
	p21:412891312089439668533 (MF)
	p325
153*	3 <sup>2</sup> .67793
	p18: 237333508084627139 (MF)
	c328
154*	11.53861
	p10:1118399729 (MF)
	c339
155*	3.41.33842293
	c347
156*	3.21961
	c355
157*	p10: 4136915059 (MF)
	c353
158*	3.31.89209
	p10: 1379633699 (MF)
	p14: 54957888020501 (MF)
	c336
159*	3.13.5669.11213.816229087
	p10:50611041883 (MF)
	c340
160*	7.942037.1223207
	p21:125729584994875519171 (AM)
100	c339
161*	3 <sup>7</sup> .7.37.67.6521.826811.6018499
L	continued

n	Factors Rsm(n)
	p23:77558900444266075256801 (MF)
	<i>c</i> 328
162*	3 <sup>4</sup> .1295113.202557967
	c361
163*	p16:1139924663537993 (MF)
	p17:17672171439068059  (MF)
	c350
164	3.193
	p24:105444241520715055381519  (AM)
	p358
165*	3
	c386
166*	p15: 396444477663149 (MF)
	p32:15221332593310506150048824812249  (AM)
	c344
167*	3.17.373.7346281.8927551.194571659
	p20:68277637362521294401  (AM)
	c347
168*	3.59.35537.68102449
	p19: 7766035514845504007 (AM)
	c362
169*	-0
170	$3^2.23.$
	p16: 3737994294192383 (MF)
	p384
171*	3 <sup>2</sup> .37
	p12: 237089136881 (MF)
	p19: 2153684224509566597 (MF)
	p21:175530075465216996787 (MF)
	p22:8105319358780665120301 (MF)
4.50	c330
172	17.29.281
	p10:4631571401 (MF)
	p11: 31981073881 (MF)
	p15: 119749047957053 (MF)
150*	p368
173*	3.1787
1,5.4	c407
174*	3.7.269.397.156894809
<u> </u>	continued

$\overline{n}$	Factors $Rsm(n)$
<del></del>	c399
175*	7.11
1.0	p10: 3763462823 (MF)
	c405
176*	3.11.47.49613
1.0	p13:2800890701267 (MF)
	p15: 315698062297249 (MF)
	p27: 880613122533775176075766757 (MF)
	c358
177	3.73.1753
	p14: 29988562180903 (MF)
	p404
178	13.47.353.644951.487703.1436731
	p12:728961984851 (MF)
	p14: 34686545199997 (MF)
	p14: 36329334000803 (MF)
	p364
179*	$3^2.23.43$
	p14:50981967790529 (MF)
	c411
180*	$3^2.29$
	p17: 33644294710009721 (MF)
	c413
181*	325251083
	p17: 57421731284347247 (MF)
	c410
182*	3.107.5568133
	p12: 139065644033 (MF)
100*	c417
183*	3.23.89
104*	c437
184*	23.19531
	p15 : 196140464783429 (MF) c424
185*	3.13.919
100	
	p11 : 32173266383 (MF) c432
186*	3.23
100	c448
	continued

n	Factors Rsm(n)
187*	61.83.103.523.3187
	p19: 1018598504636281577 (MF)
	c423
188*	3 <sup>3</sup> .7.7681.65141
1	c445
189*	3 <sup>3</sup> .7.2039.3823.9739.212453.10586519
	c433
190*	83.107.1871.25346653
	c447
191	3.809
	p18:627089953107590081 (MF)
	p444
192*	3.2549
	c464
193*	47.503.12049
	c463
194*	3.179
	p22:8000103240831609636731  (AM)
	p23:77947886830169946060329 (MF)
	c426
195*	3.79.8219
	c471
196*	19
	p16: 8982588119304797 (AM)
10=+	c463
197*	3 <sup>2</sup> .11.43.11743.125201.867619
	p11:61951529111 (MF)
	p14 : 27090970290157 (MF) c440
198	$\begin{vmatrix} 6440 \\ 3^2.11.37.2837 \end{vmatrix}$
198	
	p19: 1245013373736039779 (MF)
199*	p461 103.2377
199	103.2311 c484
200*	3.1666421
200	c485
	C400

Table 4: Factorizations of  $\operatorname{Rsm}(n)$ ,  $1 < n \le 200$ 

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The very latest up to date representation of this list can be found at the next URL: http://www.sci.kun.nl/sigma/Persoonlijk/michaf/ecm/.

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### **Smarandache Continued Fractions**

## Henry Ibstedt<sup>1</sup>

Abstract: The theory of general continued fractions is developed to the extent required in order to calculate Smarandache continued fractions to a given number of decimal places. Proof is given for the fact that Smarandache general continued fractions built with positive integer Smarandache sequences having only a finite number of terms equal to 1 is convergent. A few numerical results are given.

#### Introduction

The definitions of Smarandache continued fractions were given by Jose Castillo in the Smarandache Notions Journal, Vol. 9, No 1-2 [1].

A Smarandache Simple Continued Fraction is a fraction of the form:

$$a(1) + \frac{1}{a(2) + \frac{1}{a(3) + \frac{1}{a(4) + \frac{1}{a(5) + \cdots}}}}$$

where a(n), for n≥1, is a Smarandache type Sequence, Sub-Sequence or Function.

Particular attention is given to the Smarandache General Continued Fraction defined as

$$a(1) + \frac{b(1)}{a(2) + \frac{b(2)}{a(3) + \frac{b(3)}{a(4) + \frac{b(4)}{a(5) + \cdots}}}$$

where a(n) and b(n), for  $n \ge 1$ , are Smarandache type Sequences, Sub-Sequences or Functions.

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As a particular case the following example is quoted

$$\begin{array}{r}
 1 + \frac{1}{12 + \frac{21}{123 + \frac{321}{1234 + \frac{4321}{12345 + \cdots}}}} \\
 \hline
 1 + \frac{1}{123 + \frac{321}{12345 + \cdots}} \\
 1 + \frac{1}{123 + \frac{321}{12345 + \cdots}} \\
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 1 + \frac{1}{123 + \frac{321}{12345 + \cdots}} \\
 1 + \frac{1}{123 + \frac{321}{12345 + \cdots}} \\$$

Here 1, 12, 123, 1234, 12345, ... is the Smarandache Consecutive Sequences and 1, 21, 321, 4321, 54321, ... is the Smarandache Reverse Sequence.

The interest in Castillo's article is focused on the calculation of such fractions and their possible convergens when the number of terms approaches infinity. The theory of simple continued fractions is well known and given in most standard textbooks in Number Theory. A very comprehensive theory of continued fractions, including general continued fractions is found in *Die Lehre von den Kettenbrüchen* [2]. The symbols used to express facts about continued fractions vary a great deal. The symbols which will be used in this article correspond to those used in Hardy and Wright *An Introduction to the Theory of Numbers* [3]. However, only simple continued fractions are treated in the text of Hardy and Wright. Following more or less the same lines the theory of general continued fractions will be developed in the next section as far as needed to provide the necessary tools for calculating Smarandache general continued fractions.

#### General Continued Fractions

We define a finite general continued fraction through

$$C_{n} = q_{0} + \frac{r_{1}}{q_{1} + \frac{r_{2}}{q_{2} + \frac{r_{3}}{q_{3} + \frac{r_{4}}{q_{4} + \dots}}}} = q_{0} + \frac{r_{1}}{q_{1} + \frac{r_{2}}{q_{2} + \frac{r_{3}}{q_{3} + \frac{r_{3}}{q_{4} + \dots}}} \frac{r_{n}}{q_{n}}$$
 (1)

where  $\{q_0, q_1, q_2, \dots, q_n\}$  and  $\{r_1, r_2, r_3 \dots r_n\}$  are integers which we will assume to be positive.

The above definition is an extension of the definition of a simple continued fraction where  $r_1=r_2=\ldots=r_n=1$ . The theory developed here will apply to simple continued fractions as well by replacing  $r_k$  (k=1, 2, ...) in formulas by 1 and simply ignoring the reference to  $r_k$  when not relevant.

The formula (1) will usually be expressed in the form

$$C_{n}=[q_{0},q_{1},q_{2},q_{3},...q_{n},r_{1},r_{2},r_{3}...r_{n}]$$
(2)

For a simple continued fraction we would write

$$C_n = [q_0, q_1, q_2, q_3, \dots q_n]$$
 (2')

If we break off the calculation for m≤n we will write

$$C_{m}=[q_{0},q_{1},q_{2},q_{3},...q_{m},r_{1},r_{2},r_{3}...r_{m}]$$
(3)

Equation (3) defines a sequence of finite general continued fractions for m=1, m=2, m=3, ... Each member of this sequence is called a **convergent** to the continued fraction

Working out the general continued fraction in stages, we shall obviously obtain expressions for its convergents as quotients of two sums, each sum comprising various products formed with  $q_0, q_1, q_2, ...q_m$  and  $r_1, r_2, ...r_m$ .

If m=1, we obtain the first convergent

$$C_1 = [q_0, q_1, r_1] = q_0 + \frac{r_1}{q_1} = \frac{q_0 q_1 + r_1}{q_1}$$
(4)

For m=2 we have

$$C_2 = [q_0, q_1, q_2, r_1, r_2] = q_0 + \frac{q_2 r_1}{q_1 q_2 + r_2} = \frac{q_0 q_1 q_2 + q_0 r_2 + q_2 r_1}{q_1 q_2 + r_2}$$
 (5)

(6)

In the intermediate step the value of  $q_1 + \frac{r_2}{q_2}$  from the previous calculation has been quoted, putting  $q_1$ ,  $q_2$  and  $r_2$  in place of  $q_0$ ,  $q_1$  and  $r_1$ . We can express this by

Proceeding in the same way we obtain for m=3

 $C_2=[q_0,[q_1,q_2,r_2],r_1]$ 

$$C_{3}=[q_{0},q_{1},q_{2},q_{3},r_{1},r_{2},r_{3}]=q_{0}+\frac{(q_{2}q_{3}+r_{3})r_{1}}{q_{1}q_{2}q_{3}+q_{1}r_{3}+q_{3}r_{2}}=\frac{q_{0}q_{1}q_{2}q_{3}+q_{0}q_{1}r_{3}+q_{0}q_{3}r_{2}+q_{2}q_{3}r_{1}+r_{1}r_{3}}{q_{1}q_{2}q_{3}+q_{1}r_{3}+q_{3}r_{2}}$$

$$(7)$$

or generally

$$C_{m}=[q_{0},q_{1},\ldots q_{m-2},[q_{m-1},q_{m},r_{m}],r_{1},r_{2},\ldots r_{m-1}]$$
(8)

which we can extend to

$$C_{n}=[q_{0},q_{1},...q_{m-2},[q_{m-1},q_{m},...q_{n},r_{m},...r_{n}],r_{2},r_{2},...r_{m-1}]$$
(9)

#### Theorem 1:

Let A<sub>m</sub> and B<sub>m</sub> be defined through

$$A_0 = q_0, \quad A_1 = q_0 q_1 + r_1, \qquad A_m = q_m A_{m-1} + r_m A_{m-2} \quad (2 \le m \le n)$$

$$B_0 = 1, \quad B_1 = q_1, \qquad B_m = q_m B_{m-1} + r_m B_{m-2} \quad (2 \le m \le n)$$
(10)

then  $C_m=[q_0,q_1,...q_m,r_1,...r_m]=\frac{A_m}{B_m}$ , i.e.  $\frac{A_m}{B_m}$  is the mth convergent to the general continued fraction.

**Proof:** The theorem is true for m=0 and m=1as is seen from  $[q_0] = \frac{q_0}{1} = \frac{A_0}{B_0}$  and  $[q_0, q_1, r_1] = \frac{q_0 q_1 + r_1}{q_1} = \frac{A_1}{B_1}$ . Let us suppose that it is true for a given m<n. We will induce that it is true for m+1

$$\begin{split} [\mathbf{q}_{0},\mathbf{q}_{1},\ldots\mathbf{q}_{m+1},\mathbf{r}_{1},\ldots\mathbf{r}_{m+1}] &= [\mathbf{q}_{0},\mathbf{q}_{1},\ldots\mathbf{q}_{m-1},[\mathbf{q}_{m},\mathbf{q}_{m+1},\mathbf{r}_{m+1}],\mathbf{r}_{1},\ldots\mathbf{r}_{m}] \\ &= \frac{[q_{m},q_{m+1},r_{m+1}]A_{m-1}+r_{m}A_{m-2}}{[q_{m},q_{m+1},r_{m+1}]B_{m-1}+r_{m}B_{m-2}} \\ &= \frac{(q_{m}+\frac{r_{m+1}}{q_{m+1}})A_{m-1}+r_{m}A_{m-2}}{(q_{m}+\frac{r_{m+1}}{q_{m+1}})B_{m-1}+r_{m}B_{m-2}} \\ &= \frac{q_{m+1}(q_{m}A_{m-1}+r_{m}A_{m-2})+r_{m+1}A_{m-1}}{q_{m+1}(q_{m}B_{m-1}+r_{m}B_{m-2})+r_{m+1}B_{m-1}} \\ &= \frac{q_{m+1}A_{m-1}+r_{m+1}A_{m-1}}{q_{m+1}B_{m-1}+r_{m+1}B_{m-1}} = \frac{A_{m+1}}{B_{m+1}} \end{split}$$

The recurrence relations (10) provide the basis for an effective computer algorithm for successive calculation of the convergents  $C_m$ .

### Theorem 2:

$$A_{m}B_{m-1}-B_{m}A_{m-1}=(-1)^{m-1}\prod_{k=1}^{m}r_{k}$$
(11)

**Proof:** For m=1 we have  $A_1B_0-B_1A_0=q_0q_1+r_1-q_0q_1=r_1$ .

$$A_m B_{m-1} - B_m A_{m-1} = (q_m A_{m-1} + r_m A_{m-2}) B_{m-1} - (q_m B_{m-1} + r_m B_{m-2}) A_{m-1} = -r_m (A_{m-1} B_{m-2} - B_{m-1} A_{m-2})$$

By repeating this calculation with m-1, m-2, ..., 2 in place of m, we arrive at

$$A_{m}B_{m-1}-B_{m}A_{m-1} = \dots = (A_{1}B_{0}-B_{1}A_{0})(-1)^{m-1} \prod_{k=2}^{m} r_{k} = (-1)^{m-1} \prod_{k=1}^{m} r_{k}$$

Theorem 3:

$$A_{m}B_{m-2}-B_{m}A_{m-2}=(-1)^{m}q_{m}\prod_{k=1}^{m-1}r_{k}$$
(12)

**Proof:** This theorem follows from theorem 3 by inserting expressions for A<sub>m</sub> and B<sub>m</sub>

$$A_{m}B_{m-2}-B_{m}A_{m-2}=(q_{m}A_{m-1}+r_{m}A_{m-2})B_{m-2}-(q_{m}B_{m-1}+r_{m}B_{m-2})A_{m-2}=$$

$$q_{m}(A_{m-1}B_{m-2}-B_{m-1}A_{m-2})=(-1)^{m}q_{m}\prod_{k=1}^{m-1}r_{k}$$

Using the symbol  $C_m = \frac{A_m}{B_m}$  we can now express important properties of the number sequence  $C_m$ , m=1, 2, ..., n. In particular we will be interested in what happens to  $C_n$  as n approaches infinity.

From (11) we have

$$C_n - C_{n-1} = \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = \frac{(-1)^{n-1} \prod_{k=1}^{n} r_k}{B_{n-1} B_n}$$
(13)

while (12) gives

$$C_n - C_{n-2} = \frac{A_n}{B_n} - \frac{A_{n-2}}{B_{n-2}} = \frac{(-1)^{n-1} q_n \prod_{k=1}^{n-1} r_k}{B_{n-2} B_n}$$
(14)

We will now consider infinite positive integer sequences  $\{q_0, q_1, q_2, ....\}$  and  $\{r_1, r_2, ....\}$  where only a finite number of terms are equal to 1. This is generally the case for Smarandache sequences. We will therefore prove the following important theorem.

#### Theorem 4:

A general continued fraction for which the sequences  $q_0$ ,  $q_1$ ,  $q_2$ , .... and  $r_1$ ,  $r_2$ , .... are positive integer sequences with at most a finite number of terms equal to 1 is convergent.

**Proof:** We will first show that the product  $B_{n-1}B_n$ , which is a sum of terms formed by various products of elements from  $\{q_1, q_2, \dots, q_n, r_1, r_2, \dots, r_{n-1}\}$ , has one term which is a multiple of  $\sum_{k=2}^{n} r_k$ . Looking at the process by which we calculated  $C_1$ ,  $C_2$ , and  $C_3$ , equations

4, 5 and 7, we see how terms with the largest number of factors  $r_k$  evolve in numerators and denominators of  $C_k$ . This is made explicit in figure 1.

	C <sub>1</sub>	. C₂	C <sub>3</sub>	C₄	C <sub>5</sub>	C٤	C7	C <sub>8</sub>
Num. Am	r <sub>1</sub>	Clor <sub>2</sub>	f1f3	Q0[2[4	F1F3F5	Q0[2[4[6	[1[3[5[7	90/2/4/6/8
Den. 8 <sub>m</sub>	-	Γ <sub>2</sub>	Q1f3	f2f4	Q1F3F5	12848	Q1/3/3/7	8]a]a]S]

Figure 1. The terms with the largest number of r-factors in numerators and denominators.

As is seen from figure 1 two consecutive denominators  $B_nB_{n-1}$  will have a term with  $r_2r_3 \dots r_n$  as factor. This means that the numerator of (13) will not cause  $C_n-C_{n-1}$  to diverge. On the other hand  $B_{n-1}B_n$  contains the term  $(q_1q_2 \dots q_{n-1})^2q_n$  which approaches  $\infty$  as  $n\to\infty$ . It follows that  $\lim (C_n-C_{n-1})=0$ .

From (14) we see that

- 1. If n is odd, say n=2k+1, than  $C_{2k+1} < C_{2k-1}$  forming a monotonously decreasing number sequence which is bounded below (positive terms). It therefore has limit.  $\lim_{k\to\infty} C_{2k+1} = C_1.$
- 2. If n is even, n=2k, than  $C_{2k} > C_{2k-2}$  forming a monotonously increasing number sequence. This sequence has an upper bound because  $C_{2k} < C_{2k+1} \to C_1$  as  $k \to \infty$ . It therefore has limit.

$$\lim_{k \to \infty} C_{2k} = C_2.$$

3. Since  $\lim_{n\to\infty} (C_n - C_{n-1}) = 0$  we conclude that  $C_1 = C_2$ . Consequently  $\lim_{n\to\infty} C_n = C$  exists.

#### Calculations

A *UBASIC* program has been developed to implement the theory of Smarandache general continued fractions. The same program can be used for classical continued fractions since these correspond to the special case of a general continued fraction where  $r_1=r_2=\ldots=r_n=1$ .

The complete program used in the calculations is given below. The program applies equally well to simple continued fractions by setting all element of the array R equals to 1.

- 10 point 10 20 dim Q(25),R(25),A25),B25)
- 30 input "Number of decimal places of accuracy: ";D
- 40 input "Number of input terms for R (one more for Q)";N%
- 50 cis
- 60 for 1%=0 to N%:read Q(1%):next
- 70 data
- 80 for 1%=1 to N%:read R(1%):next
- 90 data

- 'The relevant data  $a_0$ ,  $a_1$ , ...

  'The relevant data for  $r_1, r_2$ , ...
- 100 print tab(10);"Smarandache Generalized Continued Fraction"
- 110 print tab(10);"Sequence Q:";
- 120 for I%=0 to 6:print Q(I%);:next:print " ETC"
- 130 print tab(10);"Sequence R:";
- 140 for I%=1 to 6:print R(I%);:next:print "ETC"

```
150 print tab(10);"Number of decimal places of accuracy: ";D
160 A(0)=Q(0):B(0)=1
                                                                 'Initiating recurrence algorithm
170 A(1)=Q(0)*Q(1)+R(1):B(1)=Q(1)
180 Delta=1:M=1
                                                                 'M=loop counter
190 while abs(Delta)>10^(-D)
                                                                  'Convergens check
200 inc M
210 A(M)=Q(M)*A(M-1)+R(M)*A(M-2)
                                                                 'Recurrence
220 B(M)=Q(M)*B(M-1)+R(M)*B(M-2)
230 Delta=A(M)/B(M)-A(M-1)/B(M-1)
                                                                  'Cm-Cm-1
240 wend
250 print tab(10);"An/Bn=";:print using(2,20),A(M)/B(M)
                                                                 'C<sub>n</sub> in decimalform
260 print tab(10);"An/Bn=";::print A(M);"/";B(M)
                                                                 'C<sub>n</sub> in fractional form
270 print tab(10);"Delta=";:print using(2,20),Delta;
                                                                 'Delta=Last difference
280 print " for n=";M
                                                                 'n=number of iterations
290 print
300 end
```

To illustrate the behaviour of the convergents  $C_n$  have been calculated for  $q_1=q_2=...=q_n=1$  and  $r_1=r_2=...=r_n=10$ . The iteration of  $C_n$  is stopped when  $\Delta_n=|C_n-C_{n-1}|<0.01$ . Table 1 shows the result which is illustrated in figure 2. The factor  $(-1)^{n-1}$  in (13) produces an oscillating behaviour with diminishing amplitude approaching  $\lim_{n \to \infty} C_n = C$ 

Table 1. Value of convergents C<sub>n</sub> for qs[1,1,...} and rs[10,10,...]

n	1	2	3	4	5	6	7	8	9	10	11
Cn	_11	1.91	6.24	2.6	4.84	3.07	4.26	3.35	3.99	3.51	3.85
n	12	13	14	15	16	17	18	19	20	21	22
Cn	3.6	3.78	3.65	3.74	3.67	3.72	3.69	3.71	3.69	3.71	3.7

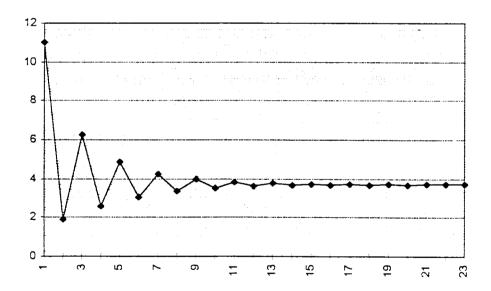


Figure 2. Cn as a function of n

A number of sequences, given below, will be substituted into the recurrence relations (10) and the convergence estimate (13).

```
\begin{split} &S_1 \! = \! \{1, 1, 1, \ldots, \} \\ &S_2 \! = \! \{1, 2, 1, 2, 1, 2, \ldots, \} \\ &S_3 \! = \! \{3, 3, 3, 3, 3, 3, \ldots, \} \\ &S_4 \! = \! \{1, 12, 123, 1234, 12345, 123456, \ldots, \} \\ &S_5 \! = \! \{1, 21, 321, 4321, 54321, 654321, \ldots, \} \\ &CS_1 \! = \! \{1, 1, 2, 8, 9, 10, 512, 513, 514, 520, 521, 522, 729, 730, 731, 737, 738, \ldots \} \\ &NCS_1 \! = \! \{1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30, \ldots \} \end{split}
```

The Smarandache CS1 sequence definition: CS1(n) is the smallest number, strictly greater than the previous one (for  $n\geq 3$ ), which is the cubes sum of one or more previous distinct terms of the sequence.

The Smarandache NCS1 sequence definition: NCS1(n) is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of one or more previous distinct terms of the sequence.

These sequences have been randomly chosen form a large number of Smarandache sequences [5].

As expected the last fraction in table 2 converges much slower than the previous one. These general continued fractions are, of course, very artificial as are the sequences on which they are based. As is often the case in empirical number theory it is not the individual figures or numbers which are of interest but the general behaviour of numbers and sequences under certain operations. In the next section we will carry out some experiments with simple continued fractions.

#### **Experiments with Simple Continued Fractions**

The theory of simple continued fractions is covered in standard textbooks. Without proof we will therefore make use of some of this theory to make some more calculations. We will first make use of the fact that

There is a one to one correspondence between irrational numbers and infinite simple continued fractions.

The approximations given in table 2 expressed as simple continued fractions would therefore show how these are related to the corresponding general continued fractions.

Table 2. Calculation of general continued fractions

Q	R	n	$\Delta_n$	C <sub>n</sub> (dec.form)	C <sub>n</sub> (fraction)
$\overline{S_1}$	S <sub>1</sub>	18	-9·10 <sup>-8</sup>	1.6180339	6765
					4181
$\overline{S_2}$	$S_1$	13	8-10-8	1.3660254	7953
					5822
$S_2$	S <sub>3</sub>	22	-9·10 <sup>-8</sup>	1.8228756	1402652240
					769472267
S <sub>4</sub>	Sı	2	-7·10 <sup>-6</sup>	1.04761	7063
					6742
		3	5·10 <sup>-12</sup>	1.04761198457	30519245
					29132203
		4	-2·10 <sup>-20</sup>	1.0476119845794017019	1657835914708
					1582490405905
S <sub>4</sub>	S <sub>5</sub>	2	-1.10-3	1.082	540
					499
		4	-7·10 <sup>-10</sup>	1.082166760	8245719435
					7619638429
		6	-1·10 <sup>-19</sup>	1.08216676051416702768	418939686644589150004
					387130433063328840289
S <sub>5</sub>	Sı	2	<b>-7</b> ·10 <sup>-6</sup>	1.04761	7063
					6742
		3	$5 \cdot 10^{-12}$	1.04761198457	30519245
					29132203
		4	$-2 \cdot 10^{-20}$	1.04761198457940170194	1657835914708
					1582490405905
$S_5$	$S_4$	2	<b>-</b> 8⋅10 <sup>-5</sup>	1.0475	2358
			2		2251
		3	7·10 <sup>-9</sup>	1.04753443	<u>2547455</u>
			20		2431858
		5	$1.10^{-20}$	1.04753443663236268392	60363763803209222
	3,001	·	-		57624610411155561
CS1	NCS1	6	-1.10-7	1.540889	1376250
		_	12		893153
		7	3.10-12	1.54088941088	1412070090
		0	1 10-70	1 5400004100070070707	916399373
		9	-1·10 <sup>-20</sup>	1.54088941088788795255	377447939426190
NCS1	CS1	16		0.6410	244954593599743
NC21	CSI	16	-5·10 <sup>-5</sup>	0.6419	562791312666017539
	-				876693583206100846

Table 3. Some general continued fractions converted to simple continued fractions

Q	R	C <sub>n</sub> (dec.form)	C <sub>n</sub> (Simple continued fraction sequence)
S₄	S <sub>5</sub>	1.08216676051416702768	1,12,5,1,6,1,1,1,48,7,2,1,20,2,1,5,1,2,1,1,9,1,
		(corresponding to 6 terms)	1,10,1,1,7,1,3,1,7,2,1,3,31,1,2,6,38,2
			(39 terms)
$S_5$	$S_4$	1.04753443663236268392	1,21,26,1,3,26,10,4,4,19,1,2,2,1,8,8,1,2,3,1,
		(corresponding to 5 terms)	10,1,2,1,2,3,1,4,1,8 (29 terms)
CS1	NCS1	1.54088941088788795255	1,1,15,1,1,1,1,2,4,17,1,1,3,13,4,2,2,2,5,1,6,2,
		(corresponding to 9 terms)	2,9,2,15,1,51 (28 terms)

These sequences show no special regularities. As can be seen from table 3 the number of terms required to reach 20 decimals is much larger than for the corresponding general continued fractions.

A number of Smarandache periodic sequences were explored in the author's book Computer Analysis of Number Sequences [6]. An interesting property of simple continued fractions is that

A periodic continued fraction is a quadratic surd, i.e. an irrational root of a quadratic equation with integral coefficients.

In terms of A<sub>n</sub> and B<sub>n</sub>, which for simple continued fractions are defined through

$$A_0 = q_0, \quad A_1 = q_0 q_1 + 1, \qquad A_n = q_n A_{n-1} + A_{n-2}$$

$$B_0 = 1, \quad B_1 = q_1, \qquad B_n = q_n B_{n-1} + B_{n-2}$$
(15)

the quadratic surd is found from the quadratic equation

$$B_{n}x^{2} + (B_{n-1} - A_{n})x - A_{n-1} = 0$$
(16)

where n is the index of the last term in the periodic sequence. The relevant quadratic surd is

$$x = \frac{A_n - B_{n-1} + \sqrt{A_n^2 + B_{n-1}^2 - 2A_n B_{n-1} - 4A_{n-1} B_n}}{2B_n}$$
 (17)

An example has been chosen from each of the following types of Smarandache periodic sequences:

1. The Smarandache two-digit periodic sequence:

<u>Definition</u>: Let N<sub>k</sub> be an integer of at most two digits. N<sub>k</sub>' is defined through

the reverse of N<sub>k</sub> if N<sub>k</sub> is a two digit integer

 $N_k' = \langle$ 

 $N_k$ -10 if  $N_k$  is a one digit integer

 $N_{k+1}$  is then determined by

$$N_{k+1} = N_k - N_k$$

The sequence is initiated by an arbitrary two digit integer N<sub>1</sub> with unequal digits.

One such sequence is  $Q=\{9, 81, 63, 27, 45\}$ . The corresponding quadratic equation is  $6210109x^2-55829745x-1242703=0$ 

# 2. The Smarandache Multiplication Periodic Sequence:

<u>Definition</u>: Let c>1 be a fixed integer and  $N_0$  and arbitrary positive integer.  $N_{k+1}$  is derived from  $N_k$  by multiplying each digit x of  $N_k$  by c retaining only the last digit of the product cx to become the corresponding digit of  $N_{k+1}$ .

For c=3 we have the sequence Q= $\{1, 3, 9, 7\}$  with the corresponding quadratic equation  $199x^2-235x-37=0$ 

## 3. The Smarandache Mixed Composition Periodic Sequence:

<u>Definition.</u> Let  $N_0$  be a two-digit integer  $a_1 \cdot 10 + a_0$ . If  $a_1 + a_0 < 10$  then  $b_1 = a_1 + a_0$  otherwise  $b_1 = a_1 + a_0 + 1$ .  $b_0 = |a_1 - a_0|$ . We define  $N_1 = b_1 \cdot 10 + b_0$ .  $N_{k+1}$  is derived from  $N_k$  in the same way.

One of these sequences is Q= $\{18, 97, 72, 95, 54, 91\}$  with the quadratic equation  $3262583515x^2-58724288064x-645584400=0$ 

and the relevant quadratic surd

$$x = \frac{58724288064 + \sqrt{3456967100707577532096}}{6525167030}$$

The above experiments were carried out with *UBASIC* programs. An interesting aspect of this was to check the correctness by converting the quadratic surd back to the periodic sequence.

There are many interesting calculations to carry out in this area. However, this study will finish by this equality between a general continued fraction convergent and a simple continued fraction convergent.

$$[1,12,123,1234,12345,123456,1234567,1,21,321,4321,54321,654321] = [1,12,5,1,6,1,1,1,48,7,2,1,20,2,1,5,1,2,1,1,9,1,1,10,1,1,7,1,3,1,7,2,1,3,31,1,2,6,38,2]$$

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# SOME CONNECTIONS BETWEEN THE SMARANDACHE FUNCTION AND THE FIBONACCI SEQUENCE

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#### I. INTRODUCTION

The Smarandache function  $S: N^* \to N^*$  is defined [9] by the condition that S(n) is the smallest positive integer k such that k! is divisible by n.

Tf

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_{21}} \cdot \dots \cdot p_r^{\alpha_r} \tag{1}$$

is the decomposition of the positive integer n into primes, then it is easy to verify that

$$S(n) = \max(S(p_i^{\alpha_i})) \tag{2}$$

One of the most important properties of this function is that a positive integer p is a fixed point of S if and only if p is a prime or p = 4.

This paper is aimed to provide generalizations of the Smarandache function. They will be constructed by means of sequences more general than the sequence of the factorials. Such sequences are monotonously convergent to zero sequences and divisibility sequences (in particular the Fibonacci sequence).

Our main result states that the Smarandache generalized function associated with every strong divisibility sequence (sequence satisfying the condition (15) from bellow) is a dual strong divisibility sequence (i.e. it satisfies the condition (26), the dual of (15)).

Note that the Smarandache function S is not monotonous. Indeed,  $n_1 \le n_2$  does not imply  $S(n_1) \le S(n_2)$ . For instance  $5 \le 12$  and S(5) = 5, S(12) = 4.

Let us denote by  $\stackrel{d}{\vee}$  the least common multiple, by  $\stackrel{\wedge}{\wedge}$  the greatest common divisor and let  $\wedge = \min$ ,  $\vee = \max$ . It is known that

$$N_0 = (N^*, \wedge, \vee)$$
 and  $N_d = (N^*, \wedge, \vee)$ 

are lattices. The order on  $N^*$  corresponding to the lattice  $N_0$  is the usual order:

$$n_1 \le n_2 \Leftrightarrow n_1 \wedge n_2 = n_1$$

and it is a total order. On the contrary, the order  $\leq d$  corresponding to the lattice  $N_d$ , defined as

$$n_1 \leq n_2 \Leftrightarrow n_1 \wedge n_2 = n_1$$

( the divisibility relation) is only a partial order.

More precisely we have

$$n_1 \leq n_2 \Leftrightarrow n_1 \text{ divides } n_2$$
.

For  $n_1 \leq n_2$  we shall also write  $n_2 \geq n_1$ . We notice that  $N_d$  has zero as the greatest element,  $N_0$  does not possess a greatest element and both lattices have 1 as the smallest element. Then it is convenient to consider in  $N_0$  the convergence to infinity and in  $N_d$ , the convergence to zero.

Let

$$n_1 = \prod p_i^{\alpha_i}$$
 and  $n_2 = \prod p_i^{\beta_i}$ 

be the decompositions into primes of  $n_1$  and  $n_2$ . Then we have

$$n_1 \stackrel{d}{\vee} n_2 = \prod p_i^{\max(\alpha_i, \beta_i)}$$
.

The definition of the Smarandache function implies that

$$S\left(n_{1} \vee n_{2}\right) = S(n_{1}) \vee S(n_{2})$$
(3)

Also we have

$$n_1 \leq n_2 \Rightarrow S(n_1) \leq S(n_2). \tag{4}$$

In order to make explicit the lattice (so, the order) on the set  $N^*$ , we shall write  $N_0$  instead of  $N^*$ , if the order on the set of the positive integers is the usual order and  $N_d$  instead of  $N^*$ , if we consider the order  $\leq$  respectively.

Then (4) shows that the Smarandache function, considered as a function

$$S: N_d \to N_0, \tag{5}$$

is an order preserving map.

From (2) it follows that the determination of S(n) reduces to the computation of  $S(p^{\alpha})$ . In addition, it is proved [1] that if the sequence

$$(p)$$
: 1,  $p$ ,  $p^2$ , ...,  $p^i$ , ... (6)

is the standard p-scale and the sequence

$$[p]: a_1(p), a_2(p), ..., a_i(p), ...$$

is the generalized numerical scale determined by the sequence

$$a_i(p) = \frac{p^i - 1}{p - 1}$$

then

$$S(p^{\alpha}) = p(\alpha_{[p]})_{(p)} . \tag{7}$$

In other words,  $S(p^{\alpha})$  can be obtained by multiplying by p the number obtained writing the exponent  $\alpha$  in the generalized scale [p] and "reading" it in the scale (p).

For instance, in order to calculate  $S(3^{99})$  let us consider the scale

Then, for  $\alpha = 99$ , we have

$$\alpha_{[3]} = 2a_4(3) + a_3(3) + a_2(3) + 2a_1(3) = 2112_{[3]}$$

and "reading" this number in the usual scale

(3) 1, 3,  $3^2$ ,  $3^3$ , ... we get  $S(3^{99}) = 3(2 \cdot 3^3 + 3^2 + 3 + 2) = 204$ . So, 204 is the smallest positive integer whose factorial is divisible by  $3^{99}$ .

We quote also the following formula used to compute  $S(p^{\alpha})$ :

$$S(p^{\alpha}) = (p-1)\alpha + \sigma_{[p]}(\alpha), \tag{8}$$

where  $\sigma_{[p]}(\alpha)$  stands for the sum of the digits of the integer  $\alpha$  written in the scale [p].

#### 2. GENERALIZED SMARANDACHE FUNCTIONS

A sequence of positive integers is a mapping  $\sigma: N^* \to N^*$  and it is usually denoted by  $(\sigma_n)_{n \in N}$  (i.e. the set of its values). Since in the sequel an essential point is to make evident the structure (the lattice) on the domain and on the range of this function respectively, we adopt the notation from (5).

Then

$$\sigma: N_0 \to N_d \tag{9}$$

shows that  $\sigma$  is a sequence of positive integers defined on the set  $N^*$ . This set was structured as a lattice by  $\wedge$  and  $\vee$  and its range has also a structure of lattice, induced by  $\wedge$  and  $\vee$ .

**Definition 2.1.** [3] The sequence (9) is a multiplicatively convergent to zero sequence (mcz) if

$$(\forall) n \in N^* \quad (\exists) \quad m_n \in N^* \quad (\forall) m \ge m_n \Rightarrow n \le \sigma(m). \tag{10}$$

In other words, a (mcz) sequence is a sequence defined as in (9), which is convergent to zero.

These sequences, satisfying in addition the condition

$$\sigma(n) \leq \sigma(n+1) \tag{11}$$

(that is  $\sigma(n)$  divides  $\sigma(n+1)$ ) were considered by G. Christol [3] in order to obtain a generalization of p – adic numbers.

As an example of a (mcz) sequence we may consider the sequence defined by  $\sigma(n) = n!$ . This sequence also satisfies the condition (11).

Remark 2.1. We find that the value S(n) of the Smarandache function at the point n is the smallest integer  $m_n$  provided by (10), whenever  $\sigma(n) = n!$ . This enables us to define a Smarandache type function for each (mcz) sequence. Indeed, for an arbitrary (mcz) sequence  $\sigma$ , we may define  $S_{\sigma}(n)$  as the smallest integer  $m_n$  given by (10).

The (mcz) sequences satisfying the extra-condition (11) generalize the factorial. Indeed, if

$$\sigma(n+1) = k_{n+1}\sigma(n) \tag{12}$$

then

$$\sigma(n) = k_1 \cdot k_2 \cdot \dots \cdot k_n$$
, with  $k_1 = 1$  and  $k_i \in N^*$  for  $i > 1$ .

Starting with the lattices  $N_0$  and  $N_d$ , we can construct sequences

$$\sigma: N_d \to N_d \quad . \tag{13}$$

Definition 2.2. A sequence (13) is called a divisibility sequence (ds) if 
$$n \le m \Rightarrow \sigma(n) \le \sigma(m)$$
 (14)

(that is if the mapping  $\sigma$  from (13) ia monotonous). The sequence (13) is called a *strong divisibility, sequence (sds)* if

$$\sigma(n \wedge m) = \sigma(n) \wedge \sigma(m) \text{ for every } n, m \in N^*.$$
 (15)

Strong divisibility sequences are considered, for instance, by N. Jensen in [5]. It is known that the Fibonacci sequence is also (sds).

For a sequence  $\sigma$  of positive integers, concepts as (usual) monotonicity, multiplicatively convergence to zero, divisibility, have been independently studied by many authors. A unifying treatement of these concepts can be achieved if we remark that they are monotonicity or convergence conditions of a given sequence  $\sigma: N^{\bullet} \to N^{\bullet}$ , for adequate lattices on  $N^{\bullet}$ .

We shall consider now all the possibilities to define a sequence of positive integers, with respect to the lattices  $N_0$  and  $N_d$ . To make briefly evident the kind of the lattice considered on the domain and on the range of  $\alpha$ , we shall use the following notation:

- (a) a sequence  $\sigma_{oo}: N_o \to N_o$  is an (oo) sequence
- (b) a sequence  $\sigma_{od}: N_o \to N_d$  is an (od) sequence
- (c) a sequence  $\sigma_{do}: N_d \to N_o$  is an (do)-sequence
- (d) a sequence  $\sigma_{dd}: N_d \to N_d$  is a (dd)-sequence

We have already seen (Remark 2.1) that, considering (mcz) sequences, the Smarandache function may be generalized.

In order to generalize the Smarandache function for each type of the above sequences, it is necessary to consider the monotonicity and the existence of a limit corresponding to each of the cases (a) - (d).

Of course, the limit is infinit for  $N_o$ -valued sequence and it is zero for the others. We have four kinds of monotonicity.

For a (do) - squence  $\sigma_{do}$ , the monotonicity reads:

$$(m_{do}) \quad (\forall) n_1, n_2 \in N^*, \quad n_1 \leq n_2 \Rightarrow \sigma_{do}(n_1) \leq \sigma_{do}(n_2)$$

and the condition of convergence to infinity is:

$$(c_{do}) \quad (\forall) n \in N^{\bullet} \quad (\exists) m_n \in N^{\bullet} \quad (\forall) m_{\geq d} m_n \Rightarrow \sigma_{do}(m) \geq n.$$

Similarly, for a (dd)-sequence  $\sigma_{dd}$ , the monotonicity reads:

$$(m_{dd})$$
  $(\forall) n_1, n_2 \in N^*, n_1 \leq n_2 \Rightarrow \sigma_{dd}(n_1) \leq \sigma_{dd}(n_2)$ 

and the convergence to zero is:

$$(c_{dd})$$
  $(\forall) n \in N^*$   $(\exists) m_n \in N^*$   $(\forall) m \geq m_n \Rightarrow \sigma_{dd}(m) \geq n$ .

**Definition 2.3.** The generalized Smarandache function associated to a sequence  $\sigma_{ii}$  satisfying the condition  $(c_{ij})$ , with  $i, j \in \{0, d\}$ , is

$$S_{ij}(n) = \min \left\{ m_n \mid m_n \text{ given by the condition } (c_{ij}) \right\}$$
 (16)

Remark that (oo)-sequences are the classical sequences of positive integers. As examples of (od)-sequences we quote the (mcz) sequences. Examples of (dd)-sequences are (ds) and (sds)-sequences. Finally, the generalized Smarandache functions  $S_{od}$  associated with (od)-sequences satisfying the condition  $(c_{od})$  are (do)-sequences.

The functions  $S_{ii}$  have the following properties:

Theorem 2.1. Every function  $S_{\infty}$  satisfies:

(i) 
$$(\forall) n_1, n_2 \in N^{\bullet}, n_1 \leq n_2 \Rightarrow S_{\infty}(n_1) \leq S_{\infty}(n_2),$$
  
that is  $S_{\infty}$  satisfies  $(m_{\infty})$ .

(ii) 
$$S_{oo}(n_1 \vee n_2) = S_{oo}(n_1) \vee S_{oo}(n_2)$$

(iii) 
$$S_{oo}(n_1 \wedge n_2) = S_{oo}(n_1) \wedge S_{oo}(n_2)$$
.

**Proof**: (i) The definition of  $S_{co}(n)$  implies that:

$$S_{oo}(n_i) = \min \{ m_{n_i} | (\forall) m \ge m_{n_i} \Rightarrow \sigma_{oo}(m) \ge n_i \}, \text{ for } i = 1, 2$$

Therefore

$$(\forall) m \geq S_{nn}(n_2) \Rightarrow \sigma_{nn}(m) \geq n_2 \geq n_1$$

and so  $S_{\infty}(n_1) \le S_{\infty}(n_2)$ . The equalities (ii) and (iii) are consequences of (i).

Theorem 2.2. Every function  $S_{od}$  has the following properties:

(iv) 
$$(\forall) n_1, n_2 \in N^*, n_1 \leq n_2 \Rightarrow S_{od}(n_1) \leq S_{od}(n_2)$$

that is  $S_{od}$  satisfies  $(m_{od})$ .

$$(v) \quad S_{od} \left( n_1 \stackrel{d}{\vee} n_2 \right) = S_{od} \left( n_1 \right) \vee S_{od} \left( n_2 \right).$$

$$(vi) \quad S_{od}(n_1 \wedge n_2) \leq S_{od}(n_1) \wedge S_{od}(n_2).$$

**Proof**: The equality (v) may be proved in the same manner as the equality (3) for the function S. Then from (v) it follows (iv).

For 
$$(vi)$$
 let us note  $u = S_{od}(n_1) \wedge S_{od}(n_2)$ . From

$$n_1 \stackrel{\wedge}{\underset{d}{\wedge}} n_2 \stackrel{<}{\underset{d}{\leq}} n_1, \quad n_1 \stackrel{\wedge}{\underset{d}{\wedge}} n_2 \stackrel{<}{\underset{d}{\leq}} n_2$$

and from (iv), it follows that
$$S_{od}(n_1 \wedge n_2) \leq S_{od}(n_1), \quad S_{od}(n_1 \wedge n_2) \leq S_{od}(n_2),$$
so  $S_{od}(n_1 \wedge n_2) \leq S_{od}(n_1) \wedge S_{od}(n_2).$ 

Theorem 2.3. The functions  $S_{do}$  satisfy:

$$(\forall ii) \quad (\forall) n_1, n_2 \in N^*, \quad n_1 \leq n_2 \Rightarrow S_{do}(n_1) \leq S_{do}(n_2).$$

$$(viii) \quad S_{do}(n_1 \vee n_2) \leq S_{do}(n_1) \overset{d}{\vee} S_{do}(n_2).$$

(ix) 
$$S_{do}(n_1 \vee n_2) = S_{do}(n_1) \vee S_{do}(n_2)$$
.

$$(x) \quad S_{do}(n_1 \wedge n_2) = S_{do}(n_1) \wedge S_{do}(n_2).$$

**Proof**: Let us note that (ix) and (x) are consequences of (vii). In our terms (vii)is just the fact that the Smarandache generalized function  $S_{do}$  associated with a (do) - sequence is (oo) - monotonous. To prove this assertion, let  $n_1 \le n_2$ . Then for every  $m \ge m_{n_2}$ , we have

$$\sigma_{do}(m) \ge n_2 \ge n_1$$

and so  $S_{do}(n_1) \leq S_{do}(n_2)$ .

(viii) For 
$$i = 1, 2$$
 we have:  
 $S_{do}(n_i) = \min \left\{ m_{n_i} | (\forall) m \ge m_{n_i} \Rightarrow \sigma_{do}(m) \ge n_i \right\}$ 

Let us suppose that  $n_1 \le n_2$ , so  $n_1 \lor n_2 = n_2$  and  $S_{do}(n_1 \lor n_2) = S_{do}(n_2)$ . If we take  $m_0 = S_{do}(n_1) \stackrel{d}{\vee} S_{do}(n_2)$ , then for every  $m \ge m_0$  it follows that  $\sigma_{do}(m) \ge n_i$ , for i = 1, 2, so  $\sigma_{do}(m) \ge n_1 \lor n$ , whence the desired inequality.

Consequence 2.1.  $S_{do}(n_1) \wedge S_{do}(n_2) \leq S_{do}(n_1) \wedge S_{do}(n_2) = S_{do}(n_1 \wedge n_2) \leq S_{do}(n_1) \wedge S_{do}(n_2) = S_{do}(n_1) \wedge S_{do}(n_2) = S_{do}(n_1) \wedge S_{do}(n_2) \leq S_{do}(n_1) \wedge S_{do}(n_2) \leq S_{do}(n_1) \wedge S_{do}(n_2) \leq S_{do}(n_1) \wedge S_{do}(n_2) \leq S_{do}(n_1) \wedge S_{do}(n_2) \leq S_{do}(n_1) \wedge S_{do}(n_2) \leq S_{do}(n_1) \wedge S_{do}(n_2) \leq S_{do}(n_1) \wedge S_{do}(n_2) \leq S_{do}(n_2) + S_{do}(n_2) \leq S_{do}(n_2) + S_{do}(n_2) \leq S_{do}(n_2) + S_{do}(n_2) \leq S_{do}(n_2) + S_{do}(n_2) \leq S_{do}(n_2) + S_{do}(n_2) \leq S_{do}(n_2) + S_{do}(n_2) + S_{do}(n_2) + S_{do}(n_2) \leq S_{do}(n_2) + S_{do$ 

$$S_{do}(n_1) \vee S_{do}(n_2) = S_{do}(n_1 \vee n_2) \leq S_{do}(n_1) \vee S_{do}(n_2).$$

**Theorem 2.4.** The functions  $S_{dd}$  satisfy:

$$(xi) \quad S_{dd} \left( n_1 \overset{d}{\vee} n_2 \right) \leq S_{dd} \left( n_1 \right) \overset{d}{\vee} S_{dd} \left( n_2 \right).$$

(xii) If 
$$n_1 \le n_2$$
 or  $n_2 \le n_1$  then

$$S_{dd}\left(n_{1} \stackrel{d}{\vee} n_{2}\right) = S_{dd}\left(n_{1}\right) \vee S_{dd}\left(n_{2}\right).$$

$$S_{dd}\left(n_{1} \stackrel{d}{\vee} n_{2}\right) \leq S_{dd}\left(n_{2}\right).$$

(xiii)  $S_{dd}(n_1 \wedge n_2) \leq S_{dd}(n_1) \wedge S_{dd}(n_2)$ .

**Proof**: The proof of (xi) is similar to the proof of (viii) and the other assertions may be easily obtained by using the definition of  $S_{dd}$  from (17) (for i = j = d).

Consequence 2.2. For all  $n_1, n_2 \in N^*$  we have

$$S_{dd}(n_1) \vee S_{dd}(n_2) \leq S_{dd}(n_1 \vee n_2) \leq S_{dd}(n_1) \vee S_{dd}(n_2).$$

This follows from the fact that

$$n_i \leq n_1 \stackrel{d}{\vee} n_2$$
 for  $i = 1, 2 \Rightarrow S_{dd}(n_i) \leq S_{dd}(n_1 \stackrel{d}{\vee} n_2)$ .

If  $\sigma_{dd}$  is a divisibility sequence, the above theorem implies that the associated Smarandache function satisfies the inequality (xi). In the following we shall see that, if the sequence  $\sigma_{dd}$  is a divisibility sequence with additional properties, namely if it is a strong divisibility sequence, then the inequality (xi) becomes equality.

Theorem 2.5: If  $\sigma_{dd}$  is a (sds) satisfying the condition ( $c_{dd}$ ), then:

$$S_{dd}\left(n_1 \stackrel{d}{\vee} n_2\right) = S_{dd}\left(n_1\right) \stackrel{d}{\vee} S_{dd}\left(n_2\right)$$
(17)

and

$$(\forall) n_1, n_2 \in N^*, \quad n_1 \leq n_2 \Rightarrow S_{dd}(n_1) \leq S_{dd}(n_2)$$

$$(18)$$

(i.e.  $S_{dd}$  satisfies the monotonicity condition  $(m_{dd})$ ).

Proof: In order to prove the equality (17), it is sufficient to show that

$$S_{dd}(n_i) \leq S_{dd}(n_1 \vee n_2)$$
, for  $i = 1, 2$ .

But if, for instance, the above inequality does not hold for  $n_i$  and we denote

$$d_O = S_{dd}(n_1) \bigwedge_d S_{dd}(n_1 \vee n_2),$$

it follows that  $d_{\scriptscriptstyle O} < S_{\scriptscriptstyle dd}(n_{\scriptscriptstyle 1})$  and taking into account that

$$\sigma_{dd}(S_{dd}(n_1)) \geq n_1$$
 and  $n_1 \leq n_1 \vee n_2 \leq \sigma_{dd}(S_{dd}(n_1 \vee n_2))$ ,

we have

$$\sigma_{dd}(d_{O}) = \sigma_{dd}\left(S_{dd}(n_{1}) \wedge S_{dd}(n_{1} \vee n_{2})\right) =$$

$$= \sigma_{dd}\left(S_{dd}(n_{1})\right) \wedge \sigma_{dd}\left(S_{dd}(n_{1} \vee n_{2})\right) \geq n_{1} \wedge n_{1} = n_{1}.$$

Thus, we obtain the contradiction

$$S_{dd}(n_1) \leq d_O < S_{dd}(n_1)$$
.

So, if the sequence  $\sigma_{dd}$  is a (sds), that is if the equality (15) holds, then the corresponding Smarandache function  $S_{dd}$  satisfies the dual equality (17).

**Example.** The Fibonacci sequence  $(F_n)_{n\in\mathbb{N}}$  is a (sds). Therefore, the generalized Smarandache function  $S_F$  associated with this sequence satisfy:

$$S_F\left(n_1 \stackrel{d}{\vee} n_2\right) = S_F\left(n_1\right) \stackrel{d}{\vee} S_F\left(n_2\right) \tag{19}$$

By means of this equality, the computation of  $S_F(n)$  reduces to the determination of  $S_F(p^{\alpha})$ , where p is a prime number. For instance

of 
$$S_F(p^a)$$
, where  $p$  is a prime number. For instance  $S_F(52) = \min \left\{ m_n | (\forall) m \ge m_n \Rightarrow 52 \le F(m) \right\} =$ 

$$= S_F(2^2)^{d} S_F(13) = 6 \overset{d}{\vee} 7 = 42$$
.

So, 42 is the smallest positive integer m such that F(m) is divisible by 52.

Also, we have

$$S_F(12) = S_F(2^2 \cdot 3) = S_F(2^2) \stackrel{d}{\vee} S_F(3) = 6 \stackrel{d}{\vee} 4 = 12,$$
 (20)

therefore n = 12 is a fixed point of  $S_F$ .

The values of  $S_F(p^\alpha)$  may be obtained by writing all  $F_n$  in the scale (p) given by (6), which is a difficult operation. At the time being, we are not able to provide a closed formula for the computation of  $S_F(p^\alpha)$ . However, we shall present some partial results in this direction. In [8] it is stated that

$$3^{k} \underset{d}{\leq} F_{n} \Leftrightarrow 4 \cdot 3^{k-1} \underset{d}{\leq} n$$
$$2^{k} \underset{d}{\leq} F_{n} \Leftrightarrow 3 \cdot 2^{k-2} \underset{d}{\leq} n, \quad for \ k \geq 3.$$

It is known (see for instance [6], [7]) that if  $\sigma$  is a non-degenerate second-order linear recurrence sequence defined by

$$\sigma(n) = A\sigma(n-1) - B\sigma(n-2) \tag{21}$$

where A and B are fixed non-zero coprime integers and  $\sigma(1)=1$ ,  $\sigma(2)=A$ , then

$$n \in Z^*, \quad n \wedge B = 1 \Rightarrow (\exists) m \in N^* \quad n \leq \sigma(m).$$
 (22)

The least index of these terms is called the rank of appearance of n in the sequence and is denoted by r(n).

If  $D = A^2 - 4B$  and (D/n) stands for the Jacobi symbol, then for  $mn \wedge BD = 1$  and p a prime we have ([6])

$$n \leq \sigma(m) \Leftrightarrow r(n) \leq m; \quad r(p) \leq p - (D/p)$$

$$r(p) \leq \frac{p - (D/p)}{2} \Leftrightarrow (B/p) = 1; \quad r(m \vee n) = r(m) \vee r(n).$$
(23)

Let us denote  $N_B^* = \{ n \in N^* \mid n \land B = 1 \}$ . Obviously, if r is considered as a function

$$r: N_B^{\bullet} \to N^{\bullet}$$
, then we can write:  
 $r(n) = \min \left\{ m \mid n \leq \sigma(m) \right\}$ .

Whence an evident parallel between the above methods described for the construction of the generalized Smarandache functions and the definition of the function r.

For the Fibonacci sequence  $(F_n)$  we have A = 1, B = -1 and so D = 5.

This implies

$$p = 5k \pm 1 \Longrightarrow (5/p) = 1 \tag{24}$$

$$p = 5k \pm 2 \Longrightarrow (5/p) = -1 \tag{25}$$

and it follows that if (24) holds, then p divides  $F_{p-1}$ . Thus  $S_F(p)$  is a divisor of p-1. In the second case p divides  $F_{p+1}$  and  $S_F(p)$  is a divisor of p+1.

From (23) we deduce

$$S_F(p) \le p - (5/p)$$

for any prime number p.

Lemma 2 from [6] implies that the fraction  $(p-(5/p))/S_F(p)$  is unbounded. We also have

$$p^{k} \leq F_{n} \Leftrightarrow S_{F}(p^{k}) \leq n$$
.

**Example.** For p=11 it follows (5/p)=1, so  $S_F(11) \le 10$ . In fact, we have precisely  $S_F(11)=11-(5/11)=10$ , but there exist prime numbers such that  $S_F(p) < p-(5/p)$ . For instance, p=17, for which p-(5/p)=18 and  $S_F(17)=9$ .

**Definition 2.4.** The sequence  $\sigma$  is a dual strong divisibility sequence (dsds) if

$$\sigma\left(n \vee m\right) = \sigma(n) \vee \sigma(m) \quad \text{for all } n, m \in N^{\bullet}. \tag{26}$$

It may be easily seen that every strong divisibility sequence is a divisibility sequence. We also have:

Proposition 2.1 Every dual strong divisibility sequence is a divisibility sequence. Proof. We have to prove that (26) implies (14). But if  $n \le m$ , it follows

 $n \vee m = m$  and then

$$\sigma(m) = \sigma\left(n \vee m\right) = \sigma(n) \vee \sigma(m)$$
so,  $\sigma(n) \leq \sigma(m)$ . (27)

Then Theorem 2.5 asserts that the Smarandache generalized function  $S_{\sigma}$  associated with any strong divisibility sequence  $\sigma$  is a dual strong divisibility sequence. Of course, in this case, both sequences  $\sigma$  and  $S_{\sigma}$  are divisibility sequences.

It would be very interesting to prove whether the converse assertion holds. That is if  $S_{dd}$  is the generalized Smarandache function associated with a (divisibility) sequence  $\sigma_{dd}$  satisfying the condition  $(c_{dd})$ , then the equality (17) implies the strong divisibility.

**Remarks.** (1) It is known that the Smarandache function S is *onto*. But given a (dd)-sequence  $\sigma_{dd}$ , even if it is a (sds), it does not follow that the associated function  $S_{dd}$  is *onto*. Indeed, the function  $S_F$  associated with the Fibonacci sequence is not *onto*, because n = 2 is not a value of  $S_F$ .

(2) One of the most interesting diophantine equations associated with a Smarandache type function is that which provides its fixed points. We remember that the fixed points for the Smarandarche function are all the primes and the composit number n=4. For the functions  $S_{dd}$  the equation providing the fixed points reads  $S_{dd}(x)=x$  and for  $S_F$  we have as solutions, for instance, n=5, n=12.

At the end of this paper we quote the following question on the Smarandache function, also related to the Fibonacci sequence:

T. Yau [10] wondered if there exist triplets of positive integers (n, n-1, n-2) such that the corresponding values of the Smarandache function satisfy the Fibonacci recurrence relation S(n) = S(n-1) + S(n-2).

He found two such triplets, namely for n = 11 and for n = 121. Indeed, we have S(9) + S(10) = S(11) and S(119) + S(120) = S(121).

Using a computer, Charles Ashbacher [2] found additional values. These are for n = 4902, n = 26245, n = 32112, n = 64010, n = 368139, n = 415664.

Recently H. Ibsent [4] proposed an algorithm permitting to find, by means of a computer, much more values. But the question posed by T. Yau "How many other triplets with the same property exist?" is still unsolved.

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# COMPUTATIVE PARADOXES IN MODERN DATA ANALYSIS

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By developing F. Smarandache thema on paradoxes in mathematics it is stated, firstly, if in measurement (natural science) experiments the best solutions are found by using methods of modern data analysis theory, then some difficulties with the interpretation of the computation results are liable to occur; secondly, one is not capable to overcome these difficulties without a data analysis theory modification, consisted in the translation of this theory from Aristotelian "binary logic" into more progressive "fuzzy logic".

Key words: data analysis, revealing outliers, confidence interval, fuzzy logic.

### 1 Introduction

As generally known from history of science, a scientific theory may have crisis in process of its development, when it disjoints in a set of fragment theories, that weak-coordinate each other and, as a whole, form a collection of various non-integrated conceptions. For instance, as we assume, F. Smarandache mathematical notions and questions <sup>1-2</sup> help us to understand quite well that a stable equilibrium, observed in mathematics at the present time, is no more than fantasy. Thus, it falls in exactly with F. Smarandache views that the finding and investigating paradoxes in mathematics is a very effective way of approximating to the truth and so at present each of scientific researches, continuing F. Smarandache thema<sup>2</sup>, should be considered as very actual one.

Let us assume that *computative paradoxes* in mathematics are mainly such computation results, obtained by using mathematical methods, which are contradicted some mathematical statements. The main goal of this paper is to demonstrate that the mentioned crisis, demanding practical action instead of debate, occurs in modern data analysis, which formally has its own developed mathematical theory, but does not capable "to cope worthily" with a large number of practical problems of quantitative processing results of measurement experiments.

Another goal of this paper is to equip the mathematicians and software designers, working in the data analysis field, with a set of examples, demonstrating dramatically that, if, for solving some problems on analysing data arrays, one uses the standard computer programmes and/or time-tested methods of modern data analysis theory, then a set of the paradoxical computative results may be obtained.

# 2 Approximative problems of data analysis

# 2.1 The main problems of regression analysis theory and standard solution methods

As generally known<sup>3-7</sup>, for found experimental dependence  $\{y_n, x_n\}$  (n = 1, 2, ..., N) and given approximative function F(A, x), in the measurement (natural science) experiments the main problems of regression analysis theory are finding estimates of A' and y' and variances of  $\delta A'$  and  $\delta (y-y')$ , where A' is an estimate of vector parameter A of the function F(A, x) and  $\{y_n'\}=\{F(A', x_n)\}$ . In particular, if  $F(A, x) = \sum_{l=1}^{L} a_l h_l(x)$   $\{F(A, x) \text{ is a linear model}\}$ , where  $h_l(x)$  are some functions on x, then in received regression analysis theory standard solution of discussed problems has form

$$\mathbf{A'} = (\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1} \mathbf{H}^{\mathrm{T}}\mathbf{Y}, \quad (\delta \mathbf{A'})^{2} = s/(N-L) \operatorname{diag}(\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}$$

$$\delta_{p}(y-y') = y' \pm t_{p} s \sqrt{1 + \mathbf{H}_{i}^{\mathrm{T}}(\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}\mathbf{H}_{i}},$$
(1)

where **H** is a matrix  $L \times N$  in size with *n*-th row  $(h_1(x_n), h_2(x_n), \ldots, h_L(x_n))$ ;  $\mathbf{H}^T$  is the transposed matrix  $\mathbf{H}$ ;  $\mathbf{Y} = \{y_n\}$ ;  $s = \sum_{n=1}^N (y_n - y_n')^2 / (N - L)$ ;  $\mathbf{H}_i = (h_1(x_i), h_2(x_i), \ldots, h_L(x_i))$ ; the value of  $t_p$  is determined by *t*-Student distribution table and generally depends on the assigned value of the significance level of p and the value of N - L (a number of freedom degree); at the assigned value of the significance level of p the notation of  $\delta_p(y-y')$  means confidence interval for possible deviations of experimental values of p from computed values  $p' = F(\mathbf{A}', x_i)$ . According to Gauss - Markov theorem p', for classical data analysis model

$$y_n = F(\mathbf{A}, x_n) + e_n \tag{2}$$

the solution (1) is the best (gives minimum value of s), if the following conditions are fulfilled:

all values of  $\{x_n\}$  are not random, mathematical expectation of random value  $\{e_n\}$  is equal to zero and random values of  $\{e_n\}$  are non-correlated and have the same dispersions  $\sigma^2$ .

**Example 1.** In table 1 we adduce an experimental data array, obtained by Russian chemist D.I.Mendeleev in 1881, when he investigated the solvability (y, relative units) of sodium nitrate (NaNO<sub>3</sub>) on the water temperature  $(x, {}^{\circ}C)$ .

Table 1. D.I.Mendeleev data array Xn Уn  $y_n - y_n'$ n Υn  $y_n - y_n'$ 15 4 80.6 0.05 7 36 99.4 0.58 85.7 71.0 21 5 -0.078 51 113.6 76.3 29 92.9 125.1

By analysing the data array  $\{y_n, x_n\}$ , presented in table 1, Y.V.Linnik<sup>3</sup> states that these data, as it was noted by D.I.Mendeleev, are well-fitted by linear model y' = 67.5 + 0.871 x ( $\delta A' = (0.5; 0.2)$ ), although the correspondence between experimental and computed on linear model values of y is slightly getting worse at the beginning and end of investigated temperature region (see the values  $\{y_n - y'_n\}$  adduced in table 1). We add that for discussed data array Y.V.Linnik<sup>3</sup> computes the confidence interval of  $\delta_p(y-y')$  from (1) at the significance level of p = 0.9:

$$\delta_{0.9} (y - y') = \pm 0.593 \sqrt{1 + (x - 26)^2 / 4511}$$
 (3)

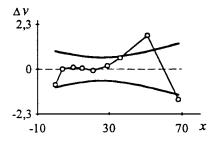


Figure 1. The plots of confidence interval of the deviation of y from y' (heavy lines) and residuals y - y' (circles) for D.I.Mendeleev data array.

We show the plots of  $\delta_{0.9}(y-y')$  on x by heavy lines in figure 1 and  $\{y_n-y_n', x_n\}$  by the circles. Since the plot of  $\{y_n-y_n', x_n\}$  steps over the heavy lines in figure 1, some computative difficulty is revealed:

the standard way (1), used by Y.V.Linnik<sup>3</sup> for determining the confidence interval of the deviations of y from y', is out of character with the discussed experimental data array.

It follows from results presented in table 1 and/or figure 1, if one assumes that  $\delta(y-y') \ge \max |y_n-y_n'| = 1.73$  then the broken connections of the confidence interval  $\delta(y-y')$  with D.I.Mendeleev data array will be pieced up. But values of  $\delta A'$ , calculated by Y.V.Linnik from (1), disagree with the values  $\delta(y-y') \ge 1.73$ , and, consequently,

standard values of  $\delta A'$  is out of character with D.I.Mendeleev data array also.

# 2.2 Alternative methods of regression analysis theory

P.Huber<sup>8</sup> noted that, as the rule, 5 - 10% of all observations in the majority of analysing experimental arrays are anomalous or, in other words, the conditions of Gauss - Markov theorem, adduced above, are not fulfilled. Consequently, in practice instead of the standard solution (1), found by "least squares (LS) method", alternative methods, developed in the frames of received regression analysis theory, should be used. In particular, if the data array  $\{y_n, x_n\}$  contains a set of

outliers, then for finding the best solution of discussed problem it is necessary<sup>6, 7</sup> or to remove all outliers from the analysing data array (strategy 1), or to compute the values of A' on the initial data array by means of M-robust estimators (strategy 2). For revealing outliers in the data array P.J.Rousseeuw and A.M.Leroy<sup>9</sup> suggest to use one of two combined statistical procedures, in which parameter estimates, minimising the median of the array  $\{(y_n - y_n')^2\}$  (the first procedure) or the sum of K first elements of the same array (the second procedure), are considered as the best ones. If F(A, x) is a linear function (see above), then the robust M-estimates of A' are obtained as result of the solving of one from two minimisation problems  $^{6-9}$ 

$$S_{\varphi}(\mathbf{A}) = \sum_{n=1}^{N} \varphi(y_n - y_n') \Rightarrow \min \quad \text{or} \quad \partial S_{\varphi} / \partial a_l = \sum_{n=1}^{N} \psi(y_n - y_n') \ h_l(x_n) = 0, \tag{4}$$

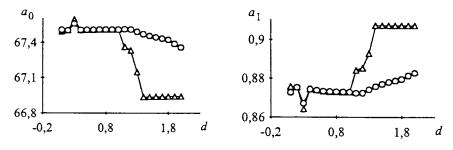
where function  $\varphi(r)$  is symmetric concerning Y-axis, continuously differentiable with a minimum at zero and  $\varphi(0) = 0$ ;  $\psi(r)$  is a derivative of  $\varphi(r)$  with respect to r.

Continued example 1. Since D.I.Mendeleev data array from table 1 contains outliers, we adduce results of quantitative processing this data by alternative methods, defined above.

- 1. Let in (4) Andrews function <sup>10</sup> be applied:  $\varphi(r) = d(1-\cos(r/d))$  if  $|r| \le d\pi$  and  $\varphi(r) = 0$  if  $|r| > d\pi$ . It is articulate in figure 2 that in this case the values of the linear model parameters  $a_0$  and  $a_1$  depend on
  - a) the values of parameter d of Andrews function  $\varphi(r)$ ;
- b) the type of the minimisation robust regression problem (solutions of the first and second minimisation problem of (4) are marked respectively by triangles and circles in figure 2).

Thus, in this case a computative paradox declares itself in the fact, that in actual practice the robust estimates are not robust and so, as K.R.Draper and H.Smith 11 wrote already,

"unreasoning application of robust estimators looks like reckless application of ridge-estimators: they can be useful, but can be improper also. The main problem is such one, that we do not know, which robust estimators and at which types of supposes about errors are effectual to applicate; but some investigations in this direction have been done..."



**Figure 2**. Dependences of parameters values of linear model  $a_0+a_1x$  on values internal parameter of robust Andrews estimator and the type of the minimisation problems (4).

2. Let us reveal outliers in D.I.Mendeleev data array by both combined statistical procedures <sup>9</sup>, mentioned above.

Our computation results show

- a) both procedures could not find the all four outliers (1, 7, 8 and 9) but the only three ones with numbers 1, 8 and 9;
- b) if a set of readings with numbers 1, 8 and 9 is deleted from D.I.Mendeleev data array, then for the truncated data array the first procedure will not find a new outlier, but the second procedure will find two outliers yet that have numbers 2 and 6 in the initial data array.

Thus, in this case the main computative paradox is exhibited in the fact, that revealing outliers problems solutions depend on a type of the used statistical procedures.

It remains for us to add, if one

a) computes y' by formula  $^{6,7}$ 

$$y'(\xi, x) = g_{\alpha}(68.12 + 0.02\xi + (0.85652 - 0.00046\xi)x),$$
 (5)

then for each n the difference  $|y_n - y_n'|$  will keep within the limit of the chosen above value for the confidence interval  $\delta(y - y')$ , where  $\alpha = 0.94$ ;  $0 \le \xi \le 35$ ;  $g_{\alpha}(y) = 2\alpha[y/(2\alpha)] + 2\alpha$  at  $|y - 2\alpha[y/(2\alpha)]| \ge \alpha$ , otherwise  $g_{\alpha}(y) = 2\alpha[y/(2\alpha)]$ , [b] means integer part of b, Thus, another computative paradox occurs:

although for each contaminated data array a family of analytical solutions exists, the only single solution of the estimation problems is found in modern regression analysis theory.

- b) puts the mentioned above extremal values of  $\xi$  in (5), one will be able to determine the exact limit of the variation for the linear model parameters  $a_0$  and  $a_1$ :  $a_0 = 67.77 \pm 0.35$  and  $a_1 = 0.865 \pm 0.008$ ;
- c) deletes a set of readings with numbers 1, 7, 8 and 9 from D.I.Mendeleev data array, one will obtain that in the truncated data array  $\{y_n, x_n\}^*$  the difference of  $|y_n y_n'|$  for each n keeps within the limit of the error  $\varepsilon$ , where  $\varepsilon$  is the measuring error for readings  $\{y_n\}^*$ :  $\varepsilon = 0.1$ . Since in this case  $\delta(y y') \le \varepsilon$ , the complete family of analytical solutions has form  $^{6,7}$

$$y'(\xi, x) = g_{\alpha}(67.566 + 0.002\xi + (0.870047 - 0.000097\xi)x), \tag{6}$$

where  $\alpha=0.07$ ;  $0 \le \xi \le 45$  and, consequently,  $a_0 = 67.521 \pm 0.045$  and  $a_1 = 0.872 \pm 0.002$ ;

d) compares solutions (5) and (6) with the standard LS-solution, one can conclude that LS-estimations of parameter  $a_0$  and  $a_1$  {A' = (67.5±0.5; 0.87±0.2)} are pretty near equal of the mean values of these parameters in the general analytical solutions (6) and (7). However,

values of variances  $\delta a_0$  and  $\delta a_1$ , computed by standard method, disagree with exact values determined by (5).

# 2.3 The main paradox of regression analysis theory

As it emerges from analysis of information presented in Sect. 2.2, the main paradox of modern regression analysis theory is exhibited in a contradiction between this theory statements, which guarantee uniqueness of data analysis problems solution, and multivarious solutions in actual practice. In this section we adduce yet several computative manifestations of this paradox.

Example 2. In table 2 a two-factors simulative data array is presented.

							Table			
Simulative data array										
n	Xn	y <sub>n</sub>	<u>n</u>	X <sub>n</sub>	Уn		n	X <sub>n</sub>	y <sub>n</sub>	
1	-1.0	0.50	8	-0.3	0.75		15	0.4	0.87	
2	-0.9	0.55	9	-0.2	0.77	•	16	0.5	0.89	
3	-0.8	0.59	10	-0.1	0.79	•	17	0.6	0.90	
4	-0.7	0.63	11	0.0	0.81		18	0.7	0.91	
5	-0.6	0.66	12	0.1	0.83	•	19	0.8	0.92	

Let the approximative model have form

$$y = (a_0 + a_1 x + a_2 x^2)/(1 + a_3 x + a_4 x^2).$$
 (7)

To find vector parameter A estimates of the model (7) on the data array from table 2 we use two different estimation methods. As the first method we choose the estimation one, involved in the software CURVE-2.0, designed AISN. In this case we obtain, that

$$A' = \{0.81; 0.008; -0.31; -0.22; -0.24\}.$$

As the second estimation method we select Marquardt method  $^{12}$ . Using the value A', found above by the first estimation method, as initial value of A we obtain that in the second case

$$A' = \{0.81; 0.55; 0.035; 0.45; 0.34\}.$$

Thus,

values of A', obtained by two different estimation methods, differ from each other.

**Example 3.** In table 3 yet one two-factors data array is presented. Let us select the model  $y = a_1x + e$  as approximative one and assume, that y is the random variable with the known density function p:

$$p = \exp\{-(y - a_1 x) / (2f(a_2))\} / \sqrt{2\pi f(a_2)}, \qquad (8)$$

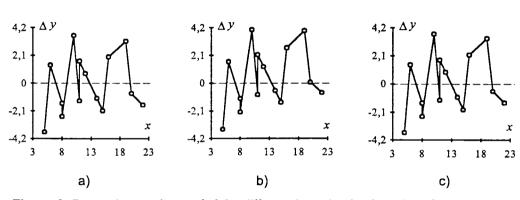
where  $f(a_2) = a$   $a_2$ ; b)  $a_2x^2$ ; c)  $a_2x$ .

We will find estimates of parameters  $a_1$  and  $a_2$  by method of maximum likelihood:

$$L = \prod_{n=1}^{N} \exp\{-(y_n - a_1 x_n) / (2f(a_2)) / \sqrt{2\pi f(a_2)} \implies \max$$
 (9)

or  $\partial \ln L / \partial a_i = 0$ , where symbol "ln" means the natural logarithm.

Table 3. Two-factors data array 



**Figure 3**. Dependences  $\{y_n - a_1'x_n\}$  for different hypothesis about Law for the random variable y variance.

Computation results of V.I.Mudrov and V.P.Kushko<sup>13</sup> show, that in the discussed case the estimates values of parameters  $a_1$  depend on hypothesis about Law for the random variable y variance: for case (a) in (8)  $a_1' = 4.938$  ( $L' = 4.995 \cdot 10^{-14}$ ); for case (b)  $a_1' = 4.896$  ( $L' = 4.421 \cdot 10^{-16}$ ) and for case (c)  $a_1' = 4.927$  ( $L' = 9.217 \cdot 10^{-15}$ ). By analysing obtained results the authors <sup>13</sup> conclude, that, since likelihood function (9) has maximum values for case (a), the more likelihood hypothesis about Law for the random variable y variance is the hypothesis (a): variance of y is the constant value.

We demonstrate in figure 3 that for cases (a), (b) and (c) dependences  $\Delta y = \{e_n\} = \{y_n - a_1'x_n\}$  have practically the same form and, consequently,

the strong distinction of values L' for all mentioned cases does not tread on infirm ground.

It should be noted that

- the very apparent expression of the discussed main computative paradox of regression analysis theory one may find also in books<sup>6, 7, 11</sup>, where, for the problem on finding the best linear multiple model, fitting Hald data array, a set of solutions,

found by various procedures and statistical tests of modern regression analysis theory, is adduced:

- the most impressive formulation of the main paradox of regression analysis theory is contained in Y.P.Adler introduction <sup>14</sup>:

"When the computation had arisen, the development of regression analysis algorithms went directly «up the stairs, being a descending road». Computer was improving and simultaneously new more advanced algorithms were yielded: whole regression method, step-by-step procedure, stepped method, etc., – it is impossible to name all methods. But again and again it appeared that all these tricks did not allow to obtain a correct solution. At least it became clear that in majority cases the regression problems belonged to a type of incorrect stated problems. Therefore either they can be regularised by exogenous information, or one must put up with ambiguous, multivarious solutions. So the regression analysis degraded ingloriously to the level of a heuristic method, in which the residual analysis and common sense of interpreter play the leading role. Automation of regression analysis problems came to a dead-lock".

# 3 Data analysis problems at unknown theoretical models

Let us assume, that a researcher is to carry out a quantitative analysis of a data array  $\{X_n\}$  in the absence of theoretical models. Further consideration will be based on the fact <sup>6, 7</sup> that the described situation demands a solution of following problems

- verification of the presence (or absence) of interconnections between analysed properties or phenomena;
- determining (in the case when the interconnection is obvious, a priori and logically plausible) in what force this interconnection is exhibited in comparison with other factors affecting the discussed phenomena;
- drawing a conclusion about the presence of a reliable difference between the selected groups of analysed objects;
- revealing object's characteristics irrelevant to analysed property or phenomenon;
- constructing a regression model describing interconnections between analysed properties or phenomena.

In following sections we consider some methods allowing to solve foregoing problems.

# 3.1 Correlation analysis

When one is to carry out a quantitative analysis of the data array  $\{X_n\}$  in the absence of theoretical models, it is usual to apply correlation analysis at the earlier

investigation stage, allowing to determine the structure and force of the connections between analysed variables<sup>15-17</sup>.

Let, for instance, in an experiment each n-th state of the object be characterised by a pair of its parameters y and x. If relationship between y and x is unknown, it is sometimes possible to establish the existence and nature of their connection by means of such simple way as graphical. Indeed, for realising this way, it is sufficient to construct a plot of the dependence  $\{y_n, x_n\}$  in rectangular coordinates y - x. In this case the plotted points determine a certain *correlation field*, demonstrating dependences x = x(y) and/or y = y(x) in a visual form.

To characterise the connection between y and x quantitatively one may use the correlation coefficient R, determined by the equation  $^{15-17}$ 

$$R_{yx} = \frac{\sum_{n=1}^{N} (y_n - \overline{y})(x_n - \overline{x})}{\sqrt{\sum_{n=1}^{N} (y_n - \overline{y})^2 \sum_{n=1}^{N} (x_n - \overline{x})^2}}.$$
 (10)

where  $\overline{y}$  and  $\overline{x}$  are the mean values of parameters y and x computed on all N readings of the array  $\{y_n, x_n\}$ . It can be demonstrated that absolute value of  $R_{yx}$  does not exceed a unit:  $-1 \le R_{yx} \le 1$ .

If variables y and x are connected by a strict linear dependence  $y = a_0 + a_1 x$ , then  $R_{yx} = \pm 1$ , where sign of  $R_{yx}$  is the same as that of the  $a_1$  parameter. This can follow, for instance, from the fact that, using  $R_{yx}$ , one can rewrite the equation for the regression line in the following form<sup>15-17</sup>

$$y = \overline{y} + R_{yx}(S_y/S_x)(x - \overline{x}), \tag{11}$$

where  $S_y$  and  $S_x$  are mean-square deviations of variables y and x respectively.

In a general case, when  $-1 < R_{yx} < 1$ , points  $\{y_n, x_n\}$  will tend to approach the line (11) more closely with increasing of  $|R_{yx}|$  value. Thus, correlation coefficient (10) characterises a linear dependence of y and x rather than an arbitrary one. To illustrate this statement we present in table 4 the values  $R_{yx} = R_{yx}(\alpha)$  for the functional dependence  $y = x^{\alpha}$ , determined on x-interval [0.5; 5.5] in 11 points uniformly.

Table 4

The values  $R_{yx} = R_{yx}(\alpha)$  for the functional dependence  $y=x^{\alpha}$ , determined on interval [0.5; 5.5]

a	$R_{yx}(\alpha = -a)$	$R_{yx}(\alpha = a)$	 а	$R_{yx}(\alpha = -a)$	$R_{yx}(\alpha = a)$	
3.0	-0.570	0.927	1.0	-0.795	1.000	
2.5	-0.603	0.951	0.5	-0.880	0.989	
2.0	-0.650	0.974	0.0	0.0	0.0	
1.5	-0.715	0.992	-	_	-	

Let us clear up a question what influence has the presence of outliers in the data array  $\{y_n, x_n\}$  on the value of correlation coefficient (10). To perform it let us analyse a data array

$$\{x_n\} = (-4; -3; -2; -1; 0; 10),$$
  
$$\{y_n\} = (2.48; 0.73; -0.04; -1.44; -1.32; 0),$$
  
(12)

where, on simulation conditions<sup>8</sup>, the reading with number 6 is an extremal outlier (such reading that contrasts sharply from others); approximative function  $F(\mathbf{A}, \mathbf{X}) = a_0 + a_1 x$  and  $\mathbf{A}_{\text{true}} = (-2; -1)$ .

By computing the values of  $R_{yx}$  of (10) and s of (1), we determine the number i of a reading, which elimination from this data array leads to the maximum absolute value of  $R_{yx}$  and, consequently, to the minimum value of s (the most simple combinatoric-parametric procedure Ps, allowing to find one outlier  $^{6,7}$  in a data array). Our calculations show that the desirable value  $R_{yx} = -0.979$  and i = 1. We note, if extremal outlier  $y_6$  is removed from the array (12),  $R_{yx} = -0.960$ , but s of (1) takes the minimum value. Presented results enable us to state that

procedure Ps loses its effectiveness when revealing the outlier is made not by test s, but by test  $R_{yx}$ .

Let us consider another case. For the array (12) the noise array  $\{e_n\} = \{-2 - x_n - y_n\} = (-0.48; 0.27; 0.04; 0.44; -0.68; -12.0)$ . We reduce by half the first 5 magnitudes of the noise array  $\{e_n\}$ :  $\{e_n\}_{\text{new}} = (-0.24; 0.14; 0.02; 0.22; -0.34; -12.0)$ ; form a new array  $\{y_n\}_{\text{new}} = \{-2 - x_n - (e_n)_{\text{new}}\}$  and determine again the number i of a reading, which elimination from the data array  $\{y_n, x_n\}_{\text{new}}$  leads to the maximum absolute value of  $R_{yx}$ . In the described case  $R_{yx}$  reaches its maximum absolute value when the reading 6 (extremal outlier) is deleted from the array  $\{y_n, x_n\}_{\text{new}}$  ( $R_{yx} = -0.989$ ). If from the array  $\{y_n, x_n\}_{\text{new}}$  we eliminate the reading 6, identified correctly by the test "the maximum absolute value of  $R_{yx}$ ", then by this test we are able to identify correctly the sequent outlier (the reading 5) in the discussed array. Thus, we obtain finally

when a dependence between the analysed variables is to a certain extent close to a linear, one may use the correlation coefficient (10) for revealing outliers, presented in data arrays.

It is known  $^{15-17}$ , when the number of analysed variables K > 2, the structure and force of the connections between variables  $x_1, x_2, ..., x_K$  are determined by computing all possible pairs of correlation coefficients  $R_{x_ix_j}$  from (10). In this case all coefficients  $R_{x_ix_j}$  are usually presented in the form of a square symmetric K by K matrix:

$$\mathbf{R} = \begin{bmatrix} R_{11} R_{12} \dots R_{1K} \\ \dots & \dots & \dots \\ R_{K1} R_{K2} \dots R_{KK} \end{bmatrix}, \tag{13}$$

which is called a correlation matrix (we note that in this matrix diagonal elements  $R_{ii}=1$ ). Finding strong-interconnected pairs of variables  $x_1, x_2, ..., x_K$  on the magnitudes of coefficients  $R_{ij}$  from the matrix **R** is a traditional use of matrix (13) in data analysis. But, obviously,

using the mentioned way, one should bear in mind all ideas presented above in outline concerning the correlation coefficient (10).

# 3.2 Discriminant analysis

Let a certain object W be characterised by a value of its vector parameter  $\mathbf{X}_{\mathbf{w}} = (x_1, x_2, \dots, x_K)$ ;  $W_1, W_2, \dots, W_p$  be p classes and the object W must be ranged in a class  $W_j$  on the value of its vector parameter  $\mathbf{X}_{\mathbf{w}}$ . In discriminant analysis the formulated problem is the *main* one  $^{18-21}$ .

The accepted technique for solving the mentioned problem entails construction of a discriminant function D(A, X). A form and coefficients  $\{a_i\}_{i=1,2,\ldots,p}$  values of this function are determined from the requirement, that values of D(A, X) must have maximum dissimilarity, if parameters of objects, belonging to different populations  $W_1, W_2, \ldots, W_p$ , are used as arguments of this function.

It seems obvious that in a general case, firstly, D(A, X) may be either linear or non-linear function on  $\{a_i\}$  and, secondly, must be some connection between the problem-solving techniques of discriminant and regression analyses. In particular, as stated  $^{18}$  -21, for solving problems of discriminant analysis one may use standard algorithms and programs of regression analysis. Thus, the similarity of techniques, used for solving problems of the regression and discriminant analyses, makes it possible in discriminant analysis to apply alternative algorithms and procedures of regression analysis and, consequently,

if data analysis problems are solved by discriminant analysis techniques then in practice the researcher may meet the same difficulties which are discussed in Sect. 2.

# 3.3 Regression analysis

In the absence of theoretical models it is usual to employ regression analysis in order to express in a mathematical form the connections existing between variables under analysis.

It happens with extreme frequency that researchers impose limitations on a type and form of approximative models or, in other words, approximative models are often chosen from a given set of ones. Evidently, in this case it is required to solve problem on finding the best approximative model from a given set of models. Let, for instance, it is required to find the best approximative multinomial with a minimal degree. With this in mind, in two examples below we consider some accepted

techniques, used for solving the mentioned problem in approximation and/or regression analysis theories.

#### Example 4. Let

$$\{x_n\} = \{-1+0.2(n-1)\}; \ f(x) = \sin x; \ \{y_n\} = \{ [kf(x_n)]/k \},$$
 (14)

where square brackets mean the integer part; n = 1, 2, ..., 11; f(x) is a given function, used for generating the array  $\{y_n\}$ ; a factor  $k = 10^3$  and its presence in (14) is necessary for "measuring" all values of  $y_n$  within error  $\varepsilon = 10^{-3}$ . It is required for the presented dependence  $\{y_n, x_n\}$  to find the best approximative multinomial with a minimal degree.

A. As well known in approximation theory  $^{22}$ , if the type of the function f(x) is given and either (m+1)-derivative of this function weakly varies on the realisation  $\{x_n\}$ , or on the x-interval [-1, 1] the function f(x) is presented in form of even-converging power series, then the problem of finding the best even-approximating multinomial for the discrete dependence  $\{y_n, x_n\}$  offers no difficulty. Indeed, in the first case the solution of problem is an interpolative multinomial  $P_M(\mathbf{B}, x)$  with a set of Chebyshev points (this multinomial is close to the best even-approximating one). In the second, case one may obtain the solution by the following economical procedure of an even-converging power series:

- 1. Choose the initial part of truncated Taylor series, approximating the function f(x) within error  $\varepsilon_M < \varepsilon$ , as the multinomial  $P_M(\mathbf{B}, x)$  (the multinomial with the degree M and vector parameter  $\mathbf{B}$ );
- 2. Replace  $\varepsilon_M$  with  $\varepsilon_M |b_M|/2^{M-1}$ , where  $b_M$  is a coefficient of the multinomial  $P_M(\mathbf{B}, x)$  at  $x^M$ ;
  - 3. If  $\varepsilon_M > 0$ , then replace the multinomial  $P_M(\mathbf{B}, x)$  with the multinomial

$$P_{M-1}(x) = P_M(x) - (b_M / 2^{M-1}) T_M(x),$$
(15)

where  $T_M(x)$  is Chebyshev multinomial:  $T_0 = 1$ ,  $T_1 = x$  and when  $M \ge 2$   $T_M = 2xT_{M-1} - T_{M-2}$ . Then decrement M by one and go to point 2. If  $\varepsilon_M \le 0$ , then go to point 4;

4. End of computations: the multinomial  $P_M(\mathbf{B}, x)$  is the desirable one.

By means of the foregoing economical procedure one may easy obtain that the multinomial with a minimal degree, even-approximating the function  $\sin x$ , given within error  $\varepsilon = 10^{-3}$  on the x-interval [-1, 1], has the following form

$$P_3(x) = (383/384) x - (5/32) x^3. (16)$$

B. Let  $1 \le M \le 9$  and in the multinomial  $P_{\lambda}(\mathbf{B}, x) = \sum_{m=0}^{M} \lambda_m b_m x^m$  all  $\lambda_m = 1$ . By determining LS-estimates of vector parameter **B** for each value of M on the formed above array  $\{y_n, x_n\}$ , we find that in all obtained approximative multinomials  $P_{\lambda}(\mathbf{B}', x)$ , as well as in Taylor series of function  $\sin x$ , the values of coefficients

 $b'_{2l} = 0$  at l = 0, 1, ..., 4. Thus, regression analysis of the discussed array  $\{y_n, x_n\}$  allows to determine a form of the best approximative multinomial:

$$P_{2l+1 \text{ opt }}(\mathbf{B}, x) = b_1 x + b_2 x^3 + \dots + b_{2l+1} x^{2l+1} + \dots$$
 (17)

We note, if l = 1, then the values of parameters  $b_1$  and  $b_3$ , computed by regression analysis method (LS-method) for model (17), coincide with ones, shown in (16).

We remind, that in classical variant of regression analysis theory the best approximative multinomial is chosen by the minimal value of the test  $s_{\lambda} = S_{\lambda}/(N-I_{\lambda})$ , where  $S_{\lambda}$  is residual sum-of-squares,  $I_{\lambda} = \sum_{m=1}^{M} \lambda_{m}$ ; N is total number of readings;  $\lambda_{m}$  is such characteristic number that  $\lambda_{m} = 0$ , if the approximative multinomial  $P_{\lambda}(\mathbf{B}, x)$  does not contain term  $b_{m}x^{m}$ , and  $\lambda_{m} = 1$ , otherwise. For approximative models (17) and l = 0, 1, 2, 3 the computation values of s are following

Since s has the minimal value at l = 2, for the discussed array in the frame of classical variant of regression analysis theory, the multinomial

$$P_5(x) = b_1 x + b_3 x^3 + b_5 x^5 (18)$$

is the best approximative one.

C. Since, for each n in the data array (14), the difference of  $|y_n - y_n'|$  must be kept within the limit of the error  $\varepsilon = 10^{-3}$ , the general solution of the discussed problem has the following form <sup>6,7</sup>

$$y'(\xi, x) = g_{\alpha} \{ (1.0012 - 0.0001\xi) x - (0.161200 - 0.000127\xi) x^{3} \},$$
 (19)

where  $\alpha = 0.001$ ;  $0 \le \xi \le 49$  and, consequently,  $b_1 = 0.9987 \pm 0.0025$  and  $b_2 = -0.1582 \pm 0.0030$ .

By analysing solutions (16), (18) and (19) we conclude that in the considered case

the solution of the problem on finding the best fitting multinomial depends on the type of the used mathematical theory.

**Example 5**. In some software products {for instance, in the different versions of software CURVE, designed by AISN} the solutions of problems on finding the best approximative models are found by the magnitude of a determination coefficient R, which value may be computed by a set of formulae

$$R_1 = \sqrt{1 - Q_r / Q}, \ Q_r = \sum_{n=1}^{N} (y_n - y_n')^2, \ Q = \sum_{n=1}^{N} (y_n - \overline{y})^2$$
 (20)

where  $\bar{y} = \sum_{n=1}^{N} y_n / N$ ,  $y_n$  is *n*-th reading of dependent variable,  $y'_n$  is *n*-th value of dependent variable, computed on the fitting model; or by formula (firstly offered by K.Pearson)

$$R_{2} = \frac{\sum_{n=1}^{N} (y_{n} - \overline{y})(y'_{n} - \overline{y}')}{\sqrt{\sum_{n=1}^{N} (y_{n} - \overline{y})^{2} \sum_{n=1}^{N} (y_{n} - \overline{y}')^{2}}}$$
(21)

{evidently, one may easy obtain formula (21) from formula (10)}.

							Table 5.				
	The simulative data array to example 5										
	n	y <sub>n</sub>	X <sub>n</sub>	_	n	y <sub>n</sub>	X <sub>n</sub>				
	1	85	11		7	205	3.8136396				
	2	105	5.6002132		8	225	3.5774037				
;	3	125	5.0984022	_	9	245	3.4193292				
	4	145	4.7047836	_	10	265	3.2903451				
	5	165	4.3936608	_	11	285	3.1026802				
- (	6	185	4.0998636		-	-	-				

There is a mathematical proof  $^{23}$  of the equivalence of formulae (20) and (21). But, if the value of coefficient  $R_2^2$  is computed within error  $\geq 10^{-8}$ ,

in actual practice, for some data arrays, firstly,  $R_2^2 \neq R_1^2$  and, secondly,  $R_2^2 > 1$ . For instance, if one fits the simulative data array, presented in table 5, by the multinomial  $P_M(\mathbf{B}, x)$  with M = 8, then software CURVE-2.0 will give the value  $R_2^2 = 1.00040$ .

# 4 Problems of quantitative processing experimental dependences found for heterogeneous objects

As it follows from the general consideration <sup>24, 25</sup>, in practice at analysis of experimental dependences found for heterogeneous objects, three various situations can be realised: the heterogeneity of investigated objects causes a) no effect; b) a removable (local) inadequacy of postulated fitting model; c) an irremovable (global) inadequacy of the postulated model. In this section we discuss some computative difficulties which may occur at analysis of the mentioned experimental dependences.

**Example 6.** As we know from Sect. 2.1 if F(A, x) is a linear model  $\{F(A, x) = \sum_{l=1}^{L} a_l h_l(x)\}$  then the value of A', minimising residual sum-of-squares S, is

computed by (1). Let rank H < L, or, in other words, there is a linear dependence between columns of matrix H:

$$c_1 h_1 + c_2 h_2 + \ldots + c_L h_L = 0,$$
 (22)

where at least one coefficient  $c_l \neq 0$ . In this case matrix  $(\mathbf{H}^T \mathbf{H})^{-1}$  does not exist, that means one cannot find  $\mathbf{A}'$  from (1). Such situation is known as *strict multicollinearity*.

In the natural science investigations values of the independent variable X are always determined with a certain round-off error, although this error may be very small. Therefore, if even strict multicollinearity is present, in practice the equation (22) is satisfied only approximately and therefore rank H = L. In such situation application of equation (1) to find the estimate of vector parameter A gives A' values drastically deviating from true coefficients values  $^{6,7,23}$ .

To correct this situation in regression on characteristics roots  $^{26}$  it is suggested to obtain the information about the grade of matrix  $\mathbf{H}^T\mathbf{H}$  conditioning from values of its eigennumbers  $\lambda_j$  and first elements  $V_{0j}$  of its eigenvector  $\mathbf{V}_j$  and to exclude from regression such j-components, whose eigennumbers  $\lambda_j$  and elements  $V_{0j}$  are small. Following values are recommended to use as critical ones:  $\lambda_{cr} = 0.05$  and  $V_{cr} = 0.1$ .

Let us demonstrate, that

in some practical computations the difference between  $A'_{CHR}$  and  $A'_{LS}$  of (1) can be explained not by the effects of multicollinearity, but by regression model inadequacy, which disappears simultaneously with the effects of multicollinearity after removing outliers.

Indeed, let data array be following

$$\{y_n, \mathbf{X}_n\} = \{1 + 0.5 \ n + 0.05 \ n^2 + 0.005 \ n^3; \ n, n^2, n^3\},\tag{23}$$

n = 1, 2, ..., 11 and we introduce two outliers in (23), by means of increasing values  $y_3$  and  $y_8$  on 0.5. For this data array we obtain the following computation results:

$$N_{\text{step}} = 1, N = 11$$
:  $\mathbf{A'}_{LS} = (1.017; 0.542; 0.0462; 0.0050),$ 
 $n_a = \{3, 8\}$   $\mathbf{A'}_{CHR} = (1.046; 0.516; 0.0516; 0.0047);$ 
 $N_{\text{step}} = 2, N = 10$ :  $\mathbf{A'}_{LS} = (0.764; 0.801; -0.0169; 0.0090),$ 
 $n_a = \{3\}$   $\mathbf{A'}_{CHR} = (0.764; 0.801; -0.0169; 0.0090),$ 

where  $N_{\text{step}}$  is a number of the step in used computative procedure;  $n_a$  is a vector to indicate the numbers of anomalous readings, contained in analysing array on the first and second steps of used computative procedure; N is the general quantity of analysing readings. In particular, after the first step of computative procedure from (23) the reading with number 8 is removed; after the second step — readings with numbers 3 and 8. And after the second step the values of A' are restored without any distortion by both examined algorithms.

By analysing obtained computation results one can conclude that

the difference between  $A'_{LS}$  and  $A'_{CHR}$  may be caused not only by multicollinearity but also, for instance, by a set of outliers presented in the data array.

**Example 7.** In table 6 we adduce an experimental data array, obtained by N.P.Bobrysheva<sup>27</sup>, when she investigated magnetic susceptibility ( $\chi$ , relative units) of polycrystalline system  $V_x Al_{1-x} O_{1,5}$  (x = 0.078) on the temperature (T, K). Let us consider some computation results of quantitative processing this temperature dependence.

Table 6. The experimental dependence of magnetic susceptibility of system  $V_x Al_{1-x} O_{1,5}$  (x = 0.078) on temperature

n_	Tn	Xn			
1	80	10.97			
2	121	8.06			
3	144	6.94			
4	182	5.56			
5	202	5.11			
6	214	4.75			
7	220	4.62			
8	267	4.00			

,		
n	Tn	χn
9	292	3.79
10	351	3.31
11	360	3.20
12	385	3.06
13	401	3.00
14	438	2.78
15	464	2.67
16	486	2.56

n	Tn	χл						
17	501	2.48						
18	512	2.46						
19	523	2.42						
20	559	2.28						
21	601	2.12						
22	651	2.02						
23	668	1.97						
-	-	-						

T

630

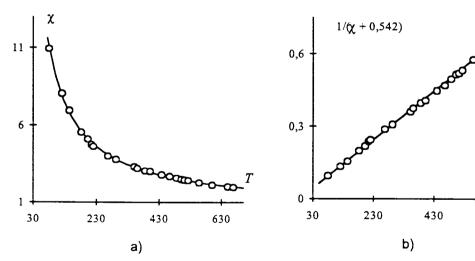


Figure 4. Experimental (circles) and analytical (continuous curves) plots of dependences  $\chi(T)$  (a) and  $1/(\chi + \chi_2) - T$  (b) for system  $V_X AI_{1-X} O_{1.5}$  (x = 0.078).

A. In figure 4(a, b) for the discussed system experimental (circles) and analytical (continuous curves) plots of dependences  $\chi(T)$  and  $1/(\chi + \chi_2) - T$  are shown. For construction of analytical (continuous) curves we use modified Curie – Weiss law <sup>6,7,24,25</sup>

$$\chi = \chi_0 + C/(T + \theta), \tag{24}$$

where  $\chi$  is the experimental magnitude of specific magnetic susceptibility; T is absolute temperature, K; C,  $\theta$  and  $\chi_0$  are parameters: C = 988;  $\theta = 14$ , K;  $\chi_0 = 0.54$ . From analysing graphical information, presented in figure 4(a, b), one can conclude, that

the magnetic behaviour of system  $V_xAl_{1-x}O_{1.5}$  (x = 0.078) is well explained by the modified Curie – Weiss law (24).

B. In figure 5(a) the dependence  $\Delta \chi = \chi - C/(T + \theta) - \chi_0$  on T for system  $V_x Al_{1-x}O_{1,5}$  (x = 0.078) is shown. Since  $\Delta \chi_{max} \approx 0.2 >> \epsilon = 0.01$ , where  $\epsilon$  is the measurement error in the discussed experiment, we obtain

in contradiction with the statement of point (A) in this case modified Curie – Weiss law (24) is an inadequate approximative model or, in other words, there is a set of outliers in the analysing experimental dependence.

C. After deleting first 5 readings from the initial data array the parameters values of modified Curie – Weiss law (24) have magnitudes C = 1386;  $\theta = 89$ , K;  $\chi_0 = 0.14$ . The plot of dependence  $\Delta \chi = \chi - C/(T + \theta) - \chi_0$  with foregoing parameters values is shown in figure 5(b).

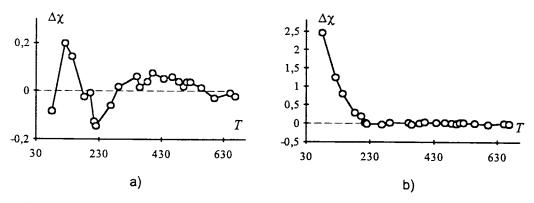


Figure 5. Plots of  $\Delta \chi = \chi - C/(T + \theta) - \chi_0$  on T for system  $V_x AI_{1-x} O_{1.5}$  (x = 0.078).

Analysing plots  $\Delta \chi(T)$ , presented in figure 5(a, b), and comparing with each other the parameters values of equation (24), mentioned in points (A) and (C), we conclude, that

neglect of the local inadequacy of the approximative model in the discussed experiment leads to distortion of both form of function  $\Delta \chi(T)$  and parameters values of the modified Curie – Weiss law (24).

Thus, if, for proving well-fitted properties of equation (24), researchers  $^{27-30}$  suggest to look at the graphic representation of dependences  $\chi(T)$  or  $1/\chi(T)$ , for the proof completeness one should ask these researchers to present information about the measurement error of values  $\chi$  and plots  $\Delta\chi(T) = \chi - C/(T+\theta) - \chi_0$ .

For the sake of convenience, the main causes, given rise to computative difficulties at analysis of experimental dependences found for heterogeneous objects, and methods of their overcoming are adduced in table 7 together. In this table all methods, overcoming computative difficulties, are marked by the symbol  $\Theta$ , if at present they are *in the rough* or absent in modern data analysis theory.

Table 7.

	Main causes and overcoming methods of computative difficulties in modern data analysis theory									
	Main causes Methods of overcoming									
1.	Impossibility to take heed of preset measurement accuracy of dependent variable values in the frame of accepted data analysis model	Θ Modification of data analysis model								
2.	Limited accuracy of computations	Increasing computation accuracy								
3.	Point estimation of parameters	Replacing the point estimation of parameters by interval one								
4.	The deficient measurement accuracy of dependent variable values	Increasing measurement accuracy of dependent variable values								
5.	Ill-conditioning of estimation problem	<ul> <li>Θ Using alternative estimations methods; Increasing measurement accuracy of dependent variable values;</li> <li>Θ Revealing and removing outliers; Designing experiments</li> </ul>								
6.	Presence of outliers in analysing data arrays	<ul> <li>Θ Revealing and removing outliers;</li> <li>Θ Robust estimation of parameters</li> </ul>								
7.	Inadequacy of approximative model	<ul> <li>Θ Eliminating inadequacy of approximative model;</li> <li>Θ Using advanced estimations methods</li> </ul>								
8.	Finding only single solution of the estimation problems for contaminated data array in the frame of modern data analysis theories	Θ Finding a family of solutions								

Using information presented in table 7, let us clear up a question, whether one is able in the frame of modern data analysis theory to obtain reliable solutions for the problems of quantitative processing of experimental dependences, found for heterogeneous objects.

Let, when an investigated object is homogeneous, a connection between characteristics y and X exist and it be close to functional one: y = F(A, X). As we said already in beginning of this section, in the discussed experiments three various situations can be realised: the structural heterogeneity

- 1) has no effect on the experimental dependence  $\{y_n, X_n\}$  or, in other words, in this case it is impossible to distinguish the homogeneous objects from heterogeneous ones on the dependence  $\{y_n, X_n\}$ ;
- 2) leads to a distortion of the dependence  $\{y_n, X_n\}$  in some small region  $\{X_{n_1}\} \subset \{X_n\}$  {the approximative model F(A, X) has removable (local) inadequacy}. In this case for extracting effects, connected with the presence of a homogeneity in the investigated objects, one may use the following way<sup>6,7</sup>
  - i) solve the problem on revealing outliers  $\{y_{n_1}, X_{n_1}\}$ ;
- *ii*) determine the value A' on readings  $\{y_n, X_n\} \setminus \{y_{n_1}, X_{n_1}\}$  {we remind, that a set of  $\{y_n, X_n\} \setminus \{y_{n_1}, X_{n_1}\}$  is to be well-fitted by the model F(A, X)};
- iii) detect a type and degree of the effects, connected with the presence of a homogeneity in the investigated objects, on the data array  $\{y_{n_1} F(\mathbf{A'}, \mathbf{X}_{n_1}), \mathbf{X}_{n_1}\}$ .

It follows from point 6 of table 7, that at solving problem (i) in actual practice some difficulties, which are unsurmountable in the frame of modern regression analysis theory, can be arisen;

3) leads to a distortion of the dependence  $\{y_n, X_n\}$  in a big region  $\{X_{n_1}\} \subseteq \{X_n\}$ : {the approximative model F(A, X) has *irremovable* (global) inadequacy}.

It follows from point 7 of table 7, that in this case it is impossible to find a reliable solution of the discussed problem in the frame of modern data analysis theory.

Summarising mentioned in points (1) - (3), we conclude

since at present the methods, marked by the symbol  $\Theta$  in table 7, are not effective for overcoming computative difficulties or absent in modern data analysis theory, one is not able to obtain reliable solutions for the problems of quantitative processing of experimental dependences found for heterogeneous objects.

From our point of view, one of possible ways, overcoming computative difficulties in modern data analysis theory, is further development of this theory by means of translation of this theory from Aristotelian "binary logic" into more progressive "fuzzy logic" <sup>6,7,24,25,31,32</sup>.

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## THE NORMAL BEHAVIOR OF THE SMARANDACHE FUNCTION

#### KEVIN FORD

Let S(n) be the smallest integer k so that n|k!. This is known as the Smarandache function and has been studied by many authors. If P(n) denotes the largest prime factor of n, it is clear that  $S(n) \ge P(n)$ . In fact, S(n) = P(n) for most n, as noted by Erdös [E]. This means that the number, N(x), of  $n \le x$  for which  $S(n) \ne P(n)$  is o(x). In this note we prove an asymptotic formula for N(x).

First, denote by  $\rho(u)$  the Dickman function, defined by

$$\rho(u) = 1 \quad (0 \le u \le 1), \qquad \rho(u) = 1 - \int_1^u \frac{\rho(v-1)}{v} \, dv \quad (u > 1).$$

For u > 1 let  $\xi = \xi(u)$  be defined by

$$u = \frac{e^{\xi} - 1}{\xi}.$$

It can be easily shown that

$$\xi(u) = \log u + \log_2 u + O\left(\frac{\log_2 u}{\log u}\right),$$

where  $\log_k x$  denotes the kth iterate of the logarithm function. Finally, let  $u_0 = u_0(x)$  be defined by the equation

$$\log x = u_0^2 \xi(u_0).$$

The function  $u_0(x)$  may also be defined directly by

$$\log x = u_0 \left( x^{1/u_0^2} - 1 \right).$$

It is straightforward to show that

(1) 
$$u_0 = \left(\frac{2\log x}{\log_2 x}\right)^{\frac{1}{2}} \left(1 - \frac{\log_3 x}{2\log_2 x} + \frac{\log 2}{2\log_2 x} + O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^2\right)\right).$$

We can now state our main result.

Theorem 1. We have

$$N(x) \sim \frac{\sqrt{\pi}(1 + \log 2)}{2^{3/4}} (\log x \log_2 x)^{3/4} x^{1 - 1/u_0} \rho(u_0).$$

There is no way to write the asymptotic formula in terms of "simple" functions, but we can get a rough approximation.

Corollary 2. We have

$$N(x) = x \exp\left\{-(\sqrt{2} + o(1))\sqrt{\log x \log_2 x}\right\}.$$

The asymptotic formula can be made a bit simpler, without reference to the function  $\rho$  as follows.

Corollary 3. We have

$$N(x) \sim \frac{e^{\gamma}(1 + \log 2)}{2\sqrt{2}} (\log x)^{\frac{1}{2}} (\log_2 x) x^{1-2/u_0} \exp\left\{ \int_0^{\frac{\log x}{u_0^2}} \frac{e^v - 1}{v} \, dv \right\},\,$$

where  $\gamma = 0.5772...$  is the Euler-Mascheroni constant.

This will follow from Theorem 1 using the formula in Lemma 2 which relates  $\rho(u)$  and  $\xi(u)$ .

The distribution of S(n) is very closely related to the distribution of the function P(n). We begin with some standard estimates of the function  $\Psi(x,y)$ , which denotes the number of integers  $n \leq x$  with  $P(n) \leq y$ .

Lemma 1 [HT, Theorem 1.1]. For every  $\epsilon > 0$ ,

$$\Psi(x,y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right), \quad u = \frac{\log x}{\log y},$$

uniformly in  $1 \le u \le \exp\{(\log y)^{3/5} \cdot \epsilon\}$ .

Lemma 2 [HT, Theorem 2.1]. For  $u \ge 1$ ,

$$\begin{split} \rho(u) &= \left(1 + O\left(\frac{1}{u}\right)\right) \sqrt{\frac{\xi'(u)}{2\pi}} \exp\left\{\gamma - \int_1^u \xi(t) \, dt\right\} \\ &= \exp\left\{-u \left(\log u + \log_2 u - 1 + O\left(\frac{\log_2 u}{\log u}\right)\right)\right\}. \end{split}$$

Lemma 3 [HT, Corollary 2.4]. If u > 2,  $|v| \leq u/2$ , then

$$\rho(u - v) = \rho(u) \exp\{v\xi(u) + O((1 + v^2)/u)\}.$$

Further, if u > 1 and  $0 \le v \le u$  then

$$\rho(u-v) \ll \rho(u)e^{v\xi(u)}.$$

We will show that most of the numbers counted in N(x) have

$$P(n) \approx \exp\left\{\sqrt{\frac{1}{2}\log x \log_2 x}\right\}.$$

Let

$$Y_1 = \exp\left\{\frac{1}{3}\sqrt{\log x \log_2 x}\right\}, \quad Y_2 = Y_1^6 = \exp\left\{2\sqrt{\log x \log_2}\right\}.$$

Let  $N_1$  be the number of n counted by N(x) with  $P(n) \leq Y_1$ , let  $N_2$  be the number of n with  $P(n) \geq Y_2$ , and let  $N_3 = N(x) - N_1 - N_2$ . By Lemmas 1 and 2,

$$N_1 \leqslant \Psi(x, Y_1) = x \exp\{-(1.5 + o(1))\sqrt{\log x \log_2 x}\}.$$

For the remaining  $n \le x$  counted by N(x), let p = P(n). Then either  $p^2|n$  or for some prime q < p and  $b \ge 2$  we have  $q^b \parallel n$ ,  $q^b \nmid p!$ . Since p! is divisible by  $q^{[p/q]}$  and  $b \le 2 \log x$ , it follows that  $q > p/(3 \log x) > p^{1/2}$ . In all cases n is divisible by the square of a prime  $\ge Y_2/(3 \log x)$  and therefore

$$N_2 \leqslant \sum_{\substack{p \geqslant \frac{Y_2}{3\log x}}} \frac{x}{p^2} \leqslant \frac{6x\log x}{Y_2} \ll x \exp\left\{-1.9\sqrt{\log x \log_2 x}\right\}.$$

Since  $q > p^{1/2}$  it follows that  $q^{[p/q]} \parallel p!$ . If n is counted by  $N_3$ , there is a number  $b \ge 2$  and prime  $q \in [p/b, p]$  so that  $q^b \mid n$ . For each  $b \ge 2$ , let  $N_{3,b}(x)$  be the number of n counted in  $N_3$  such that  $q^b \parallel n$  for some prime  $q \ge p/b$ . We have

$$\sum_{b \ge 6} N_{3,b} \ll x \left( \frac{3 \log x}{Y_1} \right)^5 \ll x \exp\left\{ -(5/3 + o(1)) \sqrt{\log x \log_2 x} \right\}.$$

Next, using Lemma 1 and the fact that  $\rho$  is decreasing, for  $3 \leq b \leq 5$  we have

$$\begin{split} N_{3,b} &= \sum_{Y_1$$

By partial summation, the Prime Number Theorem, Lemma 2 and some algebra,

$$N_{3,b} \ll \exp\left\{-(1.5 + o(1))\sqrt{\log x \log_2 x}\right\}.$$

The bulk of the contribution to N(x) will come from  $N_{3,2}$ . Using Lemma 1 we obtain

(2)

$$\begin{split} N_{3,2} &= \sum_{Y_1$$

By Lemma 3, we can write

$$\rho\left(\frac{\log x - \log p}{\log q} - 2\right) = \rho\left(\frac{\log x}{\log q} - 3\right)\left(1 + O\left(\sqrt{\frac{\log_2 x}{\log x}}\right)\right).$$

The contribution in (2) from p near  $Y_1$  or  $Y_2$  is negligible by previous analysis, and for fixed  $q \in [Y_1, Y_2/2]$  the Prime Number Theorem implies

$$\sum_{q$$

Reversing the roles of p, q in the second sum in (2), we obtain

$$N_{3,2} = \left(1 + O\left(\sqrt{\frac{\log_2 x}{\log x}}\right)\right) x \sum_{Y_1 \le p \le Y_2} \frac{1}{p^2} \left(\rho\left(\frac{\log x}{\log p} - 2\right) + \frac{\log 2}{\log p} \rho\left(\frac{\log x}{\log p} - 3\right)\right).$$

By partial summation, the Prime Number Theorem with error term, and the change of variable  $u = \log x / \log p$ ,

(3) 
$$N_{3,2} = \left(1 + O\left(\sqrt{\frac{\log_2 x}{\log x}}\right)\right) x \int_{u_1}^{u_2} \left(\frac{\rho(u-2)}{u} + \frac{\log 2}{\log x}\rho(u-3)\right) x^{-1/u} du,$$

where

$$u_1 = \frac{1}{2} \sqrt{\frac{\log x}{\log_2 x}}, \qquad u_2 = 6u_1.$$

The integrand attains its maximum value near  $u = u_0$  and we next show that the most of the contribution of the integral comes from u close to  $u_0$ . Let

$$w_0 = \frac{u_0}{100}, \quad w_1 = K\sqrt{u_0}, \quad w_2 = w_1 \left(\frac{\log_3 x}{\log_2 x}\right)^{1/2},$$

where K is a large absolute constant. Let  $I_1$  be the contribution to the integral in (3) with  $|u-u_0| > w_0$ , let  $I_2$  be the contribution from  $w_1 < |u-u_0| \leqslant w_0$ , let  $I_3$  be the contribution from  $w_2 < |u-u_0| \leqslant w_1$ , and let  $I_4$  be the contribution from  $|u-u_0| \leqslant w_2$ . First, by Lemma 2, the integrand in (3) is

$$\exp\left\{-\left(\frac{1}{c} - \frac{c}{2} + o(1)\right)\sqrt{\log x \log_2 x}\right\}, \quad c = \left(\frac{\log_2 x}{\log x}\right)u.$$

The function 1/c + c/2 has a minimum of  $\sqrt{2}$  at  $c = \sqrt{2}$ , so it follows that

$$I_1 \ll \exp\left\{-\left(\sqrt{2} + 10^{-5}\right)\sqrt{\log x \log_2 x}\right\}.$$

Let  $u = u_0 - v$ . For  $w_1 \le |v| \le w_0$ , Lemma 2 and the definition (1) of  $u_0$  imply that the integrand in (3) is

$$\leqslant \rho(u_0) \exp\left\{v\xi(u_0) - \frac{\log x}{u_0} \left(1 + \frac{v}{u_0} + \frac{v^2}{u_0^2} + \frac{v^3}{u_0^3}\right) + O\left(\frac{v^2}{u_0} + \log u_0\right)\right\} \\
\ll \rho(u_0) x^{-1/u_0} \exp\left\{-\frac{v^2}{u_0^3} \log x + O\left(\frac{v^2}{u_0} + \log u_0\right)\right\} \\
\ll \rho(u_0) x^{-1/u_0} \exp\left\{-0.9 \frac{v^2}{u_0^3} \log x\right\}$$

for K large enough. It follows that

$$I_2 \ll u_0 \rho(u_0) x^{-1/u_0} \exp\{-20 \log_2 x\} \ll (\log x)^{-10} \rho(u_0) x^{-1/u_0}$$
.

For the remaining u, we first apply Lemma 3 with v=2 and v=3 to obtain

$$I_3 + I_4 = \left(1 + O\left(\sqrt{\frac{\log_2 x}{\log x}}\right)\right) \int_{u_0 - w_1}^{u_0 + w_1} \rho(u) x^{-1/u} \left(\frac{e^{2\xi(u)}}{u} + \frac{\log 2}{\log x} e^{3\xi(u)}\right) du$$

We will show that  $I_3 + I_4 \gg \rho(u_0)x^{-1/u_0}(\log x)^{3/2}$ , which implies

$$(4) \quad N(x) = \left(1 + O\left(\sqrt{\frac{\log_2 x}{\log x}}\right)\right) \int_{u_0 - w_1}^{u_0 + w_1} \rho(u) x^{-1/u} \left(\frac{e^{2\xi(u)}}{u} + \frac{\log 2}{\log x} e^{3\xi(u)}\right) du.$$

This provides an asymptotic formula for N(x), but we can simplify the expression somewhat at the expense of weakening the error term. First, we use the formula

$$\xi(u) = \log u + \log_2 u + O\left(\frac{\log_2 u}{\log u}\right),$$

and then use  $u = u_0 + O(u_0^{1/2})$  and (1) to obtain

$$I_3 + I_4 = \left(1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right) \frac{\sqrt{2}}{4} (1 + \log 2) x (\log x)^{\frac{1}{2}} (\log_2 x)^{\frac{3}{2}} \int_{u_0 - w_1}^{u_0 + w_1} \rho(u) x^{-1/u} du.$$

By Lemma 3, when  $w_2 \leq |v| \leq w_1$ , where  $u = u_0 - v$ , we have

$$\rho(u_0 - v)x^{-\frac{1}{u_0 - v}} \ll \rho(u_0)x^{-\frac{1}{u_0}} \exp\left\{v\xi(u_0) - \frac{\log x}{u_0} \left(\frac{v}{u_0} + \frac{v^2}{u_0^2} + \frac{v^3}{u_0^3}\right)\right\} 
\ll \rho(u_0)x^{-\frac{1}{u_0}} \exp\left\{-\frac{v^2}{u_0^3}\log x\right\} 
\ll \rho(u_0)x^{-\frac{1}{u_0}} \exp\left\{-\frac{w_2^2}{u_0^3}\log x\right\} 
\ll \rho(u_0)x^{-\frac{1}{u_0}} (\log_2 x)^{-3}$$

provided K is large enough. This gives

$$\int_{w_2 \leqslant |u-u_0| \leqslant w_1} \rho(u) x^{-1/u} du \ll \rho(u_0) x^{-1/u_0} (\log x)^{1/4} (\log_2 x)^{-3.5}.$$

For the remaining v, Lemma 3 gives

$$\rho(u_0 - v)x^{-1/(u_0 - v)} = \left(1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\rho(u_0)x^{-1/u_0} \exp\left\{-\frac{v^2}{u_0^3}\log x\right\}.$$

Therefore,

$$\rho(u_0)^{-1} x^{\frac{1}{u_0}} \int_{u_0-w_2}^{u_0+w_2} \rho(u) x^{-1/u} \, du = \left(1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right) \int_{-w_2}^{w_2} \exp\left\{-v^2 \frac{\log x}{u_0^3}\right\} \, dv.$$

The extension of the limits of integration to  $(-\infty, \infty)$  introduces another factor  $1 + O((\log_2 x)^{-1})$ , so we obtain

$$I_3 + I_4 = \left(1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right) \frac{\sqrt{\pi}(1 + \log 2)}{2^{3/4}} (\log x \log_2 x)^{3/4} \rho(u_0) x^{-\frac{1}{u_0}}$$

and Theorem 1 follows. Corollary 2 follows immediately from Theorem 1 and (1). To obtain Corollary 3, we first observe that  $\xi'(u) \sim u^{-1}$  and next use Lemma 2 to write

$$\rho(u_0) \sim \frac{e^{\gamma}}{\sqrt{2\pi u_0}} \exp\left\{-\int_1^{u_0} \xi(t) dt\right\}.$$

By the definitions of  $\xi$  and  $u_0$  we then obtain

$$\begin{split} \int_{1}^{u_{0}} \xi(t) \, dt &= \int_{0}^{\xi(u_{0})} e^{v} - \frac{e^{v} - 1}{v} \, dv \\ &= e^{\xi(u_{0})} - 1 - \int_{0}^{\xi(u_{0})} \frac{e^{v} - 1}{v} \, dv \\ &= \frac{\log x}{u_{0}} - \int_{0}^{\frac{\log x}{u_{0}^{2}}} \frac{e^{v} - 1}{v} \, dv. \end{split}$$

Corollary 3 now follows from (1).

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## On values of arithmetical functions at factorials I

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1. The Smarandache function is a characterization of factorials, since S(k!) = k, and is connected to values of other arithmetical functions at factorials. Indeed, the equation

$$S(x) = k \quad (k \ge 1 \text{ given}) \tag{1}$$

has d(k!) - d((k-1)!) solutions, where d(n) denotes the number of divisors of n. This follows from  $\{x: S(x) = k\} = \{x: x|k!, x \nmid (k-1)!\}$ . Thus, equation (1) always has at least a solution, if d(k!) > d((k-1)!) for  $k \geq 2$ . In what follows, we shall prove this inequality, and in fact we will consider the arithmetical functions  $\varphi, \sigma, d, \omega, \Omega$  at factorials. Here  $\varphi(n) = \text{Euler's arithmetical function}$ ,  $\sigma(n) = \text{sum of divisors of } n$ ,  $\omega(n) = \text{number of distinct prime factors of } n$ ,  $\Omega(n) = \text{number of total divisors of } n$ . As it is well known, we have  $\varphi(1) = d(1) = 1$ , while  $\omega(1) = \Omega(1) = 0$ , and for  $1 < \prod_{i=1}^r p_i^{a_i}$  ( $a_i \geq 1$ ,  $p_i$  distinct primes) one has

$$\varphi(n) = n \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right),$$

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1},$$

$$\omega(n) = r,$$

$$\Omega(n) = \sum_{i=1}^{r} a_i,$$

$$d(n) = \prod_{i=1}^{r} (a_i + 1). \tag{2}$$

The functions  $\varphi, \sigma, d$  are multiplicative,  $\omega$  is additive, while  $\Omega$  is totally additive, i.e.  $\varphi, \sigma, d$  satisfy the functional equation f(mn) = f(m)f(n) for (m, n) = 1, while  $\omega, \Omega$  satisfy the equation g(mn) = g(m) + g(n) for (m, n) = 1 in case of  $\omega$ , and for all m, n is case of  $\Omega$  (see [1]).

2. Let  $m = \prod_{i=1}^r p_i^{\alpha_i}$ ,  $n = \prod_{i=1}^r p_i^{\beta_i}$   $(\alpha_i, \beta_i \ge 0)$  be the canonical factorizations of m and n. (Here some  $\alpha_i$  or  $\beta_i$  can take the values 0, too). Then

$$d(mn) = \prod_{i=1}^{r} (\alpha_i + \beta_i + 1) \ge \prod_{i=1}^{r} (\beta_i + 1)$$

with equality only if  $\alpha_i = 0$  for all i. Thus:

$$d(mn) \ge d(n) \tag{3}$$

for all m, n, with equality only for m = 1.

Since 
$$\prod_{i=1}^{r} (\alpha_i + \beta_i + 1) \le \prod_{i=1}^{r} (\alpha_i + 1) \prod_{i=1}^{r} (\beta_i + 1)$$
, we get the relation 
$$d(mn) \le d(m)d(n)$$
 (4)

with equality only for (n, m) = 1.

Let now m = k, n = (k - 1)! for  $k \ge 2$ . Then relation (3) gives

$$d(k!) > d((k-1)!)$$
 for all  $k \ge 2$ , (5)

thus proving the assertion that equation (1) always has at least a solution (for k = 1 one can take x = 1).

With the same substitutions, relation (4) yields

$$d(k!) \le d((k-1)!)d(k) \text{ for } k \ge 2$$
 (6)

Let k = p (prime) in (6). Since ((p-1)!, p) = 1, we have equality in (6):

$$\frac{d(p!)}{d((p-1)!)} = 2, \quad p \text{ prime.}$$
 (7)

3. Since  $S(k!)/k! \to 0$ ,  $\frac{S(k!)}{S((k-1)!)} = \frac{k}{k-1} \to 1$  as  $k \to \infty$ , one may ask the similar problems for such limits for other arithmetical functions.

It is well known that

$$\frac{\sigma(n!)}{n!} \to \infty \text{ as } n \to \infty.$$
 (8)

In fact, this follows from  $\sigma(k) = \sum_{d|k} d = \sum_{d|k} \frac{k}{d}$ , so

$$\frac{\sigma(n!)}{n!} = \sum_{d|n!} \frac{1}{d} \ge 1 + \frac{1}{2} + \ldots + \frac{1}{n} > \log n,$$

as it is known.

From the known inequality ([1])  $\varphi(n)\sigma(n) \leq n^2$  it follows

$$\frac{n}{\varphi(n)} \ge \frac{\sigma(n)}{n},$$

so  $\frac{n!}{\varphi(n!)} \to \infty$ , implying

$$\frac{\varphi(n!)}{n!} \to 0 \text{ as } n \to \infty.$$
 (9)

Since  $\varphi(n) > d(n)$  for n > 30 (see [2]), we have  $\varphi(n!) > d(n!)$  for n! > 30 (i.e.  $n \ge 5$ ), so, by (9)

$$\frac{d(n!)}{n!} \to 0 \text{ as } n \to \infty. \tag{10}$$

In fact, much stronger relation is true, since  $\frac{d(n)}{n^{\varepsilon}} \to 0$  for each  $\varepsilon > 0$   $(n \to \infty)$  (see [1]). From  $\frac{d(n!)}{n!} < \frac{\varphi(n!)}{n!}$  and the above remark on  $\sigma(n!) > n! \log n$ , it follows that

$$\limsup_{n \to \infty} \frac{d(n!)}{n!} \log n \le 1. \tag{11}$$

These relations are obtained by very elementary arguments. From the inequality  $\varphi(n)(\omega(n)+1) \geq n$  (see [2]) we get

$$\omega(n!) \to \infty \text{ as } n \to \infty$$
 (12)

and, since  $\Omega(s) \ge \omega(s)$ , we have

$$\Omega(n!) \to \infty \text{ as } n \to \infty.$$
 (13)

From the inequality  $nd(n) \ge \varphi(n) + \sigma(n)$  (see [2]), and (8), (9) we have

$$d(n!) \to \infty \text{ as } n \to \infty.$$
 (14)

This follows also from the known inequality  $\varphi(n)d(n) \geq n$  and (9), by replacing n with n!. From  $\sigma(mn) \geq m\sigma(n)$  (see [3]) with n = (k-1)!, m = k we get

$$\frac{\sigma(k!)}{\sigma((k-1)!)} \ge k \quad (k \ge 2) \tag{15}$$

and, since  $\sigma(mn) \leq \sigma(m)\sigma(n)$ , by the same argument

$$\frac{\sigma(k!)}{\sigma((k-1)!)} \le \sigma(k) \quad (k \ge 2). \tag{16}$$

Clearly, relation (15) implies

$$\lim_{k \to \infty} \frac{\sigma(k!)}{\sigma((k-1)!)} = +\infty. \tag{17}$$

From  $\varphi(m)\varphi(n) \leq \varphi(mn) \leq m\varphi(n)$ , we get, by the above remarks, that

$$\varphi(k) \le \frac{\varphi(k!)}{\varphi((k-1)!)} \le k, \quad (k \ge 2)$$
(18)

implying, by  $\varphi(k) \to \infty$  as  $k \to \infty$  (e.g. from  $\varphi(k) > \sqrt{k}$  for k > 6) that

$$\lim_{k \to \infty} \frac{\varphi(k!)}{\varphi((k-1)!} = +\infty. \tag{19}$$

By writing  $\sigma(k!) - \sigma((k-1)!) = \sigma((k-1)!) \left[ \frac{\sigma(k!)}{\sigma((k-1)!)} - 1 \right]$ , from (17) and  $\sigma((k-1)!) \to \infty$  as  $k \to \infty$ , we trivially have:

$$\lim_{k \to \infty} [\sigma(k!) - \sigma((k-1)!)] = +\infty. \tag{20}$$

In completely analogous way, we can write:

$$\lim_{k \to \infty} [\varphi(k!) - \varphi((k-1)!)] = +\infty. \tag{21}$$

4. Let us remark that for k = p (prime), clearly ((k-1)!, k) = 1, while for k = composite, all prime factors of k are also prime factors of (k-1)!. Thus

$$\omega(k!) = \begin{cases} \omega((k-1)!k) = \omega((k-1)!) + \omega(k) & \text{if } k \text{ is prime} \\ \omega((k-1)!) & \text{if } k \text{ is composite } (k \ge 2). \end{cases}$$

Thus

$$\omega(k!) - \omega((k-1)!) = \begin{cases} 1, & \text{for } k = \text{prime} \\ 0, & \text{for } k = \text{composite} \end{cases}$$
 (22)

Thus we have

$$\lim_{k \to \infty} \sup [\omega(k!) - \omega((k-1)!)] = 1$$

$$\lim_{k \to \infty} \inf [\omega(k!) - \omega((k-1)!)] = 0$$
(23)

Let  $p_n$  be the nth prime number. From (22) we get

$$\frac{\omega(k!)}{\omega((k-1)!)} - 1 = \begin{cases} \frac{1}{n-1}, & \text{if } k = p_n \\ 0, & \text{if } k = \text{composite.} \end{cases}$$

Thus, we get

$$\lim_{k \to \infty} \frac{\omega(k!)}{\omega((k-1)!)} = 1. \tag{24}$$

The function  $\Omega$  is totally additive, so

$$\Omega(k!) = \Omega((k-1)!k) = \Omega((k-1)!) + \Omega(k),$$

giving

$$\Omega(k!) - \Omega((k-1)!) = \Omega(k). \tag{25}$$

This implies

$$\lim_{k \to \infty} \sup [\Omega(k!) - \Omega((k-1)!)] = +\infty$$
 (26)

(take e.g.  $k = 2^m$  and let  $m \to \infty$ ), and

$$\liminf_{k \to \infty} [\Omega(k!) - \Omega((k-1)!)] = 2$$

(take k = prime).

For  $\Omega(k!)/\Omega((k-1)!)$  we must evaluate

$$\frac{\Omega(k)}{\Omega((k-1)!)} = \frac{\Omega(k)}{\Omega(1) + \Omega(2) + \ldots + \Omega(k-1)}.$$

Since  $\Omega(k) \leq \frac{\log k}{\log 2}$  and by the theorem of Hardy and Ramanujan (see [1]) we have

$$\sum_{n \le x} \Omega(n) \sim x \log \log x \quad (x \to \infty)$$

so, since  $\frac{\log k}{(k-1)\log\log(k-1)} \to 0$  as  $k \to \infty$ , we obtain

$$\lim_{k \to \infty} \frac{\Omega(k!)}{\Omega((k-1)!)} = 1. \tag{27}$$

5. Inequality (18) applied for k = p (prime) implies

$$\lim_{p \to \infty} \frac{1}{p} \cdot \frac{\varphi(p!)}{\varphi((p-1)!)} = 1. \tag{28}$$

This follows by  $\varphi(p)=p-1$ . On the other hand, let k>4 be composite. Then, it is known (see [1]) that k|(k-1)!. So  $\varphi(k!)=\varphi((k-1)!k)=k\varphi((k-1)!)$ , since  $\varphi(mn)=m\varphi(n)$  if m|n. In view of (28), we can write

$$\lim_{k \to \infty} \frac{1}{k} \cdot \frac{\varphi(k!)}{\varphi((k-1)!)} = 1. \tag{29}$$

For the function  $\sigma$ , by (15) and (16), we have for k = p (prime) that  $p \leq \frac{\sigma(p!)}{\sigma((p-1)!)} \leq \sigma(p) = p+1$ , yielding

$$\lim_{p \to \infty} \frac{1}{p} \cdot \frac{\sigma(p!)}{\sigma((p-1)!)} = 1. \tag{30}$$

In fact, in view of (15) this implies that

$$\liminf_{k \to \infty} \frac{1}{k} \cdot \frac{\sigma(k!)}{\sigma((k-1)!)} = 1.$$
(31)

By (6) and (7) we easily obtain

$$\limsup_{k \to \infty} \frac{d(k!)}{d(k)d((k-1)!)} = 1.$$
 (32)

In fact, inequality (6) can be improved, if we remark that for k = p (prime) we have  $d(k!) = d((k-1)!) \cdot 2$ , while for k = composite, k > 4, it is known that k|(k-1)!. We apply the following

Lemma. If n|m, then

$$\frac{d(mn)}{d(m)} \le \frac{d(n^2)}{d(n)}. (33)$$

**Proof.** Let  $m = \prod p^{\alpha} \prod q^{\beta}$ ,  $n = \prod p^{\alpha'}$   $(\alpha' \leq \alpha)$  be the prime factorizations of m and n, where n|m. Then

$$\frac{d(mn)}{d(m)} = \frac{\prod(\alpha + \alpha' + 1)\prod(\beta + 1)}{\prod(\alpha + 1)\prod(\beta + 1)} = \prod\left(\frac{\alpha + \alpha' + 1}{\alpha + 1}\right).$$

Now  $\frac{\alpha + \alpha' + 1}{\alpha + 1} \le \frac{2\alpha' + 1}{\alpha' + 1} \iff \alpha' \le \alpha$  as an easy calculations verifies. This immediately implies relation (33).

By selecting now n = k, m = (k - 1)!, k > 4 composite we can deduce from (33):

$$\frac{d(k!)}{d((k-1)!)} \le \frac{d(k^2)}{d(k)}. (34)$$

By (4) we can write  $d(k^2) < (d(k))^2$ , so (34) represents indeed, a refinement of relation (6).

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## THE AVERAGE VALUE OF THE SMARANDACHE FUNCTION

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Given a positive integer n, let P(n) denote the largest prime factor of n and S(n) denote the smallest integer m such that n divides m!

The function S(n) is known as the Smarandache function and has been intensively studied [1]. Its behavior is quite erratic [2] and thus all we can reasonably hope for is a statistical approximation of its growth, e.g., an average. It appears that the sample mean E(S) satisfies [3]

$$E(S(N)) = \frac{1}{N} \cdot \sum_{n=1}^{N} S(n) = O\left(\frac{N}{\ln(N)}\right)$$

as N approaches infinity, but I don't know of a rigorous proof. A natural question is if some other sense of average might be more amenable to analysis.

Erdös [4,5] pointed out that P(n) = S(n) for almost all n, meaning

$$\lim_{N \to \infty} \frac{\left| \left\{ n \le N : P(n) < S(n) \right\} \right|}{N} = 0 \quad \text{that is,} \quad \left| \left\{ n \le N : P(n) < S(n) \right\} \right| = o(N)$$

as N approaches infinity. Kastanas [5] proved this to be true, hence the following argument is valid. On one hand,

$$\lambda = \lim_{n \to \infty} E\left(\frac{\ln(P(n))}{\ln(n)}\right) \le \lim_{n \to \infty} E\left(\frac{\ln(S(n))}{\ln(n)}\right) = \lim_{N \to \infty} \frac{1}{N} \cdot \sum_{n=1}^{N} \frac{\ln(S(n))}{\ln(n)}$$

The above summation, on the other hand, breaks into two parts:

$$\lim_{N \to \infty} \frac{1}{N} \cdot \left( \sum_{P(n) = S(n)} \frac{\ln(P(n))}{\ln(n)} + \sum_{P(n) < S(n)} \frac{\ln(S(n))}{\ln(n)} \right)$$

The second part vanishes:

$$\lim_{N \to \infty} \frac{1}{N} \cdot \left( \sum_{P(n) < S(n)} \frac{\ln(S(n))}{\ln(n)} \right) \le \lim_{N \to \infty} \frac{1}{N} \cdot \left( \sum_{P(n) < S(n)} 1 \right) = \lim_{N \to \infty} \frac{o(N)}{N} = 0$$

while the first part is bounded from above:

$$\lim_{N\to\infty} \frac{1}{N} \cdot \left( \sum_{P(n)=S(n)} \frac{\ln(P(n))}{\ln(n)} \right) \le \lim_{N\to\infty} \frac{1}{N} \cdot \sum_{n=1}^{N} \frac{\ln(P(n))}{\ln(n)} = \lim_{n\to\infty} E\left(\frac{\ln(P(n))}{\ln(n)}\right) = \lambda$$

We deduce that

$$\lim_{n \to \infty} E\left(\frac{\ln(S(n))}{\ln(n)}\right) = \lambda = 0.6243299885...$$

where  $\lambda$  is the famous Golomb-Dickman constant [6-9]. Therefore  $\lambda \cdot n$  is the asymptotic average number of digits in the output of S at an n-digit input, that is, 62.43% of the original number of digits. As far as I know, this result about the Smarandache function has not been published before.

A closely related unsolved problem concerns estimating the variance of S.

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# On the Irrationality of Certain Constants Related to the Smarandache Function

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1. Let S(n) be the Smarandache function. Recently I. Cojocaru and S. Cojocaru [2] have proved the irrationality of  $\sum_{n=1}^{n} \frac{S(n)}{n!}$ .

The author of this note [5] showed that this is a consequence of an old irrationality criteria (which will be used here once again), and proved a result implying the irrationality of  $\sum_{i=1}^{\infty} (-1)^{n-1} \frac{S(n)}{n!}$ .

E. Burton [1] has studied series of type  $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$ , which has a value  $\in \left(e-\frac{5}{2},\frac{1}{2}\right)$ . He showed that the series  $\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!}$  is convergent for all  $r \in \mathbb{N}$ . I. Cojocaru and S. Cojocaru [3] have introduced the "third constant of Smarandache" namely  $\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)}$ , which has a value between  $\frac{71}{100}$  and  $\frac{97}{100}$ . Our aim in the following is to prove that the constants introduced by Burton and Cojocaru-Cojocaru are all irrational.

2. The first result is in fact a refinement of an old irraionality criteria (see [4] p.5):

**Theorem 1.** Let  $(x_n)$  be a sequence of nonnegative integers having the properties:

- (1) there exists  $n_0 \in \mathbb{N}^*$  such that  $x_n \leq n$  for all  $n \geq n_0$ ;
- (2)  $x_n < n-1$  for infinitely many n;
- (3)  $x_m > 0$  for an infinity of m.

Then the series  $\sum_{n=1}^{\infty} \frac{x_n}{n!}$  is irrational.

Let now  $x_n = S(n-1)$ . Then

$$\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!} = \sum_{n=3}^{\infty} \frac{x_n}{n!}.$$

Here  $S(n-1) \le n-1 < n$  for all  $n \ge 2$ ; S(m-1) < m-2 for m > 3 composite, since by  $S(m-1) < \frac{2}{3}(m-1) < m-2$  for m > 4 this holds true. (For the inequality  $S(k) < \frac{2}{3}k$  for k > 3 composite, see [6]). Finally, S(m-1) > 0 for all  $m \ge 1$ . This proves the irrationality of  $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$ .

Analogously, write

$$\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!} = \sum_{m=r+2}^{\infty} \frac{S(m-r)}{m!}.$$

Put  $x_m = S(m-r)$ . Here  $S(m-r) \le m-r < m$ ,  $S(m-r) \le m-r < m-1$  for  $r \ge 2$ , and S(m-r) > 0 for  $m \ge r+2$ . Thus, the above series is irrational for  $r \ge 2$ , too.

3. The third constant of Smarandache will be studied with the following irrationality criterion (see [4], p.8):

**Theorem 2.** Let  $(a_n), (b_n)$  be two sequences of nonnegative integers satisfying the following conditions:

- (1)  $a_n > 0$  for an infinity of n;
- (2)  $b_n \ge 2$ ,  $0 \le a_n \le b_n 1$  for all  $n \ge 1$ ;
- (3) there exists an increasing sequence  $(i_n)$  of positive integers such that

$$\lim_{n\to\infty}b_{i_n}=+\infty,\quad \lim_{n\to\infty}a_{i_n}/b_{i_n}=0.$$

Then the series  $\sum_{n=1}^{\infty} \frac{a_n}{b_1 b_2 \dots b_n}$  is irrational.

Corollary. For  $b_n \geq 2$ ,  $(b_n \text{ positive integers})$ ,  $(b_n)$  unbounded the series  $\sum_{n=1}^{\infty} \frac{1}{b_1 b_2 \dots b_n}$  is irrational.

**Proof.** Let  $a_n \equiv 1$ . Since  $\limsup_{n \to \infty} b_n = +\infty$ , there exists a sequence  $(i_n)$  such that  $b_{i_n} \to \infty$ . Then  $\frac{1}{b_{i_n}} \to 0$ , and the three conditions of Theorem 2 are verified.

By selecting  $b_n \equiv S(n)$ , we have  $b_p = S(p) = p \to \infty$  for p a prime, so by the above Corollary, the series  $\sum_{n=1}^{\infty} \frac{1}{S(1)S(2)...S(n)}$  is irrational.

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## Smarandache Magic Squares

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The objective of this article is to investigate the existence of magic squares made with Smarandache's numbers [Tabirca, 1998]. Magic squares have been studied intensively and many aspects concerning them have been found. Many interesting things about magic squares can be found at the following WEB page http://www.pse.che.tohoko.ac.jp/~msuzuki/MagicSquares.html.

**Definition 1.** A Smarandache magic square is a square matrix  $a \in M_n(N)$  with the following properties:

a) 
$$\{a_{i,j} | i, j = \overline{1, n}\} = \{S(i) | i = \overline{1, n^2}\}$$
 (1)

b) 
$$\left(\forall j = \overline{1, n}\right) \sum_{i=1}^{n} a_{i,j} = k$$
 (2)

c) 
$$\left(\forall i = \overline{1, n}\right) \sum_{j=1}^{n} a_{i,j} = k$$
 (3)

Therefore, a Smarandache magic square is a square matrix by order n that contains only the elements  $S(1), S(2), ..., S(n^2)$  [Smarandache, 1980] and satisfies the sum properties (2-3). According to these properties, the sum of elements on each row or column should be equal to the same number k. Obviously, this number satisfies the following equation

$$k = \frac{\sum_{i=1}^{n^2} S(i)}{n}.$$

**Theorem 1.** If the equation  $n \mid \sum_{i=1}^{n^2} S(i)$  does not hold, then there is not a Smarandache magic square by order n.

## **Proof**

This proof is obvious by using the simple remark  $k = \frac{\sum_{i=1}^{n^2} S(i)}{n} \in N$ . If  $a \in M_n(N)$  is a Smarandache magic square, then the equation  $n \mid \sum_{i=1}^{n^2} S(i)$  should hold. Therefore, if this equation does not hold, there is no a Smarandache magic square.

Theorem 1 provides simple criteria to find the non-existence of Smarandache magic square. All the numbers 1 < n < 101 that do not satisfy the equation  $n \mid \sum_{i=1}^{n^2} S(i)$  can be found by using a simple computation [Ibstedt, 1997]. They are  $\{2, 3, ..., 100\} \setminus \{6, 7, 9, 58, 69\}$ . Clearly, a Smarandache magic square does not exist for this numbers. If n is one of the numbers 6, 7, 9, 58, 69 then the equation  $n \mid \sum_{i=1}^{n^2} S(i)$  holds [see Table 1]. This does not mean necessarily that there is a Smarandache magic square. In this case, a Smarandache magic square is found using other techniques such us detailed analysis or exhaustive computation.

	· · · · · · · · · · · · · · · · · · ·	n <sup>2</sup>
n	S(n)	$\sum_{i=1} S(i)$
6	3	330
7	7	602
9	6	1413
58	29	1310162
69	23	2506080

**Table 1.** The values of *n* that satisfy  $n \mid \sum_{i=1}^{n^2} S(i)$ .

An algorithm to find a Smarandache magic square is proposed in the following. This algorithm uses *Backtracking* strategy to complete the matrix *a* that satisfies (1-3). The going trough matrix is done line by line from the left-hand side to the right-hand side.

## The algorithm computes:

- Go trough the matrix
  - Find an unused element of the set  $S(1), S(2), ..., S(n^2)$  for the current cell.
  - If there is no an unused element, then compute a step back.
  - ♦ If there is an used element, then
    - Put this element in the current cell.
    - Check the backtracking conditions.
    - If they are verified and the matrix is full, then a Smarandache magic square has been found.
    - If they are verified and the matrix is not full, then compute a step forward.

```
procedure Smar magic square(n);
begin
                                                    procedure forward(col, row);
col:=1; row:=1;a[col, row]:=0;
                                                    begin
while row>0 do begin
                                                       col:=col+1;
  while a[col, row] < n*n do begin
                                                       if col=n+1 then begin
        a[col,row]:=a[col,row]+1;
                                                             col:=1;row:=row+1;
       call check(col,row,n,a,cont);
                                                       end;
       if cont=0 then exit;
                                                    end;
   if cont =0 then call back(col,row);
                                                    procedure write square(n,a);
   if cont=1 and col=n and row=n
                                                    begin
          then call write square(n,a)
                                                      for i:=1 to n do begin
           else call forward(col,row);
                                                        for j:=1 to n do write (S(a[i,j]), ');
                                                        writeln:
end;
write('result negative');
                                                      end;
end;
                                                      stop;
                                                    end;
procedure back(col, row);
                                                    procedure check(col,row,n,k,a,cont);
begin
  col:=col-1;
                                                    begin
                                                    cont:=1; sum:=0;
  if col=0 then begin
                                                    for i:=1 to col do sum:=sum+S(a[i,j]);
        col:=n;row:=row-1;
                                                    if (sum>k) or (col=n and sum<>k) then begin
  end;
                                                             cont:=0;
end;
```

return;		cont:=0;
end;		return;
sum:=0	end;	
for j:=1 to row do sum:=sum+S(a[i,j]);	end;	
if (sum>k) or (row=n and sum <k) hegin<="" td="" then=""><td></td><td></td></k)>		

Figure 1. Detailed algorithm for Smarandache magic squares.

The backtracking conditions are the following:

$$(\forall j = \overline{1, n}) \sum_{i=1}^{\infty l} a_{i,j} \le k \text{ and } (\forall j = \overline{1, n}) \sum_{i=1}^{n} a_{i,j} = k$$
 (4a)

$$\left(\forall i = \overline{1, n}\right) \sum_{j=1}^{row} a_{i,j} \le k \text{ and } \left(\forall i = \overline{1, n}\right) \sum_{j=1}^{n} a_{i,j} = k.$$
 (4b)

$$(\forall (i, j) < (row, col)) a_{i, j} \neq a_{row, col}$$
(5)

These conditions are checked by the procedure check. A detailed algorithm is presented in a pseudo-cod description in Figure 1.

**Theorem 2.** If there is a Smarandache magic square by order n, then the procedure Smar\_magic\_square finds it.

#### **Proof**

This theorem establishes the correctness property of the procedure Smar\_magic\_square. The *Backtracking* conditions are computed correctly by the procedure check that verifies if the equations (4-5) hold. The correctness this algorithm is given by the correctness of the *Backtracking* strategy. Therefore, this procedure finds a Smarandache magic square.

**Theorem 3.** The complexity of the procedure complexity Smar\_magic\_square is  $O(n^{2 \cdot n^2 + 1})$ .

#### **Proof**

The complexity of the procedure Smar\_magic\_square is studied in the worst case when there is not a Smarandache magic square. In this case, this procedure computes all the checking operations for the *Backtracking* strategy. Therefore, all the values

 $S(1), S(2), ..., S(n^2)$  are gone through for each cell. For each value put into a cell, at most O(n) operations are computed by the procedure check. Therefore, the complexity is  $O\left(n\cdot\left(n^2\right)^{n^2}\right) = O\left(n^{2\cdot n^2+1}\right)$ .

Remark 1. The complexity  $O(n^{2\cdot n^2+1})$  is not polynomial. Moreover, because this is a very big complexity, the procedure Smar\_magic\_square can be applied only for small values of n. For example, this procedure computes at most  $6^{73} > 10^{56}$  operations in the case n=6. The above procedure has been translated into a C program that has been run on a Pentium MMX 233 machine. The execution of this program has taken more than 4 hours for n=6. Unfortunately, there is not a Smarandache magic square for this value of n. The result of computation for n=7 has not been provided by computer after a twelve hours execution. This reflects the huge number of operations that should be computed  $(7^{99} > 10^{83})$ .

According to these negative results, we believe that Smarandache magic squares do not exist. If n is a big number that satisfy the equation  $n \mid \sum_{i=1}^{n^2} S(i)$ , then we have many possibilities to change, to permute and to arrange the numbers  $S(1), S(2), ..., S(n^2)$  into a square matrix. In spite of that Equations (2-3) cannot be satisfied. Therefore, we may conjecture the following: "There are not Smarandache magic squares". In order to confirm or infirm this conjecture, we need more powerful method than the above computation. Anyway, the computation looks for a particular solution, therefore it does not solve the problem.

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## Some inequalities concerning Smarandache's function

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The objectives of this article are to study the sum  $\sum_{d|n} S(d)$  and to find some upper bounds for Smarandache's function. This sum is proved to satisfy the inequality  $\sum_{d|n} S(d) \le n$  at most all the composite numbers. Using this inequality, some new upper bounds for Smarandache's function are found. These bounds improve the well-

## 1. Introduction

known inequality  $S(n) \le n$ .

The object that is researched is Smarandache's function. This function was introduced by Smarandache [1980] as follows:

$$S: N^* \to N$$
 defined by  $S(n) = \min\{k \in N | \mathbf{k}! = \underline{\mathbf{M}}\mathbf{n}\} \ (\forall n \in N^*)$  (1)

The following main properties are satisfied by S:

$$(\forall a, b \in N^*)(a, b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\}. \tag{2}$$

$$(\forall a \in N^*) S(a) \le a \text{ and } S(a) = a \text{ iif } a \text{ is prim}.$$
 (3)

$$(\forall p \in N^*, p \text{ prime})(\forall k \in N^*) S(p^k) \le p \cdot k. \tag{4}$$

Smarandache's function has been researched for more than 20 years, and many properties have been found. Inequalities concerning the function S have a central place and many articles have been published [Smarandache, 1980], [Cojocaru, 1997], [Tabirca, 1997], [Tabirca, 1988]. Two important directions can be identified among these inequalities. First direction and the most important is represented by the inequalities concerning directly the function S such as upper and lower bounds. The second direction is given by the inequalities involving sums or products with the function S.

## **2. About the sum** $\sum_{d|n} S(d)$

The aim of this section is to study the sum  $\sum_{d \mid n} S(d)$ .

Let  $SS(n) = \sum_{dn} S(d)$  denote the above sum. Obviously, this sum satisfies

 $SS(n) = \sum_{1 \neq d} S(d)$ . Table 1 presents the values of S(n) and SS(n) for n < 50 [Ibstedt,

1997]. From this table, it can be seen that the inequality  $SS(n) \le n+2$  holds for all n=1, 2, ..., 50 and  $n\ne 12$ . Moreover, if n is a prim number, then the inequality becomes equality SS(n) = n.

#### Remarks 1.

- a) If n is a prime number, then SS(n) = S(1) + S(n) = n.
- b) If n > 2 is a prim number, then  $SS(2 \cdot n) = S(1) + S(2) + S(n) + S(2 \cdot n) = 2 + n + n = 2 \cdot n + 2,$

c) 
$$SS(n^2) = S(1) + S(n) + S(n^2) = n + 2 \cdot n = 3 \cdot n \le n^2$$
.

$\overline{N}$	S	SS	n	S	SS	n	S	SS	n	S	SS	n	S	SS
1	0	0	11	11	11	21	7	17	31	31	31	41	41	41
2	2	2	12	4	16	22	11	24	32	8	24	42	7	36
3	3	3	13	13	13	23	23	23	33	11	25	43	43	43
4	4	6	14	7	16	24	4	24	34	17	36	44	11	39
5	5	5	15	5	13	25	10	15	35	7	19	45	6	25
6	3	8	16	6	16	26	13	28	36	6	34	46	23	48
7	7	7	17	17	17	27	9	18	37	37	37	47	47	47
8	4	10	18	6	20	28	7	27	38	19	40	48	6	36
9	6	9	19	19	19	29	29	29	39	13	29	49	14	21
10	5	12	20	5	21	30	5	28	40	5	30	50	10	32

**Table** 1. The values of n, S, SS.

The inequality  $SS(n) \le n$  is proved to be true for the following particular values  $n = p^k, 2 \cdot p^k, 3 \cdot p^k$  and  $6 \cdot p^k$ .

**Lemma 1.** If p>2 is a prime number and k>1, then the inequality  $SS(p^k) \le p^k$  holds.

#### **Proof**

The following inequality holds according to inequality (4) and the definition of SS.

$$SS(p^k) = \sum_{i=1}^k S(p^i) \le \sum_{i=1}^k p \cdot i = p \cdot \frac{k \cdot (k+1)}{2}.$$

The inequality

$$\sum_{i=1}^{k} p \cdot i = p \cdot \frac{k \cdot (k+1)}{2} \le p^k \tag{5}$$

is proved to be true by analysing the following cases.

• 
$$k=2 \Rightarrow 3 \cdot p \le p^2$$
. (6)

• 
$$k=3 \Rightarrow 6 \cdot p \le p^3$$
. (7)

• 
$$k=4 \Rightarrow 10 \cdot p \le p^4$$
. (8)

Inequalities (6-8) are true because p>2.

•  $k \ge 4 \implies p^k \ge p \cdot p^{k-1} \ge p \cdot 2^{k-1} = p \cdot \sum_{i=0}^{k-1} \binom{k-1}{i}$ . The first and the last three terms of this sum are kept and it is found

$$p^{k} \ge p \cdot \left(2 \cdot {k-1 \choose 0} + 2 \cdot {k-1 \choose 1} + 2 \cdot {k-1 \choose 2}\right) = p \cdot \left(k^{2} - k + 2\right).$$
 The inequality

$$k^2 - k + 2 \ge \frac{k \cdot (k+1)}{2}$$
 holds because  $k > 4$ , therefore  $p^k \ge p \cdot \frac{k \cdot (k+1)}{2}$  is true.

Therefore, the inequality  $S(p^k) \le p^k$  holds.

**Remark 2.** The inequality  $S(p^k) \le p^k$  is still true for p=2 and k>3 because (8) holds for these values. Table 1 shows that the inequality is not true for p=2 and k=2,3.

**Lemma 2.** If p>2 is a prime number and k>1, then the inequality  $SS(2 \cdot p^k) \le 2 \cdot p^k$  holds.

#### Proof

The definition of SS gives the following equation

$$SS(p^k) = S(2) + \sum_{i=1}^k S(p^i) + \sum_{i=1}^k S(2 \cdot p^i).$$

Applying the inequality  $S(2 \cdot p^i) \le p \cdot i$  and (4), we have

$$SS(2 \cdot p^k) \le 2 + \sum_{i=1}^k p \cdot i + \sum_{i=1}^k p \cdot i = 2 + p \cdot k \cdot (k+1).$$
 (9)

The inequality

$$2 + p \cdot k \cdot (k+1) \le 2 \cdot p^k \tag{10}$$

is proved to be true as before.

• 
$$k=2 \Rightarrow 2+6 \cdot p \le 2 \cdot p^2$$
. (11)

• 
$$k=3 \Rightarrow 2+12 \cdot \tilde{p} \le 2 \cdot p^3$$
. (12)

• 
$$k=4 \Rightarrow 2+20 \cdot p \le 2 \cdot p^4$$
. (13)

• 
$$k=5 \Rightarrow 2+30 \cdot p \le 2 \cdot p^5$$
. (14)

• 
$$k=6 \Rightarrow 2+42 \cdot p \le 2 \cdot p^5$$
. (15)

These above inequalities (11-15) are true because p>2.

•  $k > 6 \Rightarrow p^k \ge p \cdot p^{k-1} \ge p \cdot 2^{k-1} = p \cdot \sum_{i=0}^{k-1} {k-1 \choose i}$ . The first and the last fourth terms

of this sum are kept finding

\*

$$p^{k} \ge p \cdot \left( 2 \cdot {k-1 \choose 0} + 2 \cdot {k-1 \choose 1} + 2 \cdot {k-1 \choose 2} + 2 \cdot {k-1 \choose 3} \right) \ge 2$$

$$\ge p \cdot \left( 2 \cdot {k-1 \choose 0} + 2 \cdot {k-1 \choose 1} + 2 \cdot {k-1 \choose 2} + 2 \cdot {k-1 \choose 2} \right) = 2$$

$$= p \cdot \left( 2 \cdot k^{2} - 4 \cdot k + 4 \right) \ge 2 + p \cdot (k^{2} + k)$$

The last inequality holds because k > 6, therefore  $2 \cdot p^k \ge 2 + p \cdot k \cdot (k+1)$  is true.

The inequality  $SS(2 \cdot p^k) \le 2 \cdot p^k$  holds because (10) has been found to be true.

**Remark 3.** Similarly, the inequality  $SS(3 \cdot p^k) \le 3 \cdot p^k$  can be proved for all (p > 3) and  $k \ge 1$  or (p = 2) and  $k \ge 3$ .

**Lemma 3.** If p>3 is a prime number and  $k\ge 1$ , then the inequality  $SS(6 \cdot p^k) \le 6 \cdot p^k$  holds.

#### Proof

The starting point is given by the following equation (16)

$$SS(6 \cdot p^{k}) = S(2) + S(3) + S(6) + \sum_{i=1}^{k} S(p^{i}) + \sum_{i=1}^{k} S(2 \cdot p^{i}) + \sum_{i=1}^{k} S(3 \cdot p^{i}) + \sum_{i=1}^{k} S(6 \cdot p^{i}).$$
(16)

The inequalities  $S(p^i)$ ,  $S(2 \cdot p^i)$ ,  $S(3 \cdot p^i)$ ,  $S(6 \cdot p^i) \le p \cdot i$  hold for all  $i \ge 1$  because  $p \ge 5$ . Therefore, the inequality

$$SS(6 \cdot p^{k}) \le 8 + \sum_{i=1}^{k} p \cdot i + \sum_{i=1}^{k} p \cdot i + \sum_{i=1}^{k} p \cdot i + \sum_{i=1}^{k} p \cdot i = 8 + 4 \cdot \sum_{i=1}^{k} p \cdot i$$
 (17)

holds. The inequality  $SS(6 \cdot p^k) \le 8 + 4 \cdot p^k \le 6 \cdot p^k$  is found to be true by applying (5) in (17).

\*

The following propositions give the main properties of the function SS. Let d(n) denote the number of divisors of n.

**Proposition 1.** If a is natural numbers such that  $S(a) \ge 4$ , then the inequality  $S(a) \ge 2 \cdot d(a)$  holds.

#### Proof

The proof is made directly as follows:

$$S(a) = \sum_{1 \neq d:a} S(d) = \sum_{1,n \neq d:a} S(d) + S(a) \ge \sum_{1,n \neq d:a} 2 + S(a) = 2 \cdot (d(a) - 2) + S(a) =$$

$$= 2 \cdot d(a) + S(a) - 4 \ge 2 \cdot d(a).$$

**Remark 4.** The inequality  $S(a) \ge 4$  is verified for all the numbers  $a \ge 4$  and  $a \ne 6$ .

**Proposition 2.** If a, b are two natural numbers such that (a,b)=1, then the inequality  $SS(a \cdot b) \le d(a) \cdot SS(b) + d(b) \cdot SS(a)$  holds.

#### **Proof**

This proof is made by using (2) and the simple remark that  $a, b \ge 0 \Rightarrow \max\{a, b\} \le a + b$ .

The set of the divisors of ab is split into three sets as follows:

$$\begin{cases} 1 \neq d \mid a \cdot b = \underline{M}d \end{cases} = \\ \{1 \neq d \mid a = \underline{M}d \} \cup \{1 \neq d \mid b = \underline{M}d \} \cup \{d_1 d_2 \mid a = \underline{M}d_1 \neq 1 \land b = \underline{M}d_2 \neq 1 \land (d_1, d_2) = 1 \}.$$
 (18)

The following transformations hold according to (18).

Therefore, the inequality  $SS(a \cdot b) \le d(a) \cdot SS(b) + d(b) \cdot SS(a)$  holds.

**Proposition 3.** If a, b are two natural numbers such that S(a),  $S(b) \ge 4$  and (a,b) = 1, then the inequality  $SS(a \cdot b) \le SS(a) \cdot SS(b)$  holds.

#### **Proof**

Proposition 1-2 are applied to prove this proposition as follows:

$$S(a), S(b) \ge 4 \Rightarrow S(a) \ge 2 \cdot d(a) \text{ and } S(b) \ge 2 \cdot d(b)$$
 (19)

$$(a,b) = 1 \Rightarrow SS(a \cdot b) \le d(a) \cdot SS(b) + d(b) \cdot SS(a). \tag{20}$$

The proof is completed if the inequality  $d(a) \cdot SS(b) + d(b) \cdot SS(a) \leq SS(a) \cdot SS(b)$  is found to be true. This is given by the following equivalence

$$d(a) \cdot SS(b) + d(b) \cdot SS(a) \le SS(a) \cdot SS(b) \Leftrightarrow$$

$$d(a) \cdot d(b) \le [SS(a) - d(a)] \cdot [SS(b) - d(b)].$$

This last inequality holds according to (19).

Therefore, the inequality  $SS(a \cdot b) \le SS(a) \cdot SS(b)$  is true.

**Theorem 1.** If n is a natural number such that  $n \neq 8$ , 12, 20 then

a) 
$$SS(n) = n + 2$$
 if  $(\exists p \text{ prime}) n = 2 \cdot p$ . (21)

b) 
$$SS(n) \le n$$
, otherwise. (22)

#### **Proof**

The proof of this theorem is made by using the induction on n.

Equation (21) is true according to Remark 1.a. Table 1 shows that Equation (22) holds for n < 51 and  $n \ne 8$ , 12, 20. Let n > 51 be a natural number. Let us suppose that Equation (9) is true for all the number k that satisfies k < n and k does not have the form k = 2p, p prime. The following cases are analysed:

- n is prime  $\Rightarrow SS(n) = n$ , therefore Equation (9) holds.
- n=2p, p>2 prime  $\Rightarrow SS(n)=n+2$ , therefore Equation (21) holds.
- $(n=2^k \text{ and } k>3) \text{ or } (n=p^k \text{ and } k>1) \Rightarrow SS(n) \leq n \text{ according to Lemma 1}$
- $n = 2 \cdot p^k$ , p > 2 prime number and  $k > 1 \Rightarrow SS(n) \le n$  according to Lemma 2.
- $n=3 \cdot p^k$ , (p>3 prime number and k>1) or (p=2 and k>2)  $\Rightarrow SS(n) \le n$  according to Remark 3.
- $n = 6 \cdot p^k$ , p > 3 prime number and  $k \ge 1 \implies SS(n) \le n$  according to Lemma 3.
- Otherwise  $\Rightarrow$  Let  $n = p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_s^{k_s}$  be the prime number decomposition of n with  $p_1 < p_2 < ... < p_s$ . We prove that there is a decomposition of n=ab, (a,b)=1 such that S(a),  $S(b) \ge 4$ . Let us select  $a = p_s^{k_s}$  and  $b = p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_{s-1}^{k_{s-1}}$ . It is not difficult to see that this decomposition satisfies the above conditions. The induction's hypotheses is applied for a,b < n and the inequalities  $SS(a) \le a$  and  $SS(b) \le b$  are obtained. Finally, Proposition 3 gives  $SS(n) = SS(a \cdot b) \le SS(b) \cdot SS(a) \le b \cdot a = n$ .

We can conclude that the inequality  $SS(n) \le n-2$  holds for all the natural number  $n \ne 12$ .

\*

**Remark 5.** The above analysis is necessary to be sure that the decomposition of n=ab, (a,b)=1, S(a),  $S(b)\ge 4$  exists.

Theorem 1 has some interesting consequences that are presented in the following. These establish new upper bounds for Smarandache's function.

Consequence 1. If n > 1 is a natural number, then the following inequality

$$S(n) \le n + 4 - 2 \cdot d(n) \tag{23}$$

holds.

#### Proof

The proof of this inequality is made by using Theorem 1.

Obviously, (23) is true for n=p or n=2p, p prime number.

Let  $n \neq 8$ , 12, 20 be a natural number.

We have the following transformations:

$$n \ge SS(n) = \sum_{1 \ne dn} S(d) = S(n) + \sum_{1.n \ne dn} S(d) \ge$$

$$\ge S(n) + 2 \cdot \left| \left\{ d = \overline{1, n} \mid d \ne 1, n \land d \mid n \right\} \right| = S(n) + 2 \cdot (d(n) - 2) = S(n) + 2 \cdot d(n) - 4$$

Inequality (23) is also satisfied for n=8, 12, 20.

Therefore, the inequality  $S(n) \le n + 4 - 2 \cdot d(n)$  holds.

Consequence 2. If n > 1 is a natural number, then the following inequality holds

$$S(n) \le n + 4 - \min\{p \mid p \text{ is prime and } p \mid n\} \cdot d(n) . \tag{24}$$

\*

#### **Proof**

This proof is made similarly to the proof of the previous consequence by using the following strong inequality  $S(d) \ge \min\{p \mid p \text{ is prime and } p \mid n\}$ .

# 3. Final Remark

Inequalities (23 - 24) give some generalisations of the well - known inequality  $S(n) \le n$ . More important is the fact that these inequalities reflect. When n has many divisors, the value of  $n+4-\min\{p|p \text{ is prime and } p|n\} \cdot d(n)$  is small, therefore the value of S(n) is small as well according to Inequality (24). In spite of fact that Inequalities (23 - 24) reflect this situation, we could not say that the upper bounds are the lowest possible. Nevertheless, they offer a better upper bound than the inequality  $S(n) \le n$ .

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# Smarandache's function applied to perfect numbers

#### Sebastián Martín Ruiz

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Smarandache's function may be defined as follows:

S(n) = the smallest positive integer such that S(n)! is divisible by n. [1] In this article we are going to see that the value this function takes when n is a perfect number of the form  $n = 2^{k-1} \cdot (2^k - 1)$ ,  $p = 2^k - 1$  being a prime number.

Lemma 1 Let  $n = 2^i \cdot p$  when p is an odd prime number and i an integer such that:

$$0 \le i \le E(\frac{p}{2}) + E(\frac{p}{2^2}) + E(\frac{p}{2^3}) + \dots + E(\frac{p}{2^{E(\log_2 p)}}) = e_2(p!)$$

Where  $e_2(p!)$  is the exponent of 2 in the prime number descomposition of p!. E(x) is the greatest integer less than or equal to x. One has that S(n) = p.

#### Demonstration:

Given that  $gcd(2^i, p) = 1$  (gcd = greatest common divisor) one has that  $S(n) = max\{s(2^i), S(p)\} \ge S(p) = p$ . Therefore  $S(n) \ge p$ .

If we prove that p! is divisible by n then one would have the equality.

$$p! = p_1^{e_{p_1}(p!)} \cdot p_2^{e_{p_2}(p!)} \cdot \cdot \cdot p_{\bullet}^{e_{p_s}(p!)}$$

where  $p_i$  is the i-th prime of the prime number descomposition of p!. It is clear that  $p_1 = 2$ ,  $p_s = p$ ,  $e_{p_s}(p!) = 1$  for which:

$$p! = 2^{e_2(p!)} \cdot p_2^{e_{p_2}(p!)} \cdots p_{s-1}^{e_{p_{s-1}}(p!)} \cdot p$$

From where one can deduce that:

$$\frac{p!}{n} = 2^{e_2(p!)-i} \cdot p_2^{e_{p_2}(p!)} \cdots p_{s-1}^{e_{p_{s-1}}(p!)}$$

is a positive integer since  $e_2(p!) - i > 0$ .

Therefore one has that S(n) = p

Proposition 1 If n a perfec number of the form  $n = 2^{k-1} \cdot (2^k - 1)$  with k a positive integer,  $2^k - 1 = p$  prime, one has that S(n) = p.

# Demonstration:

For the Lemma it is sufficient to prove that  $k-1 \le e_2(p!)$ .

If we can prove that

$$k - 1 \le 2^{k - 1} - \frac{1}{2} \tag{1}$$

we will have proof of the proposition since:

$$k-1 \le 2^{k-1} - \frac{1}{2} = \frac{2^k - 1}{2} = \frac{p}{2}$$

As k-1 is an integer one has that  $k-1 \le E(\frac{p}{2}) \le e_2(p!)$ 

Proving (1) is the same as proving  $k \le 2^{k-1} + \frac{1}{2}$  at the same time, since k is integer, is equivalent to proving  $k \le 2^{k-1}$  (2).

In order to prove (2) we may consider the function:  $f(x) = 2^{x-1} - x$   $x \in \mathbb{R}$ . This function may be derived and its derivate is  $f'(x) = 2^{x-1} \ln 2 - 1$ .

f will be increasing when  $2^{x-1} \ln 2 - 1 > 0$  resolving x:

$$x > 1 - \frac{ln (ln 2)}{ln 2} \simeq 1'5287$$

In particular f will be increasing  $\forall x \geq 2$ .

Therefore  $\forall \ x \geq 2 \quad f(x) \geq f(2) = 0$  that is to say  $2^{x-1} - x \geq 0 \quad \forall \ x \geq 2$  therefore

$$2^{k-1} \ge k \quad \forall \ k \ge 2 \quad integer$$

and thus is proved the proposition.

#### EXAMPLES:

$6 = 2 \cdot 3$	S(6) = 3
$28 = 2^2 \cdot 7$	S(28) = 7
$496 = 2^4 \cdot 31$	S(496) = 31
$8128 = 2^6 \cdot 127$	S(8128) = 127

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## ON THE DIVERGENCE OF THE SMARANDACHE HARMONIC SERIES

#### Florian Luca

For any positive integer n let S(n) be the minimal positive integer m such that  $n \mid m!$ . In [3], the authors showed that

$$\sum_{n>1} \frac{1}{S(n)^2} \tag{1}$$

is divergent and attempted, with limited succes, to gain information about the behaviour of the partial sum

$$A(x) = \sum_{n \le x} \frac{1}{S(n)^2}$$

by comparing it with both  $\log x$  and  $\log x + \log \log x$ .

In this note we show that none of these two functions is a suitable candidate for the order of magnitude of A(x).

Here is the result. For any  $\delta > 0$  and  $x \ge 1$  denote by

$$A_{\delta}(x) = \sum_{n \le x} \frac{1}{S(n)^{\delta}},\tag{2}$$

$$B_{\delta}(x) = \log A_{\delta}(x). \tag{3}$$

Then.

Theorem.

For any  $\delta > 0$ ,

$$B_{\delta}(x) \ge \log 2 \cdot \frac{\log x}{\log \log x} - o\left(\frac{\log x}{\log \log x}\right).$$
 (4)

What the above theorem basically says is that for fixed  $\delta$  and for arbitrary  $\epsilon > 0$ , there exists some constant C (depending on both  $\delta$  and  $\epsilon$ ), such that

$$A_{\delta}(x) > 2^{(1-\epsilon)\frac{\log x}{\log \log x}}$$
 for  $x > C$ . (5)

Notice that equation (5) asserts that  $A_{\delta}(x)$  grows much faster than any polynomial in  $\log(x)$ , so one certainly shouldn't try to approximate it by a linear in  $\log x$ .

The Proof.

In [1], we showed that

$$\sum_{n\geq 1} \frac{1}{S(n)^{\delta}} \tag{6}$$

diverges for all  $\delta > 0$ . Since the argument employed in the proof is relevant for our purposes, we reproduce it here.

Let  $t \ge 1$  be an integer and  $p_1 < p_2 < ... < p_t$  be the first t prime numbers. Notice that any integer  $n = p_t m$  where m is squarefree and all the prime factors of

m belong to  $\{p_1, p_2, ..., p_{t-1}\}$  will certainly satisfy  $S(n) = p_t$ . Since there are at least  $2^{t-1}$  such m's (the power of the set  $\{p_1, ..., p_{t-1}\}$ ), it follows that series (6) is bounded below by

$$\sum_{t\geq 1} \frac{2^{t-1}}{p_t^{\delta}} = \sum_{t\geq 1} 2^{t-1-\delta \log_2 p_t}.$$
 (7)

The argument ends noticing that since

$$\lim_{t\to\infty}\frac{p_t}{t\log t}=1,$$

it follows that the exponent  $t - 1 - \delta \log_2 p_t$  is always positive for t large enough. This proves the divergence of the series (6).

For the present theorem, the only new thing is the fact that we do not work with the whole series (6) but only with its partial sum  $A_{\delta}(x)$ . In particular, the parameter t from the above argument is precisely the maximal value of s for which  $p_1p_2...p_s \leq x$ . In order to prove our theorem, we need to come up with a good lower bound on t.

We show that for all  $\epsilon > 0$  one has

$$t > (1 - \epsilon) \frac{\log x}{\log \log x} \tag{8}$$

provided that x is enough large. Assume that this is not so. It then follows that there exists some  $\epsilon > 0$  such that

$$t < (1 - \epsilon) \frac{\log x}{\log \log x} \tag{9}$$

for arbitrarily large values of x. Since t was the value of the maximal s such that  $p_1p_2...p_s \leq x$ , it follows that

$$p_1 p_2 ... p_{t+1} > x. (10)$$

From a formula in [2], it follows easily that

$$p_i \le 2i \log i \quad \text{for } i \ge 3.$$
 (11)

It now follows, by taking logarithms in (10) and using (11), that

$$\log x < \sum_{i=1}^{t+1} \log p_i < C_1 + (t-1) \log 2 + \sum_{i=3}^{t+1} (\log i + \log \log i) < 1$$

$$C_1 + (t-1)\log 2 + \int_3^{t+2} (\log y + \log \log y) dy < C_1 + (t-1)\log 2 + (t+2)(\log(t+2) + \log \log(t+2)), \tag{12}$$

where  $C_1 = \log 6$ . Since t can be arbitrarily large (because x is arbitrarily large), it follows that one can just work with

$$\log x < t(\log t + 2\log\log t). \tag{13}$$

Indeed, the amount  $(t+2)(\log(t+2) + \log\log(t+2))$  appearing in the right hand side of (12) can be replaced by  $t(\log t + \log\log t) + f(t)$  where  $f(t) = 2\log t + 2\log\log t + \log\log t$ 

O(1) and then the sum of f(t) with the linear term from the right hand side of (12) can certainly be bounded above by  $t \log \log t$  for t large enough. Hence,

$$\log x < t(\log t + 2\log\log t). \tag{14}$$

Using inequality (9) to bound the factor t appearing in (14) in terms of x and the obvious inequality

$$t \le (1 - \epsilon) \frac{\log x}{\log \log x} < \log x$$

to bound the t's appearing inside the logs in (14), one gets

$$\log x < (1-\epsilon) \frac{\log x}{\log\log x} \Big(\log\log x + 2\log\log\log x\Big) = (1-\epsilon)\log x \Big(1 + \frac{2\log\log\log x}{\log\log x}\Big)$$

or, after some immediate simplifications,

$$\log\log x < \frac{2(1-\epsilon)}{\epsilon}\log\log\log x. \tag{15}$$

Since  $\epsilon$  was fixed, it follows that inequality (15) cannot happen for arbitrarily large values of x. This proves that indeed (8) holds for any  $\epsilon$  provided that x is large enough. We are now done. Indeed, going back to formula (7), it follows that

$$A_{\delta}(x) > 2^{t-1-\delta \log p_t}$$

or

$$B_{\delta}(x) = \log A_{\delta}(x) > \log 2(t - 1 - \delta \log p_t) > \log 2(t - 1 - \delta \log(2t \log t)), \quad (16)$$

where the last inequality in (16) follows from (11). By inequalities (8) and (16), we get

$$B_{\delta}(x) = t \log 2 - o(t) = \log 2(1 - \epsilon) \frac{\log x}{\log \log x} - o(t).$$

Since  $\epsilon$  could, in fact, be chosen arbitrarily small, we get

$$B_{\delta}(x) = \log 2 \frac{\log x}{\log \log x} - o\left(\frac{\log x}{\log \log x}\right),\tag{16}$$

which concludes the proof.

#### Remark.

We conjecture that the exact order of  $B_{\delta}(x)$  is  $\frac{\log x}{\log \log x} + O\left(\frac{\log x}{(\log \log x)^2}\right)$ .

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# A Generalisation of Euler's function

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The aim of this article is to propose a generalisation for Euler's function. This function is  $\varphi: N \to N$  defined as follows  $(\forall n \in N) \varphi(n) = |\{k = \overline{1,n} \mid (k,n) = 1\}$ . Perhaps, this is the most important function in number theory having many properties in number theory, combinatorics, *etc*. The main properties [Hardy & Wright, 1979] of this function are enumerated in the following:

$$(\forall a, b \in N)(a, b) = 1 \Rightarrow \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$$
 - the multiplicative property (1)

$$a = p_1^{m_1} \cdot p_2^{m_2} \cdot \dots \cdot p_s^{m_s} \Rightarrow \varphi(a) = a \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \dots \cdot \left(1 - \frac{1}{p_s}\right)$$
 (2)

$$\left(\forall \ a \in N\right) \sum_{d,a} \varphi(d) = a. \tag{3}$$

More properties concerning this function can be found in [Hardy & Wright, 1979], [Jones & Jones, 1998] or [Rosen, 1993].

# 1. Euler's Function by order k

In the following, we shall see how this function is generalised such that the above properties are still kept. The way that will be used to introduce Euler's generalised function is from the function's formula to the function's properties.

**Definition 1.** Euler's function by order  $k \in N$  is  $\varphi_k : N \to N$  defined by

$$\left(\forall a = p_1^{m_1} \cdot p_2^{m_2} \cdot \dots \cdot p_s^{m_s}\right) \varphi_k(a) = a^k \cdot \left(1 - \frac{1}{p_1^k}\right) \cdot \left(1 - \frac{1}{p_2^k}\right) \cdot \dots \cdot \left(1 - \frac{1}{p_s^k}\right).$$

#### Remarks 1.

1. Let us assume that  $\varphi_k(1) = 1$ .

2. Euler function by order 1 is Euler's function. Obviously, Euler's function by order 0 is the constant function 1.

In the following, the main properties of Euler's function by order k are proposed.

**Theorem 1.** Euler's function by order k is multiplicative

$$(\forall a, b \in N)(a, b) = 1 \Rightarrow \varphi_k(a \cdot b) = \varphi_k(a) \cdot \varphi_k(b). \tag{4}$$

#### Proof

This proof is obvious from the definition.

Theorem 2. 
$$(\forall a \in N) \sum_{da} \varphi_k(d) = a^k$$
. (5)

#### Proof

The function  $\overline{\varphi_k}(a) = \sum_{d,q} \varphi_k(d)$  is multiplicative because  $\varphi_k$  is a multiplicative function.

If  $a = p^m$ , then the following transformation proves (5)

$$\overline{\varphi_k}(a) = \sum_{d \in p^m} \varphi_k(d) = \sum_{i=0}^m \varphi_k(p^i) = 1 + \sum_{i=1}^m p^{k \cdot i} \left( 1 - \frac{1}{p^k} \right) = 1 + \sum_{i=1}^m \left( p^{k \cdot i} - p^{k \cdot (i-1)} \right) = 1 + p^{k \cdot m} - 1 = a^k$$

If  $a = p_1^{m_1} \cdot p_2^{m_2} \cdot \dots \cdot p_s^{m_s}$  then the multiplicative property is applied as follows:

$$\overline{\varphi_k}(a) = \overline{\varphi_k}(p_1^{m_1}) \cdot \overline{\varphi_k}(p_2^{m_2}) \cdot \dots \cdot \overline{\varphi_k}(p_s^{m_s}) = p_1^{k \cdot m_1} \cdot p_2^{k \cdot m_2} \cdot \dots \cdot p_s^{k \cdot m_s} = a^k.$$

**Definition 2.** A natural number n is said to be k-power free if there is not a prim number p such that  $p^k \mid n$ .

#### Remarks 2.

- 1. There is not a 0-power free number.
- 2. Assume that 1 is the only 1-power free number.

The combinatorial property of Euler's function by order k is given by the following theorem. This property is introduced by using the k-power free notion.

**Theorem 3.** 
$$(\forall n \in N) \varphi_k(n) = \left\{ i = \overline{1, n^k} \mid (i, n^k) \text{ is } k \text{ - power free} \right\}$$
 (6)

#### **Proof**

This proof is made using the Inclusion-Exclusion theorem.

Let  $a = p_1^{m_1} \cdot p_2^{m_2} \cdot \dots \cdot p_s^{m_s}$  be the prime number decomposition of a.

If dn, then the set  $S_d = \{i = \overline{1, a^k} \mid d^k \mid i\}$  contains all the numbers that have the divisor  $d^k$ . This set satisfies the following properties:

$$S_{d} = \left\{ d^{k}, 2 \cdot d^{k}, \dots, \frac{a^{k}}{d^{k}} \cdot d^{k} \right\} \Rightarrow \left| S_{d} \right| = \frac{a^{k}}{d^{k}}$$

$$1 \leq j_{1} < j_{2} \leq n \Rightarrow S_{p_{j_{1}}} \cap S_{p_{j_{2}}} = S_{p_{j_{1}} \cdot p_{j_{2}}}$$

$$(7)$$

 $\left\{i = \overline{1, a^k} \mid (i, a^k) \text{ is } k \text{ - power free}\right\} = \left\{i = \overline{1, a^k}\right\} - \left(S_{p_1} \cap S_{p_2} \cap \dots \cap S_{p_s}\right). \tag{8}$ 

The Inclusion-Exclusion theorem and (7-9) give the following transformations:

$$\left| \left\{ i = \overline{1, a^{k}} \mid (i, a^{k}) \text{ is } k \text{ - power free} \right\} = a^{k} - \left| \left( S_{p_{1}} \cap S_{p_{2}} \cap ... \cap S_{p_{s}} \right) \right| =$$

$$= a^{k} - \sum_{j=1}^{s} \left| S_{p_{j}} \right| + \sum_{1 \leq j_{1} < j_{2} \leq n} \left| S_{p_{j_{1}}} \cap S_{p_{j_{2}}} \right| - ... + (-1)^{s+1} \sum_{1 \leq j_{1} < j_{2} < ... < j_{s} \leq n} \left| S_{p_{j_{1}}} \cap S_{p_{j_{2}}} \cap ... \cap S_{p_{j_{s}}} \right| =$$

$$= a^{k} - \sum_{j=1}^{s} \left| S_{p_{j}} \right| + \sum_{1 \leq j_{1} < j_{2} \leq n} \left| S_{p_{j_{1}} \cdot p_{j_{2}}} \right| - ... + (-1)^{s+1} \sum_{1 \leq j_{1} < j_{2} < ... < j_{s} \leq n} \left| S_{p_{j_{1}} \cdot p_{j_{2}} \cdot ... < p_{j_{2}}} \right| =$$

$$= a^{k} - \sum_{j=1}^{s} \frac{a^{k}}{p_{j}^{k}} + \sum_{1 \leq j_{1} < j_{2} \leq n} \frac{a^{k}}{\left( p_{j_{1}} \cdot p_{j_{2}} \right)^{k}} - ... + (-1)^{s+1} \sum_{1 \leq j_{1} < j_{2} < ... < j_{s} \leq n} \frac{a^{k}}{\left( p_{j_{1}} \cdot p_{j_{2}} \cdot ... \cdot p_{j_{2}} \right)^{k}} =$$

$$= a^{k} \cdot \left( 1 - \frac{1}{p_{j}^{k}} \right) \cdot \left( 1 - \frac{1}{p_{j}^{k}} \right) \cdot ... \cdot \left( 1 - \frac{1}{p_{j}^{k}} \right) = \varphi_{k}(a)$$

Therefore, the equation (6) holds.

Ļ

# 3. Conclusion

Euler's function by order k represents a successful way to generalise Euler's function. Firstly, because the main properties of Euler's function (1-3) have been extended for Euler's function by order k. Secondly and more important, because a combinatorial property has been found for this generalised function. Obviously, many other properties can be deduced for Euler's function by order k. Unfortunately, a similar property with Euler's theorem  $a^{\varphi(n)} = 1 \mod n$  has not been found so far.

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# A result obtained using Smarandache Function

# Sebastián Martín Ruiz

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Smarandache Function is defined as followed:

S(m) =The smallest positive integer so that S(m)! is divisible by m. [1] Let's see the value which such function takes for  $m = p^{p^n}$  with n integer,  $n \ge 2$  and p prime number. To do so a Lemma is required.

Lemma 1  $\forall m, n \in \mathbb{N}$  m, n > 2

$$m^{n} = E\left[\frac{m^{n+1} - m^{n} + m}{m}\right] + E\left[\frac{m^{n+1} - m^{n} + m}{m^{2}}\right] + \dots + E\left[\frac{m^{n+1} - m^{n} + m}{m^{E[\log_{m}(m^{n+1} - m^{n} + m)]}}\right]$$

Where E(x) gives the greatest integer less than or equal to x.

#### Demonstration:

Let's see in the first place the value taken by  $E[\log_m(m^{n+1}-m^n+m)]$ .

If  $n \ge 2$ :  $m^{m+1} - m^n + m < m^{n+1}$  and therefore  $\log_m(m^{n+1} - m^n + m) < \log_m m^{n+1} = n+1$ . As a result  $E[\log_m(m^{n+1} - m^n + m)] < n+1$ .

And if  $m \ge 2$ :  $mm^n \ge 2m^n \Rightarrow m^{n+1} \ge 2m^n \Rightarrow m^{n+1} + m \ge 2m^n \Rightarrow m^{n+1} - m^n + m \ge m^n \Rightarrow \log_m(m^{n+1} - m^n + m) \ge \log_m m^n = n \Rightarrow E[\log_m(m^{n+1} - m^n + m)] \ge n$ 

As a result:  $n \leq E[\log_m(m^{n+1} - m^n + m)] < n+1$  therefore:

$$E[\log_m(m^{n+1} - m^n + m)] = n \quad if \ n, m \ge 2$$

Now let's see the value which it takes for  $1 \le k \le n$ :  $E\left[\frac{m^{n+1}-m^n+m}{m^k}\right]$ 

$$E\left[\frac{m^{n+1} - m^n + m}{m^k}\right] = E\left[m^{n+1-k} - m^{n-k} + \frac{1}{m^{k-1}}\right]$$

If 
$$k = 1$$
:  $E\left[\frac{m^{n+1} - m^n + m}{m}\right] = m^n - m^{n-1} + 1$   
If  $1 < k \le n$ :  $E\left[\frac{m^{n+1} - m^n + m}{m^k}\right] = m^{n+1-k} - m^{n-k}$ 

Let's see what is the value of the sum:

$$k = 1$$
  $m^{n}$   $-m^{n-1}$  ... ... ... +  $k = 2$   $m^{n-1}$   $-m^{n-2}$   $k = 3$   $m^{n-2}$   $-m^{n-3}$  ... ...  $m^{2}$   $-m$   $k = n$  ...  $m^{2}$   $-m$   $m$   $m$   $m$ 

Therefore:

$$\sum_{k=1}^{n} E\left[\frac{m^{n+1} - m^n + m}{m^k}\right] = m^n \quad m, n \ge 2$$

Proposition 1  $\forall p \text{ prime number } \forall n \geq 2$ :

$$S(p^{p^n}) = p^{n+1} - p^n + p$$

Demonstration:

Having  $e_p(k) = \text{exponent}$  of the prime number p in the prime numbers descomposition of k.

We get:

$$e_p(k!) = E(\frac{k}{p}) + E(\frac{k}{p^2}) + E(\frac{k}{p^3}) + \dots + E(\frac{k}{p^{E(\log_p k)}})$$

And using the Lemma we have:

$$e_p[(p^{n+1}-p^n+p)!] = E\left[\frac{p^{n+1}-p^n+p}{p}\right] + E\left[\frac{p^{n+1}-p^n+p}{p^2}\right] + \dots + E\left[\frac{p^{n+1}-p^n+p}{m^{E[\log_p(p^{n+1}-p^n+p)]}}\right] = p^n$$

Therefore:

$$\frac{(p^{n+1}-p^n+p)!}{p^{p^n}} \in \mathbb{N}$$
 and  $\frac{(p^{n+1}-p^n+p-1)!}{p^{p^n}} \notin \mathbb{N}$ 

And:

$$S(p^{p^n}) = p^{n+1} - p^n + p$$

References:

[1] C. Dumitrescu and R. Müller: To Enjoy is a Permanent Component of Mathematics. SMARANDACHE NOTIONS JOURNAL Vol. 9, No. 1-2,(1998) pp 21-26.

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# On an inequality for the Smarandache function

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1. In paper [2] the author proved among others the inequality  $S(ab) \leq aS(b)$  for all a, b positive integers. This was refined to

$$S(ab) \le S(a) + S(b) \tag{1}$$

in [1]. Our aim is to show that certain results from our recent paper [3] can be obtained in a simpler way from a generalization of relation (1). On the other hand, by the method of Le [1] we can deduce similar, more complicated inequalities of type (1).

2. By mathematical induction we have from (1) immediately:

$$S(a_1 a_2 \dots a_n) \le S(a_1) + S(a_2) + \dots + S(a_n)$$
 (2)

for all integers  $a_i \geq 1$  (i = 1, ..., n). When  $a_1 = ... = a_n = n$  we obtain

$$S(a^n) \le nS(a). \tag{3}$$

For three applications of this inequality, remark that

$$S((m!)^n) \le nS(m!) = nm \tag{4}$$

since S(m!) = m. This is inequality 3) part 1. from [3]. By the same way,  $S((n!)^{(n-1)!}) \le (n-1)!S(n!) = (n-1)!n = n!$ , i.e.

$$S((n!)^{(n-1)!}) \le n! \tag{5}$$

Inequality (5) has been obtained in [3] by other arguments (see 4) part 1.).

Finally, by  $S(n^2) \leq 2S(n) \leq n$  for n even (see [3], inequality 1), n > 4, we have obtained a refinement of  $S(n^2) \leq n$ :

$$S(n^2) \le 2S(n) \le n \tag{6}$$

for n > 4, even.

3. Let m be a divisor of n, i.e. n = km. Then (1) gives  $S(n) = S(km) \le S(m) + S(k)$ , so we obtain:

If m|n, then

$$S(n) - S(m) \le S\left(\frac{n}{m}\right). \tag{7}$$

As an application of (7), let d(n) be the number of divisors of n. Since  $\prod_{k|n} k = n^{d(n)/2}$ ,

and  $\prod_{k \le n} k = n!$  (see [3]), and by  $\prod_{k \mid n} k \mid \prod_{k \le n} k$ , from (7) we can deduce that

$$S(n^{d(n)/2}) + S(n!/n^{d(n)/2}) \ge n.$$
(8)

This improves our relation (10) from [3].

4. Let S(a)=u, S(b)=v. Then b|v! and  $u!|x(x-1)\dots(x-u+1)$  for all integers  $x\geq u$ . But from a|u! we have  $a|x(x-1)\dots(x-u+1)$  for all  $x\geq u$ . Let x=u+v+k  $(k\geq 1)$ . Then, clearly  $ab(v+1)\dots(v+k)|(u+v+k)!$ , so we have  $S[ab(v+1)\dots(v+k)]\leq u+v+k$ . Here v=S(b), so we have obtained that

$$S[ab(S(b) + 1) \dots (S(b) + k)] \le S(a) + S(b) + k.$$
(9)

For example, for k = 1 one has

$$S[ab(S(b) + 1)] \le S(a) + S(b) + 1. \tag{10}$$

This is not a consequence of (2) for n = 3, since S[S(b) + 1] may be much larger than 1.

# References

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## ON A SERIES INVOLVING $S(1) \cdot S(2) \dots \cdot S(n)$

#### Florian Luca

For any positive integer n let S(n) be the minimal positive integer m such that  $n \mid m!$ . It is known that for any  $\alpha > 0$ , the series

$$\sum_{n\geq 1} \frac{n^{\alpha}}{S(1)\cdot S(2)\cdot \dots \cdot S(n)} \tag{1}$$

is convergent, although we do not know who was the first to prove the above statement (for example, the authors of [4] credit the paper [1] appeared in 1997, while the result appears also as Proposition 1.6.12 in [2] which was written in 1996).

In this paper we show that, in fact:

#### Theorem.

The series

$$\sum_{n>1} \frac{x^n}{S(1) \cdot S(2) \cdot \dots \cdot S(n)} \tag{2}$$

converges absolutely for every x.

#### Proof

Write

$$a_n = \frac{|x|^n}{S(1) \cdot S(2) \cdot \dots \cdot S(n)}.$$
(3)

Then

$$\frac{a_{n+1}}{a_n} = \frac{|x|}{S(n+1)}. (4)$$

But for |x| fixed, the ratio |x|/S(n+1) tends to zero. Indeed, to see this, choose any positive real number m, and let  $n_m = \lfloor m|x|+1 \rfloor!$ . When  $n > n_m$ , it follows that  $S(n+1) > \lfloor m|x|+1 \rfloor > m|x|$ , or S(n+1)/|x| > m. Since m was arbitrary, it follows that the sequence S(n+1)/|x| tends to infinity.

#### Remarks.

- 1. The convergence of (2) is certainly better than the convergence of (1). Indeed, if one fixes any x > 1 and any  $\alpha$ , then certainly  $x^n > n^{\alpha}$  for n large enough.
  - 2. The convergence of (2) combined with the root test imply that

$$(S(1) \cdot S(2) \cdot \dots \cdot S(n))^{1/n}$$

diverges to infinity. This is equivalent to the fact that the average function of the logs of S, namely

$$LS(x) = \frac{1}{x} \sum_{n \le x} \log S(n)$$
 for  $x \ge 1$ 

tends to infinity with x. It would be of interest to study the order of magnitude of the function LS(x). We conjecture that

$$LS(x) = \log x - \log \log x + O(1). \tag{5}$$

The fact that LS(x) cannot be larger than what shows up in the right side of (5) follows from a result from [3]. Indeed, in [3], we showed that

$$A(x) = \frac{1}{x} \sum_{n \le x} S(n) < 2 \frac{x}{\log x}$$
 for  $x \ge 64$ . (6)

Now the fact that  $LS(x) - \log x + \log \log x$  is bounded above follows from (6) and from Jensen's inequality for the log function (or the logarithmic form of the AGM inequality). It seems to be considerably harder to prove that  $LS(x) - \log x + \log \log x$  is bounded below.

3. As a fun application we mention that for every integer  $k \geq 1$ , the series

$$\sum_{n>1} \binom{n}{k} \cdot \frac{x^n}{S(1) \cdot S(2) \cdot \dots \cdot S(n)} \tag{7}$$

is absolutely convergent. Indeed, it is a straightforward computation to verify that if one denotes by C(x) the sum of the series (2), then the series (7) is precisely

$$\frac{x^k}{k!} \cdot \frac{d^k C}{dx^k}.$$
(8)

When k = x = 1 series (7) becomes precisely series (1) for  $\alpha = 1$ .

4. It could be of interest to study the rationality of (2) for integer values of x. Indeed, if the function S is replaced with the identity in formula (2), then one obtains the more familiar  $e^x$  whose value is irrational (in fact, transcendental) at all integer values of x. Is that still true for series (2)?

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# A Congruence with Smarandache's Function

# Sebastián Martín Ruiz

# 25 September 1998

Smarandache's function is defined thus:

S(n) =is the smallest integer such that S(n)! is divisible by n. [1] In this article we are going to look at the value that has  $S(2^k - 1) \pmod{k}$  for all k integer,  $2 \le k \le 97$ .

One can observe in the following table that gives the continuation  $S(2^k-1)\equiv 1\pmod k$  in the majority of cases, there are only 4 exeptions for  $2\le k\le 97$ .

k	$S(2^{k}-1)$	$S(2^k-1) \ (mod \ k)$
2	3	1
3	7	1
4	5	1
5	31	1
6	7	1
7	127	1
8	17	1
9	73	1
10	31	1
11	89	1
12	13	1
13	8191	1
14	127	1
15	151	1
16	257	1
17	131071	1
18	73	1
19	524287	1

k	$S(2^{k}-1)$	$S(2^k-1) \ (mod \ k)$	k	$S(2^k-1)$	$S(2^k-1) \ (mod \ k)$
20	41	1	<b>5</b> 9	3203431780337	1
21	337	1	60	1321	
22	683	1	61	2305843009213693951	1
23	178481	1	62	2147483647	1
24	241	1	63	649657	1
25	1801	1	64	6700417	1
26	8191	1	65	145295143558111	1
27	262657	1	66	599479	1
28	127	15	67	761838257287	1
29	2089	1	68	131071	35
<b>3</b> 0	331	1	69	10052678938039	1
31	2147483647	1	70	122921	1
32	65537	1	71	212885833	1
33	599479	1	72	38737	1
34	131071	1	73	9361973132609	1
35	122921	1	74	616318177	1
36	109	1	75	10567201	
37	616318177	1	76	525313	1
38	524287	1	77	581283643249112959	
<b>3</b> 9	121369	1	78	22366891	
40	61681	1	79	1113491139767	
41	164511353	1	80	1 4278255361	
42	5419	1	81	97685839	
43	2099863	1	82	2 8831418697 1	
44	2113	1	83	57912614113275649087721	
45	23311	1	84	14449	1
46	2796203	1	85	9520972806333758431	1
47	13264529	1	86	2932031007403	1
48	673	1	87	9857737155463	1
49	4432676798593	1	88	2931542417	1
50	4051	1	89	618970019642690137449562111	1
51	131071	1	90	18837001	
52	8191	27	91	23140471537	
53	20394401	1	92	2796203 47	
54	262657	1	93	658812288653553079	
55	201961	1	94	165768537521	
56	15790321	1	95	30327152671	
57	1212847	1	96	22253377	1
58	3033169	1	97	13842607235828485645766393	1

One can see from the table that there are only 4 exeptions for  $2 \le k \le 97$ .

We can see in detail the 4 exeptions in a table:

$$\begin{vmatrix} k = 28 = 2^2 \cdot 7 & S(2^{28} - 1) \equiv 15 \pmod{28} \\ k = 52 = 2^2 \cdot 13 & S(2^{52} - 1) \equiv 27 \pmod{52} \\ k = 68 = 2^2 \cdot 17 & S(2^{68} - 1) \equiv 35 \pmod{68} \\ k = 92 = 2^2 \cdot 23 & S(2^{92} - 1) \equiv 47 \pmod{92} \end{vmatrix}$$

One can observe in these 4 cases that  $k=2^2\cdot p$  with p prime and moreover  $S(2^k-1)\equiv \frac{k}{2}+1\ (mod\ k)$ 

# Unsolved Question:

One can obtain a general formula that gives us, in function of k the value  $S(2^k-1) \pmod{k}$  for all pisitive integer values of k?.

# References:

[1] SMARANDACHE NOTIONS JOURNAL Vol. 9, No. 1-2,(1998) pp 21-26.

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## An Integer as a Sum of Consecutive Integers

#### Henry Ibstedt

Abstract: This is a simple study of expressions of positive integers as sums of consecutive integers. In the first part proof is given for the fact that N can be expressed in exactly d(L)-1 ways as a sum of consecutive integers, L is the largest odd factor of N and d(L) is the number of divisors of L. In the second part answer is given to the question: Which is the smallest integer that can be expressed as a sum of consecutive integers in n ways.

Introduction: There is a remarkable similarity between the four definitions given below. The first is the well known Smarandache Function. The second function was defined by K. Kashihara and was elaborated on in his book Comments and Topics on Smarandache Notions and Problems<sup>1</sup>. This function and the Smarandache Ceil Function were also examined in the author's book Surfing on the Ocean of Numbers<sup>2</sup>. These three functions have in common that they aim to answer the question which is the smallest positive integer N which possesses a certain property pertaining to a given integer n. It is possible to pose a large number of questions of this nature.

- 1. The Smarandache Function S(n):
  S(n)=N where N is the smallest positive integer which divides n!
- The Pseudo-Smarandache Function Z(n):
  Z(n)=N where N is the smallest positive integer such that 1+2+...+N is divisible by n.
- 3. The Smarandache Ceil Function of order k,  $S_k(n)$ :  $S_k(n)=N$  where N is he smallest positive integer for which n divides  $N^k$
- 4. The n-way consecutive integer representation R(n):
  R(n)=N where N is the smallest positive integer which can be represented as a sum of consecutive integer is n ways.

There may be many positive integers which can be represented as a sum of positive integers in n distinct ways - but which is the smallest of them? This article gives the answer to this question. In the study of R(n) it is found that the arithmetic function d(n), the number of divisors of n, plays an important role.

<sup>&</sup>lt;sup>1</sup> Erhus University Press, 1996

<sup>&</sup>lt;sup>2</sup> Erhus University Press, 1997

Question 1: In how many ways n can a given positive integer N be expressed as the sum of consecutive positive integers?

Let the first term in a sequence of consecutive integers be Q and the number terms in the sequence be M. We have N=Q+(Q+1)+...+(Q+M-1) where M>1.

$$N = \frac{M(2Q + M - 1)}{2} \tag{1}$$

or

$$Q = \frac{N}{M} - \frac{M-1}{2} \tag{2}$$

For a given positive integer N the number of sequences n is equal to the number of positive integer solutions to (2) in respect of Q. Let us write  $N=L\cdot 2^3$  and  $M=m\cdot 2^k$  where L and m are odd integers. Furthermore we express L as a product of any of its factors  $L=m_1m_2$ . We will now consider the following cases:

- 1. s=0, k=0
- 2. s=0, k≠0
- 3.  $s\neq 0, k=0$
- 4. s≠0, k≠0

#### Case 1. s=0, k=0.

Equation (2) takes the form

$$Q = \frac{m_1 m_2}{m} - \frac{m - 1}{2} \tag{3}$$

Obviously we must have  $m \neq 1$  and  $m \neq L$  (=N).

For m=m<sub>1</sub> we have Q>0 when m<sub>2</sub>-(m<sub>1</sub>-1)/2>0 or m<sub>1</sub><2m<sub>2</sub>+1. Since m<sub>1</sub> and m<sub>2</sub> are odd, the latter inequality is equivalent to m<sub>1</sub><2m<sub>2</sub> or, since m<sub>2</sub>=N/m<sub>1</sub>, m<sub>1</sub> <  $\sqrt{2N}$ .

We conclude that a factor m ( $\neq 1$  and  $\neq N$ ) of N (odd) for which m <  $\sqrt{2N}$  gives a solution for Q when M=m is inserted in equation (2).

#### Case 2. s=0, $k\neq 0$ .

Since N is odd we see form (2) that we must have k=1. With M=2m equation (2) takes the form

$$Q = \frac{m_1 m_2}{2m} - \frac{2m - 1}{2} \tag{4}$$

For m=1 (M=2) we find Q=(N-1)2 which corresponds to the obvious solution  $\frac{N-1}{2} + \frac{N+1}{2} = N$ .

Since we can have no solution for m=N we now consider m= $m_2$  ( $\neq 1$ ,  $\neq N$ ). We find Q=( $m_1$ - $2m_2$ +1)/2. Q>0 when  $m_1$ >2 $m_2$ -1 or, since  $m_1$  and  $m_2$  are odd,  $m_1$ >2 $m_2$  Since  $m_1$  $m_2$ =N,  $m_2$ =N/ $m_1$  we find  $m_1$ > $\sqrt{2N}$ .

We conclude that a factor m  $(\neq 1 \text{ and } \neq N)$  of N (odd) for which  $m > \sqrt{2N}$  gives a solution for Q when M=2m is inserted in equation (2).

The number of divisors of N is known as the function d(N). Since all factors of N except 1 and N provide solutions to (2) while M=2, which is not a factor of N, also provides a solution (2) we find that the number of solutions n to (2) when N is odd is

$$n=d(N)-1 \tag{5}$$

Case 3.  $s\neq 0$ , k=0.

Equation (2) takes the form

$$Q = \frac{1}{2} \left( \frac{m_1 m_2}{m} 2^{s+1} - m + 1 \right) \tag{6}$$

Q $\geq$ 1 requires m<sup>2</sup><L·2<sup>s+1</sup>. We distinguish three cases:

Case 3.1. k=0, m=1. There is no solution.

Case 3.2. k=0,  $m=m_1$ .  $Q\ge 1$  for  $m_1 < m_2 2^{s+1}$  with a solution for Q when  $M=m_1$ .

Case 3.3. k=0,  $m=m_1m_2$ .  $Q\ge 1$  for  $L<2^{s+1}$  with a solution for Q when M=L.

Case 4.  $s\neq 0$ ,  $k\neq 0$ .

Equation (2) takes the form

$$Q = \frac{1}{2} \left( \frac{m_1 m_2}{m} 2^{s-k+1} - m \cdot 2^k + 1 \right) \tag{7}$$

Q is an integer if and only if m divides L and  $2^{s-k-1}=1$ . The latter gives k=s+1.  $Q\ge 1$  gives

$$Q = \frac{1}{2} \left( \frac{m_1 m_2}{m} + 1 \right) - m \cdot 2^s \ge 1$$
 (8)

Again we distinguish three cases:

Case 4.1. k=s+1, m=1.  $Q \ge 1$  for  $L > 2^{s+1}$  with a solution for Q when  $M=2^{s+1}$ 

Case 4.2. k=s+1,  $m=m_2$   $Q \ge 1$  for  $m_1 > m_2 2^{s+1}$  with a solution for Q when  $M=m_2 2^{s+1}$ 

Case 4.3. k=s+1, m=L  $Q \ge 1$  for  $1-L \cdot 2^{s} \ge 1$ . No solution

Since all factors of L except 1 provide solutions to (2) we find that the number of solutions n to (2) when N is even is

$$n=d(L)-1 (9)$$

#### Conclusions:

- The number of sequences of consecutive positive integers by which a positive integer N=L·2<sup>s</sup> where L≡1 (mod 2) can be represented is n=d(L)-1.
- We see that the number of integer sequences is the same for N=2<sup>s</sup>L and N=L no matter how large we make s.
- When L<2<sup>s</sup> the values of M which produce integer values of Q are odd, i.e. N can in this case only be represented by sequences of consecutive integers with an odd number of terms.
- There are solutions for all positive integers L except for L=1, which means that N=2<sup>3</sup> are the only positive integers which cannot be expressed as the sum of consecutive integers.
- N=P·2<sup>s</sup> has only one representation which has a different number of terms (<p) for different s until 2<sup>s+1</sup>>P when the number of terms will be p and remain constant for all larger s.

A few examples are given in table 1.

Table 1. The number of sequences for L=105 is 7 and is independent of s in N=L·2\*.

N=105	s=0	N=210	s=1	N=3360	s=5 L>2 <sup>s+1</sup>	N=6720	s=6 L<2 <sup>s+1</sup>
Q	M	Q	M	Q	M	Q	M
34	3	69	3	1119	3	2239	3
19	5	40	5	670	5	1342	5
12	7	27	7	477	7	957	7
1	14	7	15	217	15	441	15
6	10	1	20	150	21	310	21
15	6	12	12	79	35	175	35
52	2	51	4	21	64	12	105

Question 2: Which is the smallest positive integer N which can be represented as a sum of consecutive positive integers in n different ways.

We can now construct the smallest positive integer R(n)=N which can be represented in n ways as the sum of consecutive integers. As we have already seen this smallest integer is necessarily odd and satisfies n=d(N)-1.

With the representation  $N = p_1^{\alpha_1} p_2^{\alpha_2} ... p_i^{\alpha_j}$  we have

$$d(N) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_j + 1)$$

$$n + 1 = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_j + 1)$$
(10)

The first step is therefore to factorize n+1 and arrange the factors  $(\alpha_1+1)$ ,  $(\alpha_2+1)$  ...  $(\alpha_j+1)$  in descending order. Let us first assume that  $\alpha_1>\alpha_2>\ldots>\alpha_j$  then, remembering that N must be odd, the smallest N with the largest number of divisors is

$$R(n)=N=3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}...p_j^{\alpha_j}$$

where the primes are assigned to the exponents in ascending order starting with  $p_1=3$ . Every factor in (10) corresponds to a different prime even if there are factors which are equal.

Example: 
$$n = 269$$
  
 $n+1= 2\cdot 3^3\cdot 5 = 5\cdot 3\cdot 3\cdot 3\cdot 2$   
 $R(n) = 3^4\cdot 5^2\cdot 7^2\cdot 11^2\cdot 13=156080925$ 

When n is even it is seen from (10) that  $\alpha_1, \alpha_2, \ldots \alpha_j$  must all be even. In other words the smallest positive integer which can be represented as a sum of consecutive integers in a given number of ways must be a square. It is therefore not surprising that even values of n in general generate larger smallest N than odd values of n. For example, the smallest integer that can be represented as a sum of integers in 100 ways is  $N=3^{100}$ , which is a 48-digit integer, while the smallest integer that can be expressed as a sum of integer in 99 ways is only a 7-digit integer, namely 3898125.

#### Conclusions:

or

- 3 is always a factor of the smallest integer that can be represented as a sum of consecutive integers in n ways.
- The smallest positive integer which can be represented as a sum of consecutive integers in given even number of ways must be a square.

Table 2. The smallest integer R(n) which can be represented in n ways as a sum of consecutive positive integers.

n	R(n)	R(n) in factor form
1	3	3
2	9	3 <sup>2</sup>
3	15	3.5
4	81	3 <sup>4</sup>
5	45	3 <sup>2</sup> ·5
6	729	3 <sup>6</sup>
7	105	3-5-7
8	225	3 <sup>2</sup> 5 <sup>2</sup>
9	405	3 <sup>4</sup> 5
10	59049	3 <sup>10</sup>
11	315	3 <sup>2</sup> 5:7
12	531441	3 <sup>2</sup> 5.7 3 <sup>12</sup>

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Theorem. Let  $S(m)=\min\{k \in N: m \mid k!\}$  be the Smarandache Function, and  $a_k$ ,  $b_k \in N^*$  (k=1,2,...,n), then we have the following inequality

$$S(\prod_{k=1}^{n} (a k !) ) \leq \sum_{k=1}^{n} a_k b_k$$

Proof:

$$\frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}!)} = \frac{\sum\limits_{k=1}^{n}(a_{k}!)}{\sum\limits_{k=1}^{n}(a_{k}!)} = \frac{\sum\limits_{k=1}^{n}(a_{k}!)}{\sum\limits_{k=1}^{n}(a_{k}!)} = \frac{\sum\limits_{k=1}^{n}(a_{k}!)}{\sum\limits_{k=1}^{n}(a_{k}!)} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}!)} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})!} = \frac{\sum\limits_{k=1}^{n}(a_{k}b_{k})!}{\sum\limits_{k=1}^{n}(a_{k}b_{k})$$

From this result

$$S(\prod_{k=1}^{n} (a_k!) ) \leq \sum_{k=1}^{n} a_k b_k$$

#### A FORMULA OF THE SMARANDACHE FUNCTION

#### Maohua Le

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Abstract. In this paper we give a formula expressing the Smarandache function S(n) by means of n without using the factorization of n.

For any positive integer n, let S (n) denote the Smarandache function of n. Then we have

(1) 
$$S(n) = \min\{ a \mid a \in N, n \mid a! \},$$

(See [1]). In this paper we give a formula of S(n) without using the factorization of n as follows:

Theorem. For any positive integer n, we have

(1) 
$$S(n) = n+1 - \left[ \sum_{k=1}^{n} n -(n \sin(k! \pi/n))^{2} \right]$$

Proof. Let a = S(n). It is an obvious fact that  $1 \le a \le n$ . We see from (1) that

(2) 
$$n \mid k!, k = a, a+1, ..., n.$$

It implies that

(4) 
$$n - (n \sin (k! \pi / n))^2$$
  $= n = 1, k = a, a+1, ..., n.$ 

On the other hand, since n / k! for k = 1, ..., a-1, we have  $\sin(k!\pi/n) \neq 0$  and

(5) 
$$(n \sin \frac{k! \pi}{n})^2 \ge (n \sin \frac{\pi}{n})^2 > 1, \quad k = 1, ..., a - 1.$$

Hense, by (5), we get

(6) 
$$0 < n$$
  $(n \sin (k! \pi/n))^2$   $< 1/n, k = 1, ..., a-1.$ 

Therefore, by (4) and (6), we obtain

(7) 
$$n+1-a < \sum_{k=1}^{n} n -(n \sin(k! \pi/n))^2 < n+1-a+(a-1)/n < n+2-a.$$

Thus, by (7), we get (1) immediately. The theorem is proved.

#### Reference

1. F Smarandache, A function in the number theory, Smarandache function J. 1 (1990), No.1, 3 - 17.

# ON THE DIOPHANTINE EQUATION S(n) = n

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Abstract. Let S(n) denote the Smarandache function of n. In this paper we prove that S(n) = n if and only if n = 1, 4 or p, where p is a prime.

Let N be the set of all positive integers. For any positive integer n, let S(n) denote the Smarandache function of n (see[1]). It is an obvious fact that  $S(n) \le n$ . In this paper we consider the diophantine equation

(1) 
$$S(n) = n, n \in N.$$

We prove a general result as follows:

Theorem. The equation (1) has only the solutions n = 1,4 or p, where p is a prime.

Proof. If n = 1,4 or p, then (1) holds. Let n be an another solution of (1). Then n must be a composite integer with n > 4. Since n is a composite integer, we have n = uv, where u,v are integers satisfying  $u \ge v \ge 2$ . If  $u \ne v$ , then we get  $n \mid u!$ . It implies that  $S(n) \le u = n / v < n$ , a contradiction.

If u = v, then we have  $n = u^2$  and  $n \mid (2u)!$ It implies that  $S(n) \le 2u$ . Since n > 4, we get u > 2 and  $S(n) \le 2u < u^2 = n$ , a contradiction. Thus, (1) has only the solution n = 1,4 or p. The theorem is proved.

# Reference

1. F Smarandache, A function in the number theory, Smarandache function J. 1 (1990), No.1, 3 - 17.

# ON SMARANDACHE DIVISOR PRODUCTS

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Abstract. In this paper we give a formula for Smarandache divisor products.

Let n be a positive integer. In [1, Notion 20], the product of all positive divisors of n is called the Smarandache divisor product of n and denoted by  $P_d$  (n). In this paper we give a formula of  $P_d$ (n) as follows:

Theorem. Let  $n=p_1 \ldots p_k$  be the factorization of n, and let

(1) 
$$r(n) = \begin{cases} \frac{1}{2}(r_1 + 1) \dots (r_k + 1), & \text{if n is not a square,} \\ \frac{1}{2}((r_1 + 1) \dots (r_k + 1) - 1), & \text{if n is a square.} \end{cases}$$

Then we have  $P_d(n) = n^{r(n)}$ .

Proof. Let f (n) denote the number of distinct positive divisors of n. It is a well known fact that

(2) 
$$f(n) = (r_1 + 1) \dots (r_k + 1),$$

(See [2, Theorem 273]). If n is not a square and d is a positive divisor of n, then n/d is also a positive divisor of n with  $n/d \neq d$ . It implies that

(3) 
$$P_d(n) = n^{f(n)/2}$$
.

Hence, by (1), (2) and (3), we get  $P_d(n) = n^{r(n)}$ . If n is a square and d is a positive divisor of n with  $d \neq \sqrt{n}$ , then n/d is also a positive divisor of n with  $n/d \neq d$ . So we have

(4) 
$$P_d(n) = \frac{n^{f(n)/2}}{\sqrt{n}} = n^{(f(n-1)/2}.$$

Therefore, by (1), (2) and (4), we get  $P_d(n) = n^{r(n)}$  too. The theorem is proved.

## References.

- 1. Dumitrescu and V. Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
- 2. G.H.Hardy and e.M.Wright, An Introduction to the Theory of numbers, Oxford Univ. Press, Oxford, 1938.

## ON THE SMARANDACHE N-ARY SIEVE

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Abstract. Let n be a positive integer with n > 1. In this paper we prove that the remaining sequence of Smarandache n-ary sieve contains infinitely many composite numbers.

Let n be a positive integer with n > 1. Let  $S_n$  denote the sequence of Smarandache n-ary sieve (see [1, Notions 29-31]). For example:

$$S_2=\{1,3,5,9,11,13,17,21,25,27, ...\}$$
,  
 $S_3=\{1,2,4,5,7,8,10,11,14,16,17,19,20, ...\}$ 

In [1], Dumitrescu and Seleacu conjectured that  $S_n$  contains infinitely many composite numbers. In this paper we verify the above conjecture as follows:

Theorem. For any positive integer n with n>1,

S<sub>n</sub> contains infinitely many composite numbers.

Proof. By the definition of Smarandache n-ary sieve

(see [1, Notions 29-31]), the sequence  $S_n$  contains the numbers  $n^k + 1$  for any positive integer k. If k is an odd integer with k > 1, then we have

(1) 
$$n^{k+1}=(n+1)(n^{k-1}-n^{k-2}+...+1).$$

We see from (1) that  $(n+1)|(n^k+1)$  and  $n^k+1$  is a composite number. Notice that there exist infinitely many odd integers k with k>1. Thus,  $S_n$  contains infinitely many composite numbers  $n^k+1$ . The theorem is proved.

# References.

1. Dumitrescu and V. Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.

# PERFECT POWERS IN THE SMARANDACHE PERMUTATION SEQUENCE

#### Maohua Le

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Abstract. In this paper we prove that the Smarandache permutation sequience does not contain perfect powers.

Let  $S = \{Sn\}_{n=1}^{\infty}$  be the Smarandache permutation sequence. Then we have

(1) 
$$s_1 = 12$$
,  $s_2 = 1342$ ,  $s_3 = 135642$ ,  $s_4 = 13578642$ , ...

In [1, Notion 6], Dumitrescu and Seleacu posed the following quiestion:

Question. Is there any perfect power belonging to S?
In this respect, Smarandache [2] conjectured: no! In this paper we verify the above conjecture as follows:

Theorem. The sequence S does not contain powers. Proof. Let  $s_n$  be a perfect power. Since  $2 \mid s_n$  by (1), then we have

(2)  $4 | s_n$ .

Since  $s_1 = 12$  is not a perfect power, we get n > 1. Then

from (1) we get

(3)  $s_n = 10^2 a + 42$ , where  $a_{is}$  a positive integer. Notice that  $4 \mid 10^2$  and  $4 \mid 10^2$ . We find from (3) that  $4 \mid 10^2$ , which contradicts (2). Thus, the theorem is proved.

# References

- 1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
- 2. F.Smarandache, Only Problems, not Solutions! Xiquan Pub. House, Phoenix, Chicago, 1990.

## ON SMARANDACHE PSEUDO - POWERS OF THIRD KIND

## Maohua Le

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Abstract. Let m be a positive integer with m > 1. In this paper we prove that there exist infinitely many  $m^{th}$  perfect powers which are Smarandache pseudo -  $m^{th}$  powers of third kind.

Let m be a positive integer with m > 1. For a positive integer a, if some nontrivial permutation of the digits is an  $m^{th}$  power, then a is called a Smarandache pseudo -  $m^{th}$  power. There were many questions concerning the number of Smarandache pseudo -  $m^{th}$  powers (see [1, Notions 71, 74 and 77]). In general, Smarandache [2] posed the following

Conjecture. For any positive integer m with m > 1, there exist infinitely many m powers which are Smarandache pseudo-mth powers of third kind.

In this paper we verify the above conjecture as follows.

Theorem. For any positive integer m with m > 1, there exist infinitely many m th powers are Smarandache pseudo-m th powers of third kind.

Proof. For any positive integer k, the positive integer is an  $m^{th}$  power. Notice that 0 ... 01 is a nontrivial permutation of the digits of  $10^{km}$  and 1 is also an  $m^{th}$  power. It implies that there exist infinitely many Smarandache pseudo -  $m^{th}$ 

powers of third kind. The theorem is proved.

# References

- 1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
- 2. F.Smarandache, Only Problems, not Solutions! Xiquan Pub. House, Phoenix, Chicago, 1993

## Maohua Le

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Abstract. Let a, n be positive integers. In this paper we prove that  $n (a^n - a)[n/2]!$ 

For any positive integer a and n, Smarandache [3] proved that

(1) 
$$n \mid (a^n - a)(n - 1)!$$

The above division relation is the Smarandache divisibility theorem (see [1, Notions 126]). In this paper we give an improvement on (1) as follows:

Theorem. For any positive integers a and n, we have

(2) 
$$n | (a^n - a)[n/2]!,$$

where [n/2] is the largest integer which does not exceed n/2. Proof. The division relation (2) holds for  $n \le 9$ , we may assume that n > 9. By Fermat's theorem (see [2, Theorem 71]), if n is a prime, then we have

(3) 
$$n \mid (a^n - a),$$

for any a. We see from (3) that (2) is true.

If n is a composite number, then we have n = pd, where p,d are integers satisfying  $p \ge q \ge 2$ . Further, if  $p \ne q$ , then we have n|p! It implies that  $n|(n \ne q)!$  Since  $q \ge 2$ , we get

(4) 
$$n | [n/2]!$$

If p = q, Then  $n = p^2$  and

(5) 
$$n \mid (2p)!$$

Since n > 9, we have  $n \ge 4^2$ ,  $p \ge 4$  and  $2p \le n/2$ . Hence, we see from (5) that (4) is also true in this case. The combination of (3) and (4), the theorem is proved.

#### References

- 1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
- 2. G.H.Hardy and E. M.Wright, An Introduction to the Theory of Numbers, Oxford Univ. Press, Oxforf, 1936.
- 3. F.Smarandache, Problemes avec et sans ... problemes!, Somipress, Fes, Morocco, 1983.

# ON PRIMES IN THE SMARANDACHE PIERCED CHAIN

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Abstract. Let  $C = \{c_n\}_{n=1}^{\infty}$  be the Smarandache pierced chain. In this paper we prove that if n > 2, then  $c_n / 101$  is not a prime.

For any positive integer n, let

(1) 
$$c_n = \underbrace{101*100010001...0001}_{n-1 \text{ times}}$$

Then the sequence  $C = \{c_n\}_{n=1}^{\infty}$  is colled the Smarandache percied chain (see[2, Notion 19]). In [3], Smarandache asked the following question:

Question. How many  $c_n/101$  are primes? In this paper we give a complete anser as follows: Theorem. If n > 2, then  $c_n/101$  is not a prime. Proof. Let  $\zeta_n = e^{2\pi\sqrt{-1}/n}$  be a primitive roof of unity with

Proof. Let  $\zeta_n = e^{2\pi \sqrt{-1/n}}$  be a primitive roof of unity with the degree n, and let

$$f_n(x) = \prod_{1 \leq k \leq n} (x - \zeta_n^k).$$

$$g cd(k, n) = 1$$

Then  $f_n(x)$  is a polynomial with integer coefficients. Further, it is a well known fact that if x > 2, then  $f_n(x) > 1$  (see [1]). This implies that if x is an integer with x > 2, then  $f_n(x)$  is an integer with  $f_n(x) > 1$ . On the other hand, we have

(2) 
$$x^{n} - 1 = \prod_{d \mid n} f_{d}(x).$$

We see from (1) that if n > 1, then

(3) 
$$c_n = 1 + 10 + 10 + ... + 10$$
  $= \frac{10^{4n} - 1}{10^4 - 1}$ .

By the above definition, we find from (2) and (3) that

$$\frac{c_n}{101} = \left( \prod_{d \mid 4} f_d (10) \right) / \left( \prod_{d \mid n} f_d (10) \right).$$

Since n > 2, we get 2n > 4 and 4n > 4. It implies that both 2n and 4n are divisors of 4n but not of 4. Therefore, we get from (4) that

(5) 
$$c_n$$
  
---- =  $f_{2n}$  (10)  $f_{4n}$  (10)t,

where t is not a positive integer. Notice that  $f_{2n}(10) > 1$  and  $f_{4n}(10) > 1$ . We see from (5) that  $c_n / 101$  is not a prime. The theorem is proved.

## References

- 1. G.D.Birkhoff and H.S.Vandiver, On the integral divisors of a<sup>n</sup> b<sup>n</sup>, Ann. Of Math. (2), 5 (1904), 173 180.
- 2. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
- 3. F.Smarandache, Only Problems, not Solutions!, Xiquan Pub. House, Phoenix, Chicago, 1990.

# PRIMES IN THE SEQUENCES $\{n^n + 1\}_{n=1}^n$ and $\{n^n + 1\}_{n=1}^n$

# Maohua Le

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Abstract. Let n be a positive integer. In this paper we prove that (i) if n > 2, then  $n^n - 1$  is not a prime; (ii) if n > 2 and  $n^n + 1$  is a prime, then  $n = 2^{2^n}$ , where r is a positive integer.

Let n be a positive integer. In [1, Problem 17], Smarandache posed the following questions

Question A. How many primes belong to the sequence  $\{n^n - 1\}_{n=1}^{\infty}$ ?

Question B. How many primes belong to the sequence  $\{n^n + 1\}_{n=1}^{\infty}$ ?

In this paper we prove the following results:

Theorem 1. 3 is the only prime belonging to  $\{n^n - 1\}_{n=1}^{\infty}$ .

Theorem 2. If n > 2 and  $n^n + 1$  is a prime, then we we have  $n = 2^{r-1}$ , where r is a positive integer.

Proof of Theorem 1. If n=2, then  $2^2-1=3$  is a prime. If n>2, then we have

(1) 
$$n^{n} - 1 = (n - 1) (n^{n-1} + n^{n-2} + ... + n + 1)$$

Since n-1>1 and  $(n^{n-1}+n^{n-2}+...+n+1)$  if n>2, we see from (1) that  $n^n-1$  is not a prime. The theorem is proved.

Proof of Theorem 2. Let  $n^n + 1$  be a prime with n > 2. Since  $n^n + 1$  is an even integer greater than 2 if 2 / n, we get 2 / n. Let  $n = 2^n n_1$ , where s,  $n_1$  are positive integers with  $2 / n_1$ . If  $n_1 > 1$ , then we have

(2) 
$$n + 1 = (n + 1) + 1 = (n + 1)(n + 1) = (n + 1)(n + 1) = (n + 1)(n + 1)$$

It is not a prime. So we have  $n_1 = 1$  and  $n = 2^s$ . It implies that

(3) 
$$n^{n} + 1 = 2^{s \cdot 2^{s}} + 1$$
.

By the same method, we see from (3) that if  $n^n + 1$  is a prime, then s must be a power of 2. Thus, we get  $n = 2^{r}$ . The Theorem is proved.

#### Reference

1. F.Smarandache, Only Problems, not Solutions!, Xiquan Pub. House, Phoenix, Chicago, 1990.

# ON THE SMARANDACHE PRIME ADDITIVE COMPLEMENT SEQUENCE

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Abstract. Let k be an arbitrary large positive integer. In this paper we prove that the Smarandache prime additive complement sequences includes the decreasing sequence k, k-1, ..., 1, 0.

For any positive integer n, let p(n) be the smallest prime which does not excess n. Further let d(n) = p(n) - n. Then

the sequence  $D = \{d(n)\}_{n=1}$  is called the Smarandache prime additive complement sequence. Smarandache asked that if it is possible to as large as we want but finite decreasing sequence k, k-1, ..., 1, 0 included in D? Moreover, he conjectured that the answer is negative (see [1, Notion 46]). However, we shall give a positive answer for Smarandache's questions. In this paper we prove the following result:

Theorem. For an arbitrary large positive integer k, D includes the decreasing sequence k, k-1, ..., 1, 0.

Proof. Let n = (k+1)! + 1. Since 2, 3, ..., k+1 are proper divisors of (k+1)!, then all numbers n+1, n+2, ... n+k are composite numbers. It implies that  $d(n) \ge k$ . Therefore,

D includes the decreasing sequence k, k-1, ..., 1, 0. The theorem is proved.

# Reference

1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994

# AN INEQUALITY FOR THE SMARANDACHE FUNCTION

by

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Let  $S(m) = min \{k | k \in \mathbb{N} : m | k!\}$  be the Smarandache Function. In this paper we prove the following

THEOREM: 
$$S(\prod_{k=1}^{m} m_k) \le \sum_{k=1}^{m} S(m_k)$$
.

We prove by induction. For m=1 it's true. Let m=2, then we prove  $S(m_1 \ m_2) \le S(m_1) + S(m_2)$ . We have  $m_2 \ | S(m_2)!$  and if  $r \ge S(m_1)$  then  $S(m_1)! \ | r(r-1)... \ (r-S(m_1)+1)$ . If  $t \ | S(n_1)!$  then  $t \ | r(r-1)... \ (r-S(n_1)+1)$  so  $m_1m_2 \ | S(m_2)! \ (S(m_2)+1)... \ (S(m_2)+S(m_1)) = \ (S(m_1)+S(m_2))!$  From this it results  $S(m_1m_2) \le S(m_1)+S(m_2)$ . We suppose they are true for m, and we prove for m+1.

#### REFERENCE:

[1] Smarandache, Florentin, "A function in the Number Theory", <Smarandache Function Journal> (1990), No. 1, pp. 3-17.

# ON SMARANDACHE SIMPLE FUNCTIONS

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Absatract. Let p be a prime, and let k be a positive integer. In this paper we prove that the Smarandache simple functions  $S_p(k)$  satisfies  $p \mid S_p(k)$  and  $k(p-1) \leq S_p(k) \leq kp$ .

For any prime p and any positive integer k, let  $S_p(k)$  denote the smallest positive integer such that  $p^k \mid S_p(k)!$ . Then  $S_p(k)$  is called the Smarandache simple function of p and k (see [1, Notion 121]). In this paper we prove the following result.

Theorem. For any p and k, we have  $p \mid S_{p}(k)$  and

(1) 
$$k(p-1) \le S_p(k) \le kp$$
.

Proof. Let  $a = S_p(k)$ . Then a is the smallest positive integer such that

(2) 
$$p^{k} | a!$$
.

If  $p \nmid a$ , then from (2) we get  $p^k \mid (a-1)!$ , a contradiction. So we have  $p \mid a$ .

Since  $(kp)! = 1 \dots p \dots (2p) \dots (kp)$ , we get  $p^k \mid (kp)!$ . It implies that

(3) 
$$a \le kp$$
.

On the other hand, let  $p^r \mid a!$ , where r is a positive integer. It is a well known fact that

(4) 
$$r = \sum_{i=1}^{\infty} [a/p^{i}]$$

where  $[a/p^i]$  is the greatest integer which does not exceed  $a/p^i$ . Since  $[a/p^i] \le a/p^i$  for any i, we see from (4) that

(5) 
$$r < \sum_{i=1}^{\infty} (a / p^{i}) = a / (p - 1)$$

Further, since  $k \le r$  by (2), we find from (5) that

(6) 
$$a > k (p-1)$$
.

The combination of (3) and (6) yields (1). The theorem is proved.

# Reference

1. Editor of Problem Section, Math. Mag 61 (1988), No.3, 202.

# Charles Ashbacher<sup>1</sup>

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#### and

Maohua Lel

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Abstract. Let  $A = \{a_n\}_{n=1}^n$  be a Smarandache type sequence. In this paper we show that if A is a positive integer sequence, then the simple continued fraction  $[a_1, a_2, ...]$  is convergent.

Let  $A = \{a_n\}_{n=1}^{\infty}$  be a Smarandache type sequence. Then The simple continued frction

(1) 
$$a_{1} + \frac{1}{a_{2} + \frac{1}{a_{3} + \cdots}}$$

is called the Smarandache simple continued fraction associated A (See [1]). By the usually symbol (see [2, Notion 10.1]), the continued frction (1) can be written as [a<sub>1</sub>, a<sub>2</sub>, a<sub>3</sub>, ...]. Recently, Castillo [1] posed the following guestion:

Question. Is the continued fraction (1) convergent? In particular, is the continued fraction [1, 12, 123, ...] convergent? In this paper we give a positive answer as follows.

Theorem. If A is a positive integer sequence, then the

<sup>&</sup>lt;sup>1</sup>Editor's Note (M.L.Perez): This article has been done by each of the above authors independently.

continued fraction (1) is convergent.

Proof. If A is a positive integer sequence, then (1) is a usually simple continued fraction and its quotient are positive integers. Therefore, by [2,Theorem165], it is convergent. The Theorem is proved.

On applying [2, Theorems 165 and 176], we get a further result immediately.

Theorem 2. If A is an infinite positive integer sequence, then (1) is equal to an irrational number  $\alpha$ . Further, if A is not periodic, then  $\alpha$  is not an algebraic number of degree two.

## References

- 1. J.Castillo, Smarandache continued fractions, Smarandache Notions J., to appear. Vol. 9, No. 1-2, 40-42, 1998.
- 2. G.H.Hardy and E.M.Wright, An Introduction to the Theory of Numbers, Oxford Univ. Press, Oxford, 1938.

#### NOTES ON PRIMES SMARANDACHE PROGRESSIONS

#### Maohua Le

Department of Mathematics, Zhanjiang Normal College Zhanjiang, Guangdong, P.R.China.

Abstract. In this note we discuss the primes in Smarandache progressions.

For any positive integer n, let p<sub>n</sub> denote the n<sup>th</sup> prime.

~

For the fixed coprime positive integers a,b, let  $P(a,b)=\{ap_n+b\}_{n=1}$ . Then P(a,b) is called a Smarandache progression.

In [1, Problem 17], Smarandache possed the following questions:

Questions. How many primes belonng to P(a,b)?

It would seen that the answers of Smarandache's question is different from pairs (a,b). We now give some observable examples as follows:

Example 1. If a,b are odd integers, then ap  $_n$  +b is an even integer for n>1. It implies that P(a,b) contains at most one prime. In particular, P(1,1) contains only the prime 3.

Exemple 2. Under the assumption of twin prime conjecture that there exist infinitely many primes p such that p+2 is also a prime, then the progression P(1,2) contains infinitely

many primes.

Example 3. Under the assumption of Germain prime conjecture that there exist infinitely many primes p such that 2p+1 is also a prime, then the progression P(2,1) contains infinitely many primes.

# Reference

1. F.Smarandache, Only Problems, not Solutions!, Xiquan Pub. House, Phoenix, Chicago, 1990.

# THE PRIMES p WITH $\lg(p) = 1$

#### Maohua Le

Department of Mathematics, Zhanjiang Normal College Zhanjiang, Guangdong, P.R.China.

Abstract. In this paper we prove that if  $p = a_k ... a_1 a_0$  is a prime satisfying p>10 and lg(p)=1, then  $a_k=...=a_1=a_0=1$  and k+1 is a prime.

Let  $n = \overline{a_k \dots a_1 a_0}$  be a decimal integer. Then the number of distinct digits of n is called the length of Smarandache generalized period of n and denoted by lg(n) (see [1, Notion 114]). In this paper we prove the following result.

Theorem. If  $p = \overline{a_k \dots a_1 a_0}$  is a prime satisfying p > 10 and lg(p)=1, then we have  $a_k = \dots = a_1 = a_0 = 1$  and k+1 is a prime.

Proof. Since  $\lg(p)=1$ , we have  $a_k=...=a_1=a_0$ . Let  $a_0=a$ , where a is an integer with  $0 \le a \le 9$ . Then we have  $a \mid p$ . Since p is a prime and  $p \ge 10$ , we get a=1 and

(1) 
$$p=1...11=10^{k}+...+10+1=....$$

where k is a positive integer. Since k+1>1, if k+1 is not a prime, then k+1 has a prime factor q such that (k+1)/q>1.

Hence, we see from (1) that

$$p = \frac{10^{k+1} - 1}{10-1} \frac{10^{q} - 1}{10-1} \frac{10^{k+1} - 1}{10^{q} - 1} = \frac{q-1}{(10 + ... + 10 + 1)(10} + ... + 10 + 1).$$

It implies that p is not a prime, a contradiction. Thus, if p is a prime, then k+1 must be a prime. The theorem is proved.

# Reference

1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994

# SOME SOLUTIONS OF THE SMARANDACHE PRIME EQUATION

## Maohua Le

Department of Mathematics, Zhanjiang Normal College Zhanjiang, Guangdong, P.R.China.

Abstract. Let k be a positive integer with k>1. In this paper we give some prime solutions  $(x_1,x_2,...,x_k,y)$  of the diophantine equation  $y=2x_1x_2...x_k+1$  with  $2< x_1< x_2<...< x_k< y$ .

Let k be a positive integer with k>1. In [4, Problem 11], Smarandache conjectured that the equation

(1) 
$$y=2x_1 x_2...x_k+1, 2 < x_1 < x_2 < ... < x_k$$

has infinitely many prime solutions  $(x_1, x_2, ..., x_k, y)$  for any k. This is a very dificult problem. The equation (1) is call the Smarandache prime equation (see [3, Notion 123]), while the authors gave solutions of (1) as follows.

$$k=2$$
,  $(x_1, x_2, y) = (17,19,647)$ ;  
 $k=3$ ,  $(x_1, x_2, x_3, y) = (3,5,19,571)$ 

For any positive integer n, let  $p_n$  be the n th odd prime, and let  $q_n = 2 p_1 p_2 ... p_n + 1$ . In this paper, by the calculating

result of [1] and [2], we give nine other solutions as follows.

$$(x_1,x_2,...,x_k,y) = (p_1,p_2,...,p_k,q_k)$$

where k=4,10,66,138,139,311,368,495,514.

# References

- 1. J.P.Buhler, R.E.Crandall, M.A.Penk, Primes of The form n!±1 and 2\*3\*5... p±1, Math. Comp. 38 (1982), 639-643.
- 2. H.Dubner, factorial and primorial primes, J.Recreational Math. 19 (1987), 197-203.
- 3. Dumitrescu and Seleacu, Some notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994
- 4. F.Smarandache, Only Problems, not Solutions!, Xiquan Pub. House, Phoenix, Chicago, 1990.

### SMARANDACHE SERIES CONVERGES

by Charles Ashbacher Charles Ashbacher Technologies Hiawatha, Iowa

The Smarandache Consecutive Series is defined by repeatedly concatenating the positive integers on the right side of the previous element.

1, 12, 123, 1234, . . . , 123456789, 12345678910, 1234567891011, . . .

The Smarandache Reverse Sequence is defined by repeatedly concatenating the positive integers on the left side of the previous element.

- 1, 21, 321, 4321, . . . , 987654321, 10987654321, 1110987654321, . . .
- a) Consider the series formed by summing the inverses of the Smarandache Consecutive Series  ${\sf Series}$

 $1/1 + 1/12 + 1/123 + 1/1234 + \dots$ 

It is a simple matter to prove that this series is convergent. Forming the series

 $1/1 + 1/10 + 1/100 + 1/1000 + \dots$ 

where it is well-known that this series is convergent to the number 10/9. Furthermore, the elements of the two series matched in the following correspondence

1/1 <= 1/1, 1/12 <= 1/10, 1/123 <= 1/100, ...

Therefore, by the ratio test, the sum of the inverses of the Smarandache Consecutive Series is also convergent.

b) Consider the series formed by taking the ratios of the terms of the consecutive sequence over the reverse sequence.

 $1/1 + 12/21 + 123/321 + 1234/4321 + \dots$ 

In this case, it is straightforward to show that the series is divergent.

Consider an arbitrary element of the sequence

where the digit ak = 9. Clearly, e(n) > 1/10, as the numerator and denominator of this ratio have the same number of digits. Since there are an infinite number of such terms, the series contains an infinite number of terms all greater than 1/10. This forces divergence.

## Maohua Le

Department of Mathematics, Zhanjiang Normal College Zhanjiang, Guangdong, P.R.China.

Abstract. Let a,b be integers such that gcd(a,b)=1

and  $a \ne -1$ , 0 or 1. Let  $U(a,b) = \{a^n + b\}_{n=1}^{\infty}$  In this note we discuss the primes in U(a,b).

Let a,b be integers such that gcd(a,b)=1 and  $a \ne -1$ , 0 or 1.

Let  $U(a,b)=\{a^n+b\}_{n=1}^\infty$  In [1, Problem 17], Smarandache posed the following questions:

Question. How many primes belong to U(a,b)?

It would seem that the answers of this questions is different from different pairs (a,b). We now give some observable examples as follows:

Example 1. If a,b are odd integers, then a "+b is either an even integer or zero. It implies that U(a,b) contains at most one prime. In particular, U(3,-1) contains only the prime 2, U(3,1) does not contain any prime.

Example 2. If a>2 and b=-1, then we have

(1) 
$$a^{n}+b=a^{n}-1=(a-1)(a^{n-1}+a^{n-2}+...+1).$$

We see from (1) that  $a^n + b$  is not a prime if n>1. It implies that U(a,b) contains at most one prime. In particular, U(4,-1) contains only the prime 3, U(10,-1) does not contain any prime.

Example 3. Under the assumption of Mersenne prime conjecture that there exist infinitely many primes with the form 2 <sup>n</sup> -1, then the sequence U(2,-1)contains infinitely many primes.

# Reference

1. F.Smarandache, Only Problems, not Solutions!, Xiquan Pub. House, Phoenix, Chicago, 1990.

# THE PRIMES IN THE SMARANDACHE SYMMETRIC SEQUENCE

# Maohua Le

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Abstract. Let  $S=\{s_n\}_{n=1}^{\infty}$  Be the Smarandache symmetric sequence. In this paper we prove that if n is an even integer and  $n/2 \neq 1 \pmod{3}$ , then  $s_n$  is not a prime.

Let S= $\{s_n\}_{n=1}^{\infty}$  be the Smarandache symmetric sequence, where

(1) 
$$s_1=1$$
,  $s_2=11$ ,  $s_3=121$ ,  $s_4=1221$ ,  $s_5=12321$ ,  $s_6=123321$ ,  $s_7=1234321$ ,  $s_8=12344321$ , ...

Smarandache asked how many primes are there among S? (See [1, Notions 3]). In this paper we prove the following result:

Theorem. If n is an iven integer and  $n/2 \ne 1 \pmod{3}$ , then  $s_n$  is not a prime.

Proof. If n is an even integer, then n=2k, where k is a positive integer. We see from (1) that

(2) 
$$s_n = \overline{12 ... kk...21}$$

It implies that

(3) 
$$s_n = 1*10 + 2*10 + ... + k*10 + k*10 + ... + 2*10 + 1*10$$
,

where  $t_1$ ,  $t_2$ , ...,  $t_{2k}$  are nonnegative integers. Since 10  $^t \equiv 1 \pmod{3}$  for any nonnegative integer t, we get from (3) that

(4)  $s_n = 1+2+...+k+k+...+2+1 = k(k+1) \pmod{3}$ . If  $k \neq 1 \pmod{3}$ , then either  $k = 0 \pmod{3}$  or  $k = 2 \pmod{3}$ . In both cases, we have  $k(k+1) = 0 \pmod{3}$  and  $3 \mid s_n$  by (4). Thus,  $s_n$  is not a prime. The theorem is proved.

## Reference

1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.

# ON SMARANDACHE GENERAL CONTINUED FRACTIONS

#### Maohua Le

Department of Mathematics, Zhanjiang Normal College Zhanjiang, Guangdong, P.R.China.

Abstract. Let  $A=\{a_n\}_{n=1}^{\infty}$  and  $B=\{b_n\}_{n=1}^{\infty}$  be two Smarandache type sequences. In this paper we prove that if  $a_{n+1} \ge b_n > 0$  and  $b_{n+1} \ge b_n$  for any positive integer n, the continued fraction

(2) 
$$a_1 + \dots a_2 + a_3 + \dots$$
 is convergent.

Let  $A=\{a_n\}_{n=1}$  and  $B=\{b_n\}_{n=1}$  be two Smarandache type sequences. Then the continued fraction

(1) 
$$a_{1} + \frac{b_{1}}{a_{2} + \frac{b_{2}}{a_{3} + \cdots}}$$

is called a Smarandache general continued fraction associated with A and B (see [1]). By using Roger's symbol, the continued fraction (1) can be written as

Recently, Castillo [1] posed the following question:

Question. Is the continued fractions 
$$1 + \frac{1}{21} = \frac{321}{21} = \frac{321}{21} = \frac{1}{21} = \frac{321}{21} = \frac{1}{21} = \frac{321}{21} = \frac{1}{21} = \frac{321}{21} = \frac{321}{21} = \frac{1}{21} = \frac{321}{21} $

convergent?

In this paper we prove a general result as follows.

Theorem. If  $a_{n+1} \ge b_n > 0$  and  $b_{n+1} \ge b_n$  for any positive integer n, then the continued fraction (2) is convergent.

Proof. It is a well known fact that (2) is equal to the simple continued fraction

(2) 
$$a_1 + \frac{1}{c_1 + c_2 + \dots}$$

where

(4) 
$$c_{2t-1} = \frac{b_2 b_4 \dots b_{2t-2}}{b_1 b_3 \dots b_{2t-1}} a_{2t}$$

$$c_{2t} = \frac{b_1 b_3 \dots b_{2t-1}}{b_2 b_4 \dots b_{2t}}$$
  $t = 1, 2, \dots$ 

Since  $a_{n+1} \ge b_n > 0$  and  $b_{n+1} \ge b_n$  for any positive n, we see from (4) that  $c_n \ge 1$  for any n. It implees that the simple continued fraction (3) is convergent. Thus, the Smarandache general continued fraction (2) is convergent too. The theorem is proved.

#### Reference

1. J.Castillo, Smarandache continued fractions, Smarandache Notions J., Vol. 9, No. 1-2, 40-42, 1998.

# THE LOWER BOUND FOR THE SMARANDACHE COUNTER C(0,n!)

## Maohua Le

Department of Mathematics, Zhanjiang Normal College Zhanjiang, Guangdong, P.R.China.

Abstract. In this paper we prove that if n is an integer with  $n \ge 5$ , then the Smarandache counter C(0,n!) satisfies  $C(0,n!) \ge (1-5^{-k})(n+1)/4-k$ , where  $k=\lfloor \log n / \log 5 \rfloor$ .

Let a be an integer with  $0 \le a \le 9$ . For any positive decimal integer m, the number of a in digits of m is called the Smarandache counter of m with a. It is denoted by C(a,m) (see [1,Notion 132]). Let n be a positive integer. In this paper we give a lower bound for C(0,n!) as follows:

Theorem. If  $n \ge 5$ , then we have

(1) 
$$C(0,n!)>1/4(1-5^{-k})(n+1)-k$$
,

where k=[log n / log 5]
Proof. Let

(2) 
$$n! = \overline{a_{s} a_{s-1} \dots a_{1} a_{0}}$$
.

If  $n \ge 5$ , then we have 10 | n! and  $a_0 = 0$ . Further, let  $2^u | n!$  and  $5^v | n!$ . By [2, Theorem 1\*11\*1], we get

(3) 
$$u = \sum_{r=1}^{\infty} [n/2^r], \quad v = \sum_{r=1}^{\infty} [n/5^r]$$
.

We see from (3) that  $u \ge v$ . It implies that there exist continuous v zeros  $a_0 = a_1 = ... = a_{w1} = 0$  in (2). So we have

(4) 
$$C(0,n!) \ge v$$
.

Let  $k=[\log n/\log 5]$ . Since  $[n/5]^r=0$  if r>k, we see from (3) that

(5) 
$$v = \sum_{r=1}^{\infty} [n/5^r]$$

Since  $[n/5] \ge n/5$ , we get from (5)that

(6) 
$$v \ge \sum_{r=1}^{k} (n/5^r - (5^r - 1)/5^r) = 1/4(1-5^{-k})(n+1)-k$$

Substitute (6) into (4) yelds (1). The theorem is proved.

## Reference

- 1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
- 2. L.-K.Hua, Introduction to Number Theory, Springer, Berlin, 1982.

#### Maohua Le

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Abstract. In this paper we prove that there exist infinitely many Smarandache pseudo-primes of second kind.

Let n be a composite number. If some permutation of the digits of n is a prime, then n is called a Smarandache pseudoprime of second kind (see[1,Notion 65]). In this paper we prove the following result:

Theorem. There exist infinitely many Smarandache pseudoprimes of second kind. ∞

Proof. Let the sequence  $P=\{100r+1\}_{r=1}$ . By Dirichlet's theorem (see[2,Theorem 15]), P contains infinitely many primes. Let

(1) 
$$p = \overline{a_k \dots a_2 a_1 a_0}$$

be a prime belonging to P. Then we have  $a_0 = 1$  and  $a_1 = 0$ . Further let

(2) 
$$n = \overline{a_k ... a_2 a_0 a_1}$$

Then we have  $10 \mid n$ , since  $a_1 = 0$ . Therefore, n is a composite number. Moreover, by (1) and (2), some permutation of the digits of n is prime p. It implies that n is a Smarandache pseudo-prime of second kind. Thus, there exist infinitely many Smarandache pseudo-primes of second kind. The theorem is proved.

#### Reference

- 1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
- 2. G.H.Hardy and E.M.Wright, An Introduction to the Theory of Numbers, Oxford Univ. Press, Oxford, 1938.

### SUMMARY OF

# THE FIRST INTERNATIONAL CONFERENCE ON SMARANDACHE TYPE NOTIONS IN NUMBER THEORY (UNIVERSITY OF CRAIOVA, AUGUST 21-24 1997)

by Henry Ibstedt

The First International Conference on Smarandache Notions in Number Theory was held in Craiova, Romania, 21-22 August 1997. The Organizing Committee had spared no effort in preparing programme, lodging and conference facilities. The Conference was opened by the professor Constantin Dumitrescu<sup>1</sup>, chairman of the Organizing Committee and the initiator of the conference and a leading personality in Number Theory research. He welcomed all participants. Unfortunately professor Dumitrescu's state of health did not permit him to actively lead the conference, although he delivered his first paper later in the day and was present during most sessions. He requested the author of these lines to chair the first day of the conference, a task for which I was elected to continue for the rest of the conference.

In view of the above it is appropriate that I express mine and the other participants gratitude to the organizers and in particular to the Dumitrescu family who assisted throughout with social and arrangements and the facilities required for the smooth running of the conference. I would like to pay special tribute to professor Dumitrescu's son Antoniu Dumitrescu who presented his father's second paper on his behalf.

Unfortunately not all those who intended to participate in the conference were able to come. Their contributions which were submitted in advance have been gratefully received and are included in these proceedings.

A pre-conference session was held with professor V. Seleacu the day before the conference. This was held in french with Mrs Dumitrescu as interpreter. Prof. Seleacu showed some interesting work being conducted by the research group at Craiova University. Mrs Dumitrescu also acted actively during the conference to bridge language difficulties.

Special thanks were expressed at the conference to Dr. F. Luca, USA, who helped during sessions when translation from the romanian language to english was needed. In this context thanks are also due to my wife Anne-Marie Rochard-Ibstedt who made my participation possible by helping me drive from Sweden to Paris and then across Europe to Craiova. She was also active during the conference in taking photos and distributing documents.

Although united through the international language of Mathematics it was not always possible to penetrate presentations in such detail that extended discussions could take place after each session. Informal contacts between participants proved important and opportunities for this was given during breaks and joint dinners.

In the concluding remarks the chairman thanked the organizers and in particular professor Dumitrescu for having very successfully organized this conference. It was noted that the presentations were not made as an end in itself but as sources for further thought and research in this particular area of Number Theory, n.b. the very large number of open problems and notions formulated by Florentin Smarandache. The hope was expressed that the conference had linked together researchers for continuing exchange of views with our modern means of communication such as electronic mail and high speed personal computers.

Professor Dumitrescu thanked the chairman for his work.

Paris 26 March 1998

<sup>&</sup>lt;sup>1</sup> 1949-1997, Obituary in Vol. 8 of the Smarandache Notions Journal.

### PREAMBLE TO

THE FIRST INTERNATIONAL CONFERENCE ON SMARANDACHE TYPE NOTIONS IN NUMBER THEORY (UNIVERSITY OF CRAIOVA, AUGUST 21-24 1997)

## by Henry Ibstedt

Ladies and gentlemen,

It is for me a great honour and a great pleasure to be here at this conference to present some of the thoughts I have given to a few of the ideas and research suggestions given by Florentin Smarandache. In both of my presentations we will look at some integer sequences defined by Smarandache. As part of my work on this I have prepared an inventory of Smarandache sequences, which is probably not complete, but nevertheless it contains 133 sequences. I welcome contributions to complete this inventory, in which an attempt is also made to classify the sequences according to certain main types.

Before giving my first presentation I would like to say a few words about what eventually brought me here.

When I was young my interest in Mathematics began when I saw the beauty of Euclidean geometry - the rigor of a mathematical structure built on a few axioms which seemed the only ones that could exist. That was long before I heard of the Russian mathematician Lobachevsky and hyperbolic geometry. But my facination for Mathematics and numbers was awoken and who can dispute the incredible beauty of a formula like

$$e^{i\pi} + 1 = 0$$

and many others. But there was also the disturbing fact that many important truths can not be expressed in closed formulas and that more often than not we have to resort to approximations and descriptions. For a long time I was fascinated by classical mechanics. Newton's laws provided an ideal framework for a great number of interesting problems. But Einstein's theory of relativity and Heisenberg's uncertainty relation put a stop to living and thinking in such a narrow world. Eventually I ended up doing computer applications in Atomic Physics. But also my geographical world became too narrow and I started working in developing countries in Africa, the far East and the Caribbean, far away from computers, libraries and contact with current research. This is when I returned to numbers and Number Theory. In 1979, when micro computers had just started making an impact, I bought one and brought it with me to the depths of Africa. Since then Computer Analysis in Number Theory has remained my major intellectual interest and stimulant.

With these words I would now like to proceed to the subject of this session.

# The Smarandache Sequence Inventory

# Compiled by Henry Ibstedt, July 1997

A large number of sequences which originate from F. Smarandache or are of similar nature appear scattered in various notes and papers. This is an attempt bring this together and make some notes on the state of the art of work done on these sequences. The inventory is most certainly not exhaustive. The sequences have been identified in the following sources where <u>Doc. No.</u> refers the list of Smarandache Documents compiled by the author. Nearly all of the sequences listed below are also found in Doc. No. 7: *Some Notions and Questions in Number Theory*, C. Dumitrescu and V. Seleacu, with, sometimes, more explicit definitions than those given below. Since this is also the most comprehensive list of Smarandache Sequences the paragraph number where each sequence is found in this document is included in a special column "D/S No"

Source	Seq. No.	Doc. No.
Numeralogy or Properties of Numbers	1-37	1
Proposed Problems, Numerical Sequences	38-46	2
A Set of Conjectures on Smarandache Sequences	47-57	16
Smarandache's Periodic Sequences	58-61	1 <i>7</i>
Only Problems, Not Solutions	62-118	4
Some Notions and Questions in Number Theory	119-133	7

Classification of sequences into eight different types (T):

The classification has been done according to what the author has found to be the dominant behaviour of the sequence in question. It is neither exclusive nor absolutely conclusive.

Recursive:	l R	$t_n=f(t_{n-1})$ , iterative, i.e. $t_n$ is a function of $t_{n-1}$ only. $t_n=f(t_i,t_j,)$ , where i,j <n, i<math="">\neqj and f is a function of at least two variables.</n,>
Non-Recursive:	F	$t_k=f(n)$ , where $f(n)$ may not be defined for all $n$ , hence $k \le n$ .
<u>Concatenation</u>	С	Concatenation.
<u>Elimination:</u>	E	All numbers greater than a given number and with a certain property are eliminated.
<u>Arrangement:</u>	Α	Sequence created by arranging numbers in a prescribed way.
Mixed operations:	М	Operations defined on one set (not necessarily N) to create another set.
Permutation:	Р	Permutation applied on a set together with other formation rules.

					State of the
Seq.	D/S	T	Name ·	Definition (intuitive and/or analytical)	Art
No.	No.				References
1		f	Reverse Sequence	1, 21, 321, 4321, 54321, 10987654321,	
2		R	Multiplicative	2, 3, 6, 12, 18, 24, 36, 48, 54, For arbitrary n <sub>1</sub> and n <sub>2</sub> :	
_			Sequence	n <sub>k</sub> =Min(ni·nj), where k≥3 and j≤k, i≤k, i≠j.	
3		R	Wrong Numbers	$n=\underline{a_1a_2a_k}$ , $k\ge 2$ (where $\underline{a_1a_2a_k}=\underline{a_1\cdot 10^{k-1}}+\underline{a_2\cdot 10^{k-2}}++\underline{a_k}$ ).	Reformulated
				For n>k the terms of the sequence $a_1, a_2, a_n$ are	
ĺ				<u>n-1</u>	
				defined through $a_n = \prod_i a_i$ . n is a wrong number if	
				<i>i=n-k</i>	
		_		the sequence contains n.	
4		f	Impotent Numbers	2, 3, 4, 5, 7, 9, 13, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47, 49,	
				53, 59, 61, A number n whose proper divisors product is less than n, i.e. {p, p²; where p is prime}	·
5		Ε	Random Sieve	1,5,6,7,11,13, 17, 19, 23, 25, General definition:	
,		-	Kanaom sieve	Choose a positive number u <sub>1</sub> at random; -delete all	
				multiples of all its divisors, except this number; choose	
				another number u <sub>2</sub> greater than u <sub>1</sub> among those	
				remaining; -delete all multiples of all its divisors, except	
		<u> </u>		this number; and so on.	
6		F	Cubic Base	0,1,2,3,4,5,6,7,10,11,12,13,14,15,16,17,20,21,22,23,24,25,26,	
		-	Anti Compositorio	Each number n is written in the cubic base.	
7		1	Anti-Symmetric Sequence	11,1212,123,123,12341234,   ,123456789101112123456789101112,	
8		R	ss2(n)	1,2,5,26,29,677,680,701, ss2(n) is the smallest number,	Ashbacher, C.
٥		"	332(11)	strictly greater than the previous one, which is the	Doc.14, p 25.
				squares sum of two previous distinct terms of the	
				sequence.	
9		R	ss l (n)	1,1,2,4,5,6,16,17,18,20, ss1(n) is the smallest number,	
				strictly greater than the previous one (for n≥3), which is	
				the squares sum of one ore more previous distinct terms	
10		_	0(-)	of the sequence.	A-bb-a-b-a-C
10		R	nss2(n)	1,2,3,4,6,7,8,9,11,12,14,15,16,18, nss2(n) is the smallest number, strictly greater than the previous one, which is	Ashbacher, C. Doc.14, p 29.
	1			NOT the squares sum of two previous distinct terms of	DOC.14, p 27.
				the sequence.	
11		R	nss l (n)	1,2,3,6,7,8,11,12,15,16,17,18,19, nss1(n) is the smallest	
				number, strictly greater than the previous one, which is	
	1			NOT the squares sum of one ore more previous distinct	
				terms of the sequence.	
12		R	cs2(n)	1,2,9,730,737,389017001, 389017008,389017729, cs2(n)	Ashbacher, C.
				is the smallest number, strictly greater than the previous	Doc.14, p 28.
		1		one, which is the cubes sum of two previous distinct terms of the sequence	
13		R	cs1(n)	1,1,2,8,9,10,512,513,514,520, cs1(n) is the smallest	
. •		``	1.77	number, strictly greater than the previous one (for n≥3),	
				which is the cubes sum of one ore more previous	
				distinct terms of the sequence.	
14		R	ncs2(n)	1,2,3,4,5,6,7,8,10,11,12,13,14,15, ncs2(n) is the smallest	Ashbacher, C.
				number, strictly greater than the previous one, which is	Doc.14, p 32.
				NOT then cubes sum of two previous distinct terms of the	
15	<del>                                     </del>	R	ncs1(n)	sequence. 1,2,3,4,5,6,7,10,,26,29, ncs1(n) is the smallest number.	
13		"	11031(11)	strictly greater than the previous one, which is NOT the	
				cubes sum of one or more previous distinct terms of the	
				sequence.	
16		R	SGR, General	Let k≥j be natural numbers, and a1, a2,, ak given	
			Recurrence Type	elements, and R a j-relationsship (relation among j	
			Sequence	elements). Then: 1) The elements $a_1, a_2,, a_k$ belong to	
		1		SGR. 2) If m <sub>1</sub> , m <sub>2</sub> ,, m <sub>i</sub> belong to SGE, then R(m <sub>1</sub> , m <sub>2</sub> ,,	
				m <sub>i</sub> ) belongs to SGR too. 3) Only elements, obtained by	
				rules 1) and/or 2) applied a finite number of times, belong to SGR.	
17		F	Non-Null Squares,	1,1,1,2,2,2,3,4,4,, The number of ways in which n can	
''		l '	ns(n)	be written as a sum of non-null squares. Example:	
	í				

				405	Doc. 19, p. 9.
		1	Sequence	2, 0, 0, 7, 20,07,00,70	Doc.14, p 48. Ibstedt, H.
47			Sequence Prime-Digital Sub-	2, 3, 5, 7, 23,37,53,73	Doc.14, p 46. Ashbacher, C.
46		E	Sequence Cube-Digital Sub-	0, 1, 8, 1000, 8000,	Doc.14, p 45. Ashbacher, C.
45		Ε	Square-Digital Sub-	1, 2, 1,3,4,2, 1,3,5,6,4,2, 1,3,5,7,8,6,4,2,1, 0, 1, 4, 9, 49, 100, 144, 400, 441,	Ashbacher, C.
44	+	-	Sub-S Permutation Sub-S	1 2 1342 135642 135784421	
43		•	Sub-S Decresc. Symmetric	1.1. 2.1.1.2, 3,2,1,1,2,3, 4,3,2,1,1,2,3,4,	
42		A	Sub-S Cresc. Symmetric	1, 1, 2,1,1,2, 3,2,1,1,2,3, 1,2,3,4,4,3,2,1	
41		A	Sub-S Decresc. Pyramidal	1, 2.1.2, 3,2.1.2,3, 4,3.2,1,2,3,4,	
40			Cresc. Pyramidal	1, 1,2,1 3,2,1 4,3,2,1 5,4,3,2,1 1, 1,2,1 1,2,3,2,1, 1,2,3,4,3,2,1	
39	<del>  -</del>		Decrescendo Sub-S.		
38		A	Sequence (Generalization) Crescendo Sub-Seq.	s1s <sub>2</sub> , s1s <sub>2</sub> s <sub>3</sub> , s1s <sub>2</sub> s <sub>3</sub> s <sub>n</sub> is called the concatenated S- sequence. 1, 1,2 1,2,3 1,2,3,4 1,2,3,4,5	
37		С	Concatenated S-	Let $s_1, s_2, s_3, \dots s_n$ be an infinite integer sequence. Then $s_1,$	200. 17, p. 12.
36		С	Concatenated Even Sequence	2,24,246,2468,246810,24681012,	lbstedt, H. Doc. 19, p. 12.
			Sequence		Doc. 19, p. 12
35		С	Prime Sequence Concatenated Odd	1,13,135,1357,13579,1357911,135791113	Doc. 19, p. 13. Ibstedt, H.
34		С	Sequence Concatenated	digits from n. 2.23,235,2357, 235711, 23571113,	Ibstedt, H.
33		F	Sequence No Square Digits	from n. 2,3,4,6,7,8,2,3,5,6,7,8,2,2,22,23,2,25, Take out all square	
32		F	No Prime Digits	1,4,6,8,9,10,11,1,1,14,1,16,1,18, Take out all prime digits	
31		F	Unary Sequence	11, 111, 11111, 1111111, 1111111111, $u_n=111$ , $p_n$ digits of "1", where $p_n$ is the n-th prime.	
				the smallest number such that no 3-term geometric progression is in the sequence.	
			Sequence	be the first two term of the sequence, then $m_k$ , for $k \ge 3$ , is	
30		R	Non-Geometric	1,2,3,5,6,7,8,10,11,13,14,15, Definition: Let m <sub>1</sub> and m <sub>2</sub>	
			Sequence (Generalization)	define a U-sequence as follows: $U_n=1+u_1u_2u_n$ .	
29		R	U-Product	k-th factorial number. Let un, n≥1, be a positive integer sequence. Then we	
28		R	Factorial Product Sequence	1,3,13,289,34561,24883201, $F_n=1+f_1f_2f_n$ , where $f_k$ is the	
۷/		Λ.	Sequence	$2.9.217,13825,1728001,373248001,$ $C_n = 1 + C_1C_2C_n$ , where $C_k$ is the k-th cubic number.	
27		R	Sequence Cubic Product	s <sub>k</sub> is the k-th square number. 2,9,217,13825,1728001,373248001, C <sub>n</sub> =1+c <sub>1</sub> c <sub>2</sub> c <sub>n</sub> ,	Doc. 19, p. 7.
26		R	Square Product	2,5,37,577,14401,518401,25401601, S <sub>n</sub> =1+s <sub>1</sub> s <sub>2</sub> s <sub>n</sub> , where	lbstedt, H.
25		R	Prime Product Sequence	$2.7,31,211,2311,30031,510511, p_n=1+p_1p_2p_n$ , where $p_k$ is the k-th prime.	lbstedt, H. Doc. 19, p.4.
				arithmetic progression is in the sequence.	
24		R	Non-Arithmetic Sequence	If $m_1$ , $m_2$ are the first k two terms of the sequence, then $m_{1k}$ for $k \ge 3$ , is the smallest number such that no 3-term	Ibstedt, H. Doc. 19, p. 1.
				number not equal to the product of the k previous terms.	
20			Sequence	sequence, where k≥2; then m <sub>i</sub> , for i≥k+1, is the smallest	
23		R	Base Non-Multiplicative	Let m <sub>1</sub> , m <sub>2</sub> , m <sub>k</sub> be the first k given terms of the	
22		F	Double Factorial	triangular base, defined as follows: t <sub>n</sub> =n(n+1)/2 for n≥1. 1,10,100,101,110,200,201,1000,	
21		F	Triangular Base	1,2,10,11,12,100,101,102,110,1000, Numbers written in	
20		С	Concatenate Seq.	1,22,333,4444,55555,666666,	
				$n=R(f(n_1),f(n_2),,f(n_k))$ for some k and $n_1, n_2, n_k$ such that $n_1+n_2++n_k=n$ ?}.	
			Sequence	numbers. (How many times can n be written in the form:	
19		F F	Non-Null Cubes General Partition	1,1,1,1,1,1,1,2,2,2,2,2,2,2,(8)3,(3)4, Let f be an arithmetic function, and R a relation among	<del> </del>
18	, ,		I Nian Niull Culnac	11111110000000000000000	1

Digital Sub-Seq.   partitioned into groups of digits which are perfect squares		T	T	12		·
Foundation   Sub-Sequence   Sub-Sequence   Sub-Sequence   Sub-Sequence   Prime-Partial-Digital   Sub-Sequence   Sub-Sequence   Prime-Partial-Digital   Sub-Sequence   Sub	48		E	Square-Partial-	49, 100, 144, 169, 361, 400, 441, Squares which can be	Ashbacher, C.
Sub-Sequence   Prime-ParticloBjtal   23, 37, 53, 73, 113, 197, 173, 193, 197, Primes which can be portifined into groups of digits which are also primes. Sub-Sequence   Sub-Sequence   Prime-ParticloBjtal   123, (1+2-3, where 1,2 and 3 are Lucas numbers)   Abhabcher, C. Doc. 14, p. 94, 97, 97, 97, 97, 97, 97, 97, 97, 97, 97				Digital Sub-Seq	squares	Doc.14, p 44.
E   Prime-Partiol-Digital   23.37, 53, 73.113, 137, 173, 193, 197, Primes which can   Abbacher, C.   Doc.14, p. 49	49		E		1000, 8000, 10648, 27000,	
be partitioned into groups of digits which are also  Doc.14, p. 49.  F   Lucas-Portial Digital   123   1+2=3, where 1,2 and 3 are Lucas numbers   Sequence   f   Eligibial Sub-  Sequence   f   Sub-Sequence   f   Sub-Se	50		E		23, 37, 53, 73, 113, 137, 173, 193, 197, Primes which can	
Sub-Sequence				,	be partitioned into groups of digits which are also	1
Sequence  of nj, then the f-digital sub-sequence at solatined by screening the sequence and selecting only those terms which can be partitioned into two groups of digits grand garfign).  53  E Even-Digital Sub-S.  12, 24, 36, 48, 510, 612, 714, 816, 918, 1020, 1122, 1224, Ashbacher, C. Doc. 14, p. 43.  54  E Lucy-Digital Sub-S.  37, 49, (i.e. 37 can be partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned in the partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 3 and 7, and lg-7; the lucky numbers are partitioned as 4, and 7, and 19, and 6, and 7, and 19, and 6, and 7, and 19, and 6, and 7, and 19, and 6, and 7, and 19, and 6, and 7, and 19, and 6, and 7, and 19, and 6, and 7, and 19, and 6, and 7, and 19, and 6, and 7, and 19, and 6, and 7, and 19, and 6, and 7, and 19, and 1	51		F	Sub-Sequence	123, (1+2=3, where 1,2 and 3 are Lucas numbers)	
E   Even-Digital Sub-S.   12, 24, 38, 48, 510, 612, 714, 816, 918, 1020, 1122, 1224.   Ashbacher, C. Dec.14, p. 43.	52		E	1 - 3	of n), then the f-digital subsequence is obtained by screening the sequence and selecting only those terms which can be partitioned into two groups of digits g <sub>1</sub>	
E   Lucy-Digital Sub-S.   37, 49, (i.e. 37 can be partioned as 3 and 7, and ls=7; Ashbacher, C. Doc.14, p 51.	53		E	Even-Digital Sub-S.		
base B. Then: multiples of n, written with digits d <sub>1</sub> , d <sub>2</sub> , d. only (but all r of them), in base B, increasingly ordered, are called the uniform S.    M   Operation   Lef E be an ordered set of elements, E={e <sub>1</sub> , e <sub>2</sub> ,}, and θ as et of binary operations well defined for these elements. Then: all e <sub>1</sub> , e <sub>2</sub> ,}, and = min(e <sub>1</sub> , θ <sub>1</sub> , e <sub>2</sub> ,, and θ as et of binary operations well defined for these elements. Then: all e <sub>1</sub> , e <sub>2</sub> ,}, and = min(e <sub>1</sub> , θ <sub>1</sub> , e <sub>2</sub> ,, and θ as et of binary operations well defined for these elements. Then: all e <sub>1</sub> , e <sub>2</sub> ,}, and = min(e <sub>1</sub> , θ <sub>1</sub> , e <sub>2</sub> ,, and θ as et of binary operations well defined for these elements. Then: all e <sub>1</sub> , e <sub>2</sub> ,, β <sub>2</sub> , and e <sub>3</sub> elements. Then: all e <sub>4</sub> , e <sub>2</sub> ,, β <sub>2</sub> , and e <sub>4</sub> elements. Then: all e <sub>4</sub> , e <sub>2</sub> ,, β <sub>2</sub> , and e <sub>4</sub> elements. Then: all e <sub>4</sub> e <sub>4</sub> , e <sub>2</sub> ,, β <sub>3</sub> , and e <sub>4</sub> elements. Then: all e <sub>4</sub> e <sub>4</sub> , e <sub>5</sub> ,, β <sub>4</sub> , e <sub>4</sub> , e <sub>5</sub> , a.s., β <sub>5</sub> , e <sub>6</sub> , e <sub>7</sub>			E		the lucky numbers are 1,3,7,9,113,15,21,25,31,33, <u>37</u> ,43,49,51,63,	Ashbacher, C.
Sequence   a set of binary operations well defined for these elements. Then: a lack {e1, e2,} an=min{e} 1 de 1 e2, e2, de   e1 e1 e2, e2, de   e1 e1 e2, e2, de   e1 e1 e2, e2, de   e1 e1 e2, e2, de   e1 e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e1 e2, e2, de   e2, e2, e2, e2, e2, e3, e4, e3, e3, e2, e4, e3, e3, e2, e4, e3, e4, e3, e4, e3, e4, e3, e4, e4, e4, e4, e4, e4, e4, e4, e4, e4	55		M	Uniform Sequence	base B. Then: multiples of n, written with digits $d_1$ , $d_2$ , $d_r$ only (but all r of them), in base B, increasingly ordered,	
Maintain   Random Operation   Let E be an ordered set of elements, E=(e_1, e_2,) and θ a set of binary operations well defined for these elements. Then: ose(e_1, e_2,), an=(e_1 ⊕1 e_2 ⊕2, ⊕4, e_1)≥0. for n≥1.	56		М		a set of binary operations well defined for these elements. Then: $a_1 \in \{e_1, e_2,\}$ , $a_{n+1} = min\{e_1 \theta_1 e_2 \theta_2 \theta_A\}$	
58         M         N-digit Periodic Sequence         42.18,63.27,45.09.81,63.27, Start with a positive integer N with not all its digits the same, and let N' be its digital reverse. Put N₁=l N·N¹ and let N' be its digital reverse of N Put N₂=l N₁-N·1], and so on.         Ibstedt, H.         Doc. 20, p. 3.           59         M         Subtraction Periodic Sequence         52.24.41.13.30.02.19.90.08.79.96.68.85.57.74.46.63.35.52 Let c be a fixed positive integer. Start with a positive integer N and let N' be its digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N₁-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N¹-d, and let N₁' be the digital reverse. Put N₁=l N₁-d, and let N₁' be the digital reverse. Put N₁=l N₁-d, and let N₁' be the digital reverse. Put N₁=l N₁-d, and let N₁' be the digital reverse. Put N₁=l N₁-d, and let N₁' be the digital reverse. Put N₁=l N₁-d, and let N₁' be the digital reverse. Put N₁=l N₁-d, and let N₁' be the digital reverse. Put N₁=l N₁-d, and let N₁' be the digital reverse. Put N₁=l N₁-d, and let N₁' be the digital reverse. Put N₁=l N₁-d, and let N₁-d, and let N₁-d, and let N₁-d, and let N₁-d, and let N₁-d, and let N₁-d, and let N₁-d, and let N₁-d, and let N₁-d, and let N₁	57		М		Let E be an ordered set of elements, E={e <sub>1</sub> , e <sub>2</sub> ,} and $\theta$ a set of binary operations well defined for these elements. Then: a <sub>1</sub> $\epsilon$ { e <sub>1</sub> , e <sub>2</sub> }, a <sub>m+1</sub> ={e <sub>1</sub> $\theta$ <sub>1</sub> e <sub>2</sub> $\theta$ <sub>2</sub> $\theta$ <sub>A</sub>	
59         M         Subtraction Periodic Sequence         52.24.41.13.30.02.19.90.08.79.96,68.85.57.74.46,63.35.52	58		М		42,18,63,27,45,09,81,63,27, Start with a positive integer N with not all its digits the same, and let N' be its digital reverse. Put $N_1 = N - N'$ and let $N_1$ ' be the digital reverse	1
60 M Multiplication Periodic Sequence	59		М		52,24,41,13,30,02,19,90,08,79,96,68,85,57,74,46,63,35,52, Let c be a fixed positive integer. Start with a positive integer N and let N' be its digital reverse. Put $N_1 = N' - cl$ and let $N_1'$ be the digital reverse of $N_1$ . Put $N_2 = N_1' - cl$ ,	· ·
61         M         Mixed Composition Periodic Sequence         75,32,51,64,12,31,42,62,84,34,71,86,52,73,14,53,82,16,75, Ibstedt, H. Doc. 20, p. 8.           62         1         1         Consecutive Seq.         1, 12, 123, 12345, 123456, 123456, 1234567,         Kashihara, K. Doc. 15, p. 25.           63         2         I/P         Circular Sequence         1, (12, 21), (123, 231, 312), (1234, 2341, 3412, 4123), Kashihara, K. Doc. 15, p. 25.         Kashihara, K. Doc. 15, p. 25.           64         3         A         Symmetric Sequence         1, 11, 121, 1221, 12321, 123321, 123321, 1234321, 12344321, Ashbacher, C. Doc. 14, p 57.         Ashbacher, C. Doc. 14, p 57.           65         4         A         Deconstructive Sequence         1, 23, 456, 7891, 23456, 789123, 4567891, 23456789, 23456789, 23456789, 23456789, 12	60		М		$68,26,42,84,68, \dots$ Let c>1 be a fixed integer. Start with a positive integer N, multiply each digit x of N by c and replace that digit by the last digit of cx to give N <sub>1</sub> , and	
62         1         I         Consecutive Seq.         1, 12, 123, 12345, 123456, 1234567,         Kashihara, K. Doc. 15, p. 25.           63         2         I/P         Circular Sequence         1, (12, 21), (123, 231, 312), (1234, 2341, 3412, 4123),         Kashihara, K. Doc. 15, p. 25.           64         3         A         Symmetric Sequence         1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 12344321, 12344321, 12344321, 12344321, 12344321, 12344321, 12344321, 12344321, 12344321, 12344321, 12344321, 12344321, 12344321, 123444321, 123444321, 1234444321, 1234444321, 1234444321, 12344444321, 12344444321, 1234444444, 123444444, 12344444, 12344444, 12344444, 12344444, 12344444, 12344444, 12344444, 1234444, 12344444, 1234444, 1234444, 1234444, 1234444, 123444, 123444, 123444, 123444, 123444, 123444, 123444, 123444, 123444, 123444, 123444, 123444, 123444, 123444, 123444, 123444, 12344, 12344, 123444, 12344, 123444, 12344, 12344, 123444, 12344, 12344, 1234			М	,	75,32,51,64,12,31,42,62,84,34,71,86,52,73,14,53,82,16,75, Let N be a two-digit number. Add the digits, and add them again if the sum is greater then 10. Also take the absolute value of their difference. These are the first and	
63         2         I/P         Circular Sequence         1, (12, 21), (123, 231, 312), (1234, 2341, 3412, 4123),         Kashihara, K. Doc. 15, p. 25.           64         3         A         Symmetric Sequence         1, 11, 121, 1221, 12321, 123321, 1234321, 123444321, 12344321, 123444321, 123444321, 123444321, 123444321, 123444321, 123444321, 123444321, 123444321, 123444321, 123444321, 123444321, 123444321, 123444321, 123444321, 12344444, 123444444, 123444444, 12344444, 123444444, 12344444, 123444444, 123444444, 123444444, 1234444, 123			1		1, 12, 123, 12345, 123456, 1234567,	
64       3       A Symmetric Sequence       1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 12344321, 120c.14, p 57.       Ashbacher, C. Doc.14, p 57.         65       4       A Deconstructive Sequence       1, 23, 456, 7891, 23456, 789123, 4567891, 23456789, 123456789, 123456789, 1234567891, 1234567891, 1234567891, 1234567891, 1234567891, 1234567891, 1234567891, 1234567891, 1234567891, 1234567891, 123456, 123456, 123456, 123456, 123456, 123456, 1234567891, 1234567891, 1234567891, 1234567891, 123456,			I/P		1, (12, 21), (123, 231, 312), (1234, 2341, 3412, 4123),	•
65         4         A         Deconstructive Sequence         1, 23, 456, 7891, 23456, 789123, 4567891, 23456789, 23456789, 23456789, 234567891, 2345	64	3	Α	Sequence	1, 11, 121, 1221, 12321, 123321, 1234321, 12344321,	Ashbacher, C.
66         5         A         Mirror Sequence         1, 212, 32123, 4321234, 543212345, 65432123456,         Ashbacher, C. Doc.14, p 59.           67         6         A/P Permutation Sequence Gen. in doc. no. 7         12, 1342, 135642, 13578642, 13579108642, 13579108642, 135791112108642, 1357911131412108642,         Ashbacher, C. Doc.14, p 5.           68*         M Digital Sum         (0,1,2,3,4,5,6,7,8,9), (1,2,3,4,5,6,7,8,9,10), (2,3,4,5,6,7,8,9,10), (2,3,4,5,6,7,8,9,0,11), (d <sub>s</sub> (n) is the sum of digits)         Kashihara, K. Doc. 15, p.6.           69*         M Digital Products         0,1,2,3,4,5,6,7,8,9,0,1,2,3,4,5,6,7,8,9,0,2,4,6,8,19,12,14,16,18, (3,6,9,12,15,18,21,24,27,0,4,8,12,16,20,24,28,32,36,0,5,10,1) 5,20,25, d <sub>p</sub> (n) is the product of digits         Doc. 15, p.7.	65	4	A	1		Kashihara, K.
67 6 A/P Permutation Sequence Gen. in doc. no. 7 12. 1342. 135642, 13578642, 13579108642,	66	5	A			Ashbacher, C.
(2.3.4.5,6,7,8,9,10,11), (d <sub>s</sub> (n) is the sum of digits) Doc. 15, p.6.  M Digital Products 0,1,2,3,4,5,6,7,8,9,0,1,2,3,4,5,6,7,8,9,0,2,4,6,8,19,12,14,16,18, Kashihara, K. 0,3,6,9,12,15,18,21,24,27,0,4,8,12,16,20,24,28,32,36,0,5,10,1 Doc. 15, p.7. 5,20,25, d <sub>p</sub> (n) is the product of digits	_			Sequence Gen. in doc. no. 7	135791112108642,1357911131412108642,	Ashbacher, C.
69 * M Digital Products 0,1.2,3,4,5,6,7,8,9,0,1,2,3,4,5,6,7,8,9,0,2,4,6,8,19,12,14,16,18, Kashihara, K. 0,3,6,9,12,15,18,21,24,27,0,4,8,12,16,20,24,28,32,36,0,5,10,1 Doc. 15, p.7. 5,20,25, d <sub>p</sub> (n) is the product of digits			М		(2.3,4,5,6,7,8,9,10,11), (d <sub>s</sub> (n) is the sum of digits)	·
	69 *		М	Digital Products	0.1.2.3.4.5.6.7.8,9.0.1.2.3.4.5.6.7.8,9.0.2.4.6.8,19.12,14,16,18, 0.3.6.9.12.15.18.21.24.27.0.4.8,12.16.20.24.28,32,36,0.5,10.1	Kashihara, K.
	70	15	F	Simple Numbers	2,3,4,5,6,7,8,9,10,11,13,14,15,17, A number is called a	Ashbacher, C.

				· · · · · · · · · · · · · · · · · · ·	
				simple number if the product of its proper divisors is less than or equal to n.	Doc.14, p20.
71	19		Pierced Chain	101, 1010101, 101010101010, 101010101010	Ashbacher, C.
'	.,	·	riciced criami	c(2)=101*10001, c(3)=101*100010001, etc	Doc.14, p 60.
		İ		Qn. How many c(n)/100 are primes?	Kashihara, K.
				differentially chily too are planes.	Doc. 15, p. 7.
72	20	F	Divisor Products	1,2,3,8,5,36,7,64,27,100,11,1728,13,196,225,1024,17, p <sub>d</sub> (n)	Kashihara, K.
/2	20	Г	Divisor Floducis		i e
70		-	B B	is the product of all positive divisors of n.	Doc. 15, p. 8.
73	21	F	Proper Divisor	1,1,1,2,1,6,1,8,3,10,1,144,1,14,15,64,1.324, p <sub>d</sub> (n) is the	Kashihara, K.
	<u> </u>		Products	product of all positive divisors of n except n.	Doc. 15, p. 9.
74	22	F	Square	1,2,3,1,5,6,7,2,1,10,11,3,14,15,1,17,2,19,5,21,22,23,6,1,26,	Ashbacher, C.
1			Complements	For each integer n find the smallest integer k such that	Doc.14, p 9.
				nk is a perfect square.	Kashihara, K.
					Doc. 15, p. 10.
75	23	F	Cubic	1,4,9,2,25,36,49,1,3,100,121,18,169,196,225,4,289, For	Ashbacher, C.
	24		Complements	each integer n find the smallest integer k such that nk is	Doc.14, p 9.
			Gen. to m-power	a perfect cube.	Kashihara, K.
		l	complements in		Doc. 15, p. 11.
			doc. no. 7		
76	25	E	Cube free sieve	2,3,4,5,6,7,9,10,11,12,13,14,15,17,18,19,20,21,22,23,24,25,26	
	26		Gen. in doc. no. 7	,28,	
77	27	Ε	Irrational Root Sieve	2.3.5.6.7.10.11.12.13.14.15.17. Eliminate all ak, when a is	
				squarefree.	
78	37	F	Prime Part (Inferior)	2,3,3,5,5,7,7,7,11,11,13,13,13,13,17,17,19,19,19,19,19,23,23,2	Kashihara, K.
				3,23,23,23, For any positive real number n $p_p(n)$ equals	Doc. 15, p. 12.
		İ		the largest prime less than or equal to n.	
79	38	F	Prime Part (Superior)	2,2,2,3,5,5,7,7,11,11,11,11,13,13,17,17,17,17,19,19,23,23,23,	Kashihara, K.
1 1		'	· ············	23, For any positive real number n $p_p(n)$ equals the	Doc. 15, p. 12.
				smallest prime number greater than or equal to n.	230. 10, p. 12.
80	39	F	Square Part (Inferior)	0,1,1,1,4,4,4,4,9,9,9,9,9,9,9, The largest square less	Kashihara, K.
	3/	'	square rain (innerior)	than or equal to n.	Doc. 15, p. 13.
81	40	F	Square Part	0,1,4,4,4,9,9,9,9,9, The smallest square greater than or	Kashihara, K.
01	40	'	(Superior)	equal to n.	
82	41	F	Cube Part (Inferior)	0,1,1,1,1,1,1,1,8,8,8,8,8,8,8,8,8,8,8,8,	Doc. 15, p. 13.
02	41	'	Cobe Full (intelior)		
92	42	F	Cuba Part (Superior)	largest cube less than or equal to n.	
83	42	Г	Cube Part (Superior)	0,1,8,8,8,8,8,8,8, The smalest cube greater than or	
94	43	F	Factorial Part	equal to n.	
84	43	r		1,2,2,2,2, (18)6, $F_p(n)$ is the largest factorial less than or	
		-	(Inferior)	equal to n.	
85	44	F	1	1,2, (4)6, (18)24, (11)120, f <sub>p</sub> (n) is the smallest factorial	
	15		(Superior)	greater than or equal to n.	
86	45	F	Double Factorial	1,1,1,2,3,8,15,1,105,192,945,4,10395,46080,1,3,2027025,	
			Complements	For each n find the smallest k such that nk is a double	
				factorial, i.e. nk=1·3·5·7·9·n (for odd n) and	
		<u> </u>		nk=2·4·6·8··n (for even n)	
87	46	F	Prime additive	1,0,0,1,0,1,0,3,2,1,0,1,0,3,3,2, t <sub>n</sub> =n+k where k is the	Ashbacher, C.
	·		complements	smallest integer for which n+k is prime (reformulated).	Doc.14, p 21.
					Kashihara, K.
					Doc. 15, p. 14.
88		F	Factorial Quotients	1,1,2,6,24,1,720,3,80,12,3628800, t <sub>n</sub> =nk where k is the	Kashihara, K.
				smallest integer such that nk is a factorial number	Doc. 15, p. 16.
				(reformulated).	
89 *		F	Double Factorial	1,2,3,4,5,6,7,4,9,10,11,6, d <sub>f</sub> (n) is the smallest integer	
			Numbers	such that d <sub>fin)</sub> !! is a multiple of n.	
90	55	F	Primitive Numbers	2,4,4,6,8,8,8,10,12,12,14,16,16,16,16, S <sub>2</sub> (n) is the smallest	Important
			(of power 2)	integer such that $S_2(n)!$ is divisible by $2^n$ .	•
91	56	F	Primitive Numbers	3,6,9,9,12,15,18,18, S <sub>3</sub> (n) is the smallest integer such	Kashihara, K.
	57		(of power 3)	that $S_3(n)!$ is divisible by $3^n$ .	Doc. 15, p. 16.
			Gen. to power p, p	* *	· ·
			prime.		
92		М	Sequence of	Definition: Unsolved problem: 55	
-	l		Position	2 2	
93	58	F	Square Residues	1,2,3,2,5,6,7,2,3,10,11,6, S <sub>r</sub> (n) is the largest square free	
			1.05.2005	number which divides n.	
94	59	F	Cubical Residues	1,2,3,4,5,6,7,9,10,11,12,13, C <sub>r</sub> (n) is the largest cube free	
· ·	60	'	Gen. to m-power	number which divides n.	
1	55		residues.	Hornbor Willeri divides II.	
95	61	F		0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4, e <sub>2</sub> (n)=k if 2* divides n but	Ashbach C
	01		Exponents to bower	0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4, e2(n)=k ii 2* aivides n but	Ashbacher, C.

	1	T -	To:	Out that I	T=
- 01	1	F	2)	2 <sup>k+1</sup> if it does not.	Doc.14, p 22.
96	62 63		Exponents (of power 3). Gen. to exp. of	$0.0,1,0,0,1,0,0,2,0,0,1,0,0,1,0,0,2,$ $e_2(n)=k$ if $3^k$ divides $n$ but $3^{k+1}$ if it does not.	Ashbacher Doc.14, p 24.
97	64	F/P	Pseudo-Primes of	0.2 5 7 11 12 14 14 17 10 00	12 1 11 12
"	65	'''	first kind. Ext. to	2.3,5,7,11,13,14,16,17,19,20, A number is a pseudo- prime if some permutation of its digits is a prime	Kashihara, K.
	66		second and third	(including the identity permutation).	Doc. 15, p. 17.
	00		kind in doc. no. 7.	the identity permutation.	
98	69	F/P	<del></del>	1,4,9,10,16,18,25,36,40, A number is a pseudo-square if	Ashbacher, C.
	70	'''	first kind. Ext. to	some permutation of its digits is a perfect square	Doc.14, p 14.
	71		second and third	(including the identity permutation).	Kashihara, K.
	į		kind in doc. no. 7.	January, period 2001,	Doc. 15, p. 18.
99	72	F/P	Pseudo-Cubes of	1.8, 10,27,46,64,72,80,100, A number is a pseudo-cube	Ashbacher, C.
	73		first kind. Ext. to	if some permutation of its digits is a cube (including the	Doc.14, p 14.
ļ	74		second and third	identity permutation).	Kashihara, K.
1	75		kind in doc. no. 7.		Doc. 15, p. 18.
	76		(Gen. Pseudo-m-		
	77	ļ	powers)		
100	78	F/P	1	1,2,6,10,20,24,42,60,100,102,120, A number is a pseudo-	
	79		first kind. Ext. to	factorial if some permutation of its digits is a factorial	
İ	80		second and third	number (including the identity permutation).	
101	81	F/P	kind in doc. no. 7. Pseudo-Divisors of	1 10 100 1 0 10 00 100 000 1 0 10 00	
'0'	82	''	first kind. Ext. to	1,10,100,1,2,10,20,100,200,1,3,10,30, A number is a pseudo-divisor of n if some permutation of its digits is a	
	83		second and third	divisor of n (including the identity permutation).	
			kind in doc. no. 7.	divisor of it (incloding the identity permutation).	
102	84	F/P		1,3,5,7,9,10,11,12,13,14,15,16,17, A number is a pseudo-	Ashbacher, C.
	85	'	Numbers of first	odd number if some permutation of its digits is an odd	Doc.14, p 16.
	86		kind. Ext. to second	number.	DOC.14, p 16.
		Ì	and third kind in		
			doc. no. 7.		
103	87	F/P	Pseudo-Triangular	1,3,6,10,12,15,19,21,28,30,36, A number is a pseudo-	
			Numbers	triangular number if some permutation of its digits is a	
				triangular number.	
104	88	F/P	<del>-</del>	0,2,4,6,8,10,12,14,16,18,20,21,22,23, A number is a	Ashbacher, C.
	89		Numbers of first	pseudo-even number if some permutation of its digits is	Doc.14, p17.
	90		kind. Ext. to second and third kind in	an even number.	
			doc. no. 7.		
105	91	F/P	Pseudo-Multiples (of	0.5,10,15,20,25,30,35,40,45,50,51, A number is a	4.11
, , , ,	92	' '	5) of first kind. Ext.	pseudo-multiple of 5 if some permutation of its digits is a	Ashbacher, C. Doc.14, p19.
	93		to second and third	multiple of 5 (including the identity permutation).	DOC.14, D19.
	94		kind in doc. no. 7.	the state of the s	
	95		(Gen. to Pseudo-		
	96		multiples of p.)		
106	100	F	Square Roots	$0,1,1,1,2,2,2,2,3,3,3,3,3,3,3,3,\ldots$ $s_{\alpha}(n)$ is the superior integer	
				part of the square root of n.	
107	101	F	Cubical Roots	0.1,1,1,1,1,1,1, 19(2), 37(3), c <sub>q</sub> (n) is the superior integer	
	102		Gen. to m-power	part of the cubical root of n.	
108	47	F	roots m <sub>c</sub> (n)	0.1.10.100.101.1000.100	
100	47			0,1,10,100,101.1000,1001,10000,10001,10010, See	Kashihara, K.
109	48	F		Unsolved problem: 90	Doc. 15, p. 32.
,0,	40 49	r	_ '	0.1,2,3,10,11,12,13,20,100,101, See Unsolved problem:	
İ	7/		base and gen.	7.1	
	ļ		base (Unsolved		
		I	problem 93)		
110	28	М		7,13,19,23,25,31,33,37,43, All odd numbers that are not	
				equal to the difference between two primes.	
111	29	Е	Binary Sieve	1,3,5,9,11,13,17,21,25, Starting to count on the natural	Ashbacher, C.
]	1		ļ	numbers set at any step from 1: -delete every 2-nd	Doc.14, p 53.
	[			numbers; -delete, from the remaining ones, every 4-th	, ,
į		1		numbers and so on: delete, from the remaining ones,	
110				every 2 <sup>k</sup> -th numbers, k=1,2,3,	
112	30		Trinary Sieve	1,2,4,5,7,8,10,11,14,16,17, (Definition equiv. to 114)	Ashbacher, C.
113	31		Gen. to n-ary sieve	1250111701000045	Doc.14, p 54.
113	ی∠	=		1,3,5,9,11,17,21,29,33,41,47,57, From the natural	Ashbacher, C.
				numbers: - keep the first number, delete one number out	Doc.14, p 55.

	,				
				of 2 from all remaining numbers; - keep the first	
				remaining number, delete one number out of 3 from the	
		-		next remaining numbers; and so on	
114	33	E	General-Sequence	Let u;>1, for i=1, 2, 3,, be a strictly increasing integer	
		İ	Sieve	sequence. Then: From the natural numbers: -keep one	
		1		number among 1,2,3,, u <sub>1</sub> -1 and delete every u <sub>1</sub> -th	
				numbers; -keep one number among the next u <sub>2</sub> -1	
	ļ			remaining numbers and delete every u2th numbers; and	
				so on, for step k ( $k \ge 1$ ): keep one number among the next $u_k = 1$ remaining numbers and delete every $u_k = 1$ numbers:	
115	36	M	General Residual	$(x+C_1)$ $(x+C_{F[m]})$ ,m=2, 3, 4,, where $C_i$ , $1 \le i \le F[m]$ ,	Kashihara, K.
113	30	~``	Sequence	forms a reduced set of residues mod m. x is an integer	
			3equence	and f is Euler's totient.	Doc. 15, p. 11.
116	<del>                                     </del>	м	Table:(Unsolved	6,10,14,18,26,30,38,42,42,54,62,74,74,90, t <sub>n</sub> is the largest	Kashihara, K.
110		,,,,	103)	even number such that any other even number not	Doc. 15, p. 19.
			100)	exceeding it is the sum of two of the first nodd primes.	100C. 13, p. 19.
117	<del>                                     </del>	м	Second Table	9.15,21,29,39,47,57,65,71,93,99,115,129,137, v <sub>n</sub> is the	Kashihara, K.
'''		'''	Second Table	largest odd number such that any odd number $\geq 9$ not	Doc. 15, p. 20.
				exceeding it is the sum of three of the first nodd primes.	DOC. 13, p. 20.
118		M	Second Table	0.0.0,0,1,2,4,4,6,7,9,10,11,15,17,16,19,19,23, q <sub>2x+1</sub>	Kashihara, K.
	1	"``	Sequence	represents the number of different combinations such	Doc. 15, p. 20.
	1			that 2k+1 is written as a sum of three odd primes.	200. 10, p. 20.
119	34	E	More General-	Let u>1, for i=1, 2, 3,, be a strictly increasing integer	
	1	-	Sequence Sieve	sequence, and vizio; another positive integer sequence.	
				Then: From the natural numbers: -keep the vi-th number	
				among 1,2,3,, u <sub>1</sub> -1 and delete every u <sub>1</sub> -th numbers; -	
		ĺ		keep the vzth number among the next uz1 remaining	
				numbers and delete every u <sub>2</sub> -th numbers; and so on, for	
				step k ( $k \ge 1$ ): -keep the $v_k$ -th number among the next $u_k$ -1	
				remaining numbers and delete every ukth numbers;	
120	35	F	Digital Sequences	In any number base B, for any given infinite integer or	
			Special case:	rational sequence s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> ,, and any digit D from 0 to	
			Construction	B-1, build up a new integer sequence which associates	
			sequences	to $s_1$ the number of digits of D of $s_1$ in base B, to $s_2$ the	
				number of digits D of $s_2$ in base B, and so on.	
121	50	F	Factorial Base	0,1.10,11,20,21,100,101,110,111,120,121,200,201,210,211,	
				(Each number n written in the Smarandache factorial	
		1		base.)(Smarandache defined over the set of natural	
100				numbers the following infinite base: for k≥1, f <sub>k</sub> =k!)	
122	51	F	Generalized Base	(Each number in written in the Smarandache	
				generalized base.)(Smarandache defined over the set	
				of natural numbers the following infinite base: $1=g_0 < g_1 < g_2 < g_1 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_2 < g_$	•
123	52	F	S	<g<sub>k&lt;)</g<sub>	
123	32		Smarandache	1,2,3,4,5,3,7,4,6,5, S(n) is the smallest integer such that	
124	53	F	Numbers Smarandache	s(n)! is divisible by n.	
124	JJ	"	Quotients	1,1,2,6,24,1,720,3,80,12,3628800, For each n find the	
125	54	F	Double Factorial	smallest k such that nk is a factorial number.	
,25	54	'	Numbers	1,2,3,4,5,6,7,4,9,10,11,6,13, d <sub>f</sub> (n) is the smallest integer such that d <sub>f</sub> (n)!! is a multiple of n.	
126	67	R	Smarandache	+	
120	٠,	"	almost Primes of the	a <sub>1</sub> ≥2, for n≥2 a <sub>n</sub> = the smallest number that is not divisible	
1		Ì	first kind	by any of the previous terms.	
127	68	R	Smarandache	$a_1 \ge 2$ , for $n \ge 2$ $a_n =$ the smallest number that is coprime	
/		١,	almost Primes of the	with all the previous terms.	
j			second kind	The previous lettes.	
128	97	С	Constructive Set S	1: 1,2 belong to S	
		R	(of digits 1 and 2)	II: if a and b belong to S, then <u>ab (</u> concatenation)	
ŀ				belongs to S	
1				III: Only elements obtained be applying rules I and II a	
				finite number of times belong to S	
129	98	С	Constructive Set S	I: 1,2,3 belong to \$	
Ì	99	R	(of digits 1,2 and 3)	II: if a and b belong to S, then <u>ab (</u> concatenation)	
-			Gen. Constructive	belongs to S	
	ļ		Set (of digits d1, d2,	III: Only elements obtained be applying rules I and II a	
			d <sub>m</sub> ) 1≤m≤9.	finite number of times belong to S	
130	104	F	Goldbach-	6,10,14,18,26,30,38,42,42,54,t(n) is the largest even	
			Smarandache Table	number such that any other even number not	
				exceeding it is the sum of two of the first n odd primes.	
				190	

131	105	F	Smarandache- Vinogradov Table	9,15,21,29,39,47,57,65,71,93, V(n) is the largest odd number such that any odd number ≥9 not exceeding it is the sum of three of the first n odd primes.	
132	106	F	Smarandache- Vinogradov Sequence	0,0,0,0,1,2,4,4,6,7,9,10,a(2k+1) represents the number of different combinations such that 2k+1 is written as a sum of three odd primes.	
133	115	F	Sequence of Position	Let $\{x_n\}$ , $n \ge 1$ , be a sequence of integers and $0 \le k \le 9$ a digit. The Smarandache sequence of position is defined as $U_n^{(k)} = U^{(k)} \{x_n\} = \max\{i\}$ if k is the 10-th digit of $x_n$ else -1.	

# List of Smarandache Documents

# Compiled by Henry Ibstedt, July 1997

Doc.	Title	Author/Ref.	Year	ISBN nr
No				
1	Numeralogy or Properties of Numbers	Smarandache, F.	1975	Univ. Craiova Archives
2	Proposed Problems, Numerical Sequences	Smarandache, F.	1975	Univ. Craiova
3	Smarandache Function Journal	Vol. 1	1992	Ś
4	Only Problems, Not Solutions (Edition?)	Smarandache, F.	1993	1-879585-00-6
5	Only Problems, Not Solutions, 4 <sup>th</sup> ed.	Smarandache, F.	1993	1-879585-00-6
6	Smarandache Function Journal	Vol. 2-3, No. 1	1993	1053-4792
7	Some Notions and Questions in	Dumitrescu, C.	1994	1-879585-48-0
•	Number Theory	Seleacu, V.	1,,,,	1 077000 40 0
8	Smarandache Function Journal	Vol. 4-5, No. 1	1994	1053-4792
9	Smarandache Function Journal	Vol. 6, No. 1	1995	1053-4792
10	An Introduction to the	Ashbacher, C.	1995	1-879585-49-9
	Smarandache Function			
11	Smarandache Notions Journal	Vol. 7, No. 1-2-3	1996	1084-2810
12	The Most Paradoxxist	Le Charles, T.	1996	1-879585-52-9
10	Mathematician of the World			
13	Collected Papers	Smarandache, F.	1996	973-9205-02-X
14	Collection of Problems on Smarandache Notions	Ashbacher, C.	1996	1-879585-50-2
15	Comments and Topics on Smarandche Notions and	Kashihara, K.	1996	1-897585-55-3
	Problems			
16	A Set of Conjectures on	Smith, Sylvester	1996	Bulletin of
. •	Smarandache Sequences	Siriiri, Sylvesiei	1770	Pure and Applied Sciences
17	Smarandache's Periodic	Popov, M.R.	1996	Mathematical
	Sequences (Sequences of	Smarandache, F.	,,,,	Spectrum, Vol
	Numbers)	0,110,0,10,0,1.		29, No 1
	,			(Univ. Craiova
				Conf. 1975)
18	Surfing on the Ocean of Numbers - a Few Smarandache Notions	lbstedt, H.	1997	1-879585-57-X
	and Similar Topics			
19	A Few Integer Sequences	lbstedt, H.	1997	
20	On Smarandache's Periodic	lbstedt, H.	1997	
21	Sequences The Smarandache Function	C. Dumitrescu,	1007	1 070505 47 0
41		V.Seleacu	1996	1-879585-47-2 EUP
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# HISTORY OF THE

## SM ARANDACHE FUNCTION

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## 1 Introduction

This function is originated from the Romanian professor Florentin Smarandache. It is defined as follows:

For any non-null integer n,  $S(n)=\min\{m\mid m! \text{ is divisible by }n\}$ . So we have S(1)=0,  $S(2^5)=S(2^6)=S(2^7)=8$ .

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_t^{\alpha_t} \tag{1}$$

is the decomposition of n into primes, then

$$S(n) = \max S(p_i^{\alpha_i}) \tag{2}$$

and moreover, if [m, n] is the smallest common multiple of m and n then

$$S([m,n]) = \max\{S(m), S(n)\}\tag{3}$$

Let us observe that if  $\bigwedge = \min$ ,  $\bigvee = \max$ ,  $\bigwedge_d =$  the greatest common divizor,  $\bigvee^d =$  the smallest common multiple then S is a function from the lattice  $\left(\mathbf{N}^\star, \bigwedge_d, \bigvee^d\right)$  into the lattice  $\left(\mathbf{N}, \bigwedge, \bigvee\right)$  for which

$$S\left(\bigvee_{i=\overline{1,s}}^{d} m_{i}\right) = \bigvee_{i=\overline{1,s}} S\left(m_{i}\right) \tag{4}$$

# 2 The calculus of S(n)

From (2) it results that to calculate S(n) is necessary and sufficient to know  $S(p_i^{\alpha_i})$ . For this let p be an arbitrary prime number and

$$a_n(p) = \frac{p^n - 1}{p - 1}$$
  $b_n(p) = p^n$  (5)

If we consider the usual numerical scale

$$(p): b_0(p), b_1(p), \ldots, b_k(p), \ldots$$

and the generalised numerical scale

$$[p]: a_1(p), a_2(p), \ldots, a_n(p), \ldots$$

then from the Legendre's formula

$$\alpha! = \prod_{p_i < \alpha} p_i^{E_{p_i}(\alpha)} \tag{6}$$

where  $E_p(\alpha) = \sum\limits_{j \geq 1} \left[ \frac{\alpha}{p^j} \right]$  it results that

$$S\left(p^{a_n(p)}\right) = b_n(p)$$

and even that: if

$$\alpha = k_{\nu} a_{\nu}(p) + k_{\nu-1} a_{\nu-1}(p) + \ldots + k_1 a_1(p) = \overline{k_{\nu} k_{\nu-1} \ldots k_1}_{[p]}$$
 (7)

is the expression of  $\alpha$  in the generalised scale [p] then

$$S(p^{\alpha}) = k_{\nu}p^{\nu} + k_{\nu-1}p^{\nu-1} + \ldots + k_{1}p$$
 (8)

The right hand in (8) may be written as  $p\left(\alpha_{[p]}\right)_{(p)}$ . That is  $S\left(p^{\alpha}\right)$  is the number obtained multiplying by p the exponent  $\alpha$  written in the scale [p] and "read" it in the scale (p). So, we have

$$S\left(p^{\alpha}\right) = p\left(\alpha_{[p]}\right)_{(p)} \tag{9}$$

For example to calculate  $S\left(3^{100}\right)$  we write the exponent  $\alpha=100$  in the scale

$$[3]: 1, 4, 13, 40, 121, \dots$$

We have  $a_{\nu}(p) \leq p \Leftrightarrow (p^{\nu}-1)/(p-1) \leq \alpha \Leftrightarrow \nu \leq \log_p((p-1)\alpha+1)$  and so  $\nu$  is the integer part of  $\log_p((p-1)\alpha+1)$ ,

$$\nu = [\log_p((p-1)\alpha + 1)]$$

For our example  $\nu = [\log_3 201] = 4$ . Then the first difft of  $\alpha_{[3]}$  is  $k_4 = [\alpha/a_4(3)] = 2$ . So  $100 = 2a_4(3) + 20$ . For  $\alpha_1 = 20$  it results  $\nu_1 = [\log_3 41] = 3$  and  $k_{\nu_1} = [20/a_3(3)] = 1$  so  $20 = a_3(3) + 7$  and we obtain  $100_{[3]} = 2a_4(3) + a_3(3) + a_2(3) + 3 = 2113_{[3]}$ .

From (8) it results  $S(3^{100}) = 3(2113)_{(3)} = 207$ .

Indeed, from the Legendre's formula it results that the exponent of the prime p in the decomposition of  $\alpha!$  is  $\sum_{i>1} \left[\frac{\alpha}{p^j}\right]$ , so the exponent of 3 in the decom-

position of 207! is  $\sum_{j \ge 1} \left[ \frac{207}{3^j} \right] = 69 + 23 + 7 + 2 = 101$  and the exponent of 3 in the decomposition of 206! is 99.

Let us observe that, as it is shown in [1], the calculus in the generalised scale [p] is essentially different from the calculus in the standard scale (p), because

$$a_{n+1}(p) = pa_n(p) + 1$$
 and  $b_{n+1}(p) = pb_n(p)$ 

Other formulae for the calculus of  $S(p^{\alpha})$  have been proved in [2] and [3]. If we note  $S_p(\alpha) = S(p^{\alpha})$  then it results [2] that

$$S_{p}(\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \tag{10}$$

where  $\sigma_{[p]}(\alpha)$  is the sum of the digits of  $\alpha$  written in the scale [p]

$$\sigma_{[p]}(\alpha) = k_{\nu} + k_{\nu-1} + \dots + k_1$$

and also

$$S_p(\alpha) = \frac{(p-1)^2}{p} \left( E_p(\alpha) + \alpha \right) + \frac{p-1}{p} \sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha)$$

where  $\sigma_{(p)}(\alpha)$  is the sum of digits of  $\alpha$  written in the scale (p), or

$$S_p(\alpha) = p\left(\alpha - \left[\frac{\alpha}{p}\right] + \left[\frac{\sigma_{[p]}(\alpha)}{p}\right]\right)$$

As a direct application of the equalities (2) and (8) in [16] is solved the following problem:

"Which are the numbers with the factorial ending in 1000 zeros?"

The solution is

 $S(10^{1000}) = S(2^{1000}5^{1000}) = \max\{S(2^{1000}), S(5^{1000})\} =$ =  $\max\{2(1000_{[2]})_{(2)}, 5(1000_{[5]})_{(5)}\} = 4005$ . 4005 is the smallest natural number with the asked propriety.

4006, 4007, 4008, and 4009 verify the proprety but 4010 does not, because  $4010! = 4009! \cdot 4010$  has 1001 zeros.

In [11] it presents an another calculus formula of S(n):

$$S(n) = n + 1 - \left[ \sum_{k=1}^{n} n^{-\left(n \sin\left(k! \frac{\pi}{n}\right)\right)^{2}} \right]$$

3 Solved and unsolved problems concerning the Smarandache Function

In [16] there are proposed many problems on the Smarandache Function. M. Mudge in [12] discuses some of these problems. Many of them are unsolved until now. For example:

Problem (i): Investigate those sets of consecutive integers  $i, i+1, i+2, \ldots, i+x$  for which S generates a monotonic increasing (or indeed monotonic decreasing) sequence. (Note: For 1,2,3,4,5,S generates the monotonic increasing sequence 0,2,3,4,5).

Problem (ii): Find the smallest integer k for which it is true that for all n less than some given  $n_0$  at least one of S(n), S(n+1), ..., S(n-k+1) is

- (A) a perfect square
- (B) a divisor of  $k^n$
- (C) a factorial of a positive integer

Conjecture what happens to k as  $n_0$  tends to infinity.

Problem (iii): Construct prime numbers of the form  $\overline{S(n)S(n+1)\dots S(n+k)}$ . For example  $\overline{S(2)S(3)}=23$  is prime, and  $\overline{S(14)S(15)S(16)S(17)}=75617$  also prime.

The first order forward finite differences of the Smarandache function are defined thus:

$$D_s(x) = |S(x+1) - S(x)|$$
  

$$D_s^{(k)}(x) = D(D(\dots \text{ k times } D_s(x) \dots)$$

Problem (iv): Investigate the conjecture that  $D_s^{(k)}(1) = 1$  or 0 for all k

greater than or equal to 2.

- J. Duncan in [7] has proved that for the first 32000 natural numbers the conjecture is true.
- J. Rodriguez in [14] poses the question than if it is possible to construct an increasing sequence of any (finite) length whose Smarandache values are strictly decreasing. P. Gronas in [9] and K. Khan in [10] give different solution to this question.
- T. Yau in [17] ask the question that:

For any triplets of consecutive positive integers, do the values of S satisfy the Fibonacci relationship S(n) + S(n+1) = S(n+2)?

Checking the first 1200 positive integers the author founds just two triplets for which this holds:

S(9) + S(10) = S(11), S(119) + S(120) = S(121).

That is S(11-2) + S(11-1) = S(11) and  $S(11^2-2) + S(11^2-1) = S(11^2)$  but we observe that  $S(11^3-2) + S(11^3-1) \neq S(11^3)$ .

More recently Ch. Ashbacher has anounced that for n between 1200 and 1000000 there exists the following triplets satisfying the Fibonacci relationship:

S(4900) + S(4901) = S(44902); S(26243) + S(26244) = S(26245);

S(32110) + S(32111) = S(32112); S(64008) + S(64009) = S(64010);

S(368138) + S(368139) = S(368140); S(415662) + S(415663) = S(415664); but it is not known if there exists an infinity family of solutions.

The function  $C_s: \mathbb{N}^* \mapsto \mathbb{Q}$ ,  $C_s(n) = \frac{1}{n}(S(1) + S(2) + \cdots + S(n))$  is the sum of Cesaro concerning the function S.

Problem (v): Is there  $\sum_{n\geq 1} C_s^{-1}(n)$  a convergent series? Find the smallest k

for which  $\left(\underbrace{C_s \circ C_s \circ \cdots \circ C_s}_{k \text{ times}}\right) (m) \geq n$ .

Problem (vi): Study the function  $S_{\min}^{-1}: \mathbb{N} \setminus \{1\} \mapsto \mathbb{N}$ ,  $S_{\min}^{-1}(n) = \min S^{-1}(n)$ , where  $S^{-1}(n) = \{m \in \mathbb{N} | S(m) = n\}$ .

M. Costewitz in [6] has investigated the problem to find the cardinal of  $S^{-1}(n)$ .

In [2] it is shown that if for n we consider the standard decomposition (1) and  $q_1 < q_2 < \cdots < q_s < n$  are the primes so that  $p_i \neq q_j$ ,  $i = \overline{1,t}$ ,  $j = \overline{1,s}$ , then if we note  $e_i = E_{p_i}(n)$ ,  $f_k = E_{q_k}(n)$  and  $\hat{n} = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ ,  $\hat{n}_0 = \hat{n}/n$ ,

 $q = q_1^{f_1} q_2^{f_2} \cdots q_s^{f_s}$ , it result

card 
$$S^{-1}(n) = (d(\hat{n}) - d(\hat{n}_0)) d(q)$$
 (11)

where d(r) is the number of divisors of r.

The generating function  $F_S: \mathbb{N}^* \mapsto \mathbb{N}$  associated to S is defined by  $F_S(n) = \sum_{d/n} S(d)$ . For example  $F_S(18) = S(1) + S(2) + S(3) + S(6) + S(9) + S(18) = S(1) + S$ 

S(18) = 20.

P. Gronas in [8] has proved that the solution of the diophantine equation  $F_S(n) = n$  have the solution  $n \in \{9, 16, 24\}$  or n prime.

In [11] is investigated the generating function for  $n = p^{\alpha}$ . It is shown that

$$F_S(p^{\alpha}) = (p-1)\frac{\alpha(\alpha+1)}{2} + \sum_{j=1}^{\alpha} \sigma_{[p]}(j)$$
 (12)

and it is given an algorithm to calculate the sum in the right hand of (12).

Also it is proved that  $F_S(p_1p_2\cdots p_t)=\sum\limits_{i=1}^t 2^{i-1}p_i$ . Diophantine equations are given in [14] (see also [12]).

We mentione the followings:

- (a) S(x) = S(x+1) conjectured to have no solution
- (b) S(mx + n) = x
- (c) S(mx+n) = m + nx
- (d) S(mx + n) = x!
- (e)  $S(x^m) = x^n$
- (f) S(x) + y = x + S(y), x and y not prime
- (g) S(x + y) = S(x) + S(y)
- (h) S(x+y) = S(x)S(y)
- (i) S(xy) = S(x)S(y)

In [1] it is shown that the equation (f) has as solution every pair of composite numbers x = p(1+q), y = q(1+p), where p and q are consecutive primes, and that the equation (i) has no solutions x, y > 1.

Smarandache Function Journal, edited at the Department of Mathematics from the University of Craiova, Romania and published by Number Theory Publishing Co, Glendale, Arizona, USA, is a journal devoted to the study of Smarandache function. It publishes original material as well as reprints some that has appeared elsewhere. Manuscripts concerning new results, including computer generated are actively solicited.

# 4 Generalizations of the Smarandache

Function'

In [4] are given three generalizations of the Smarandache Function, namely the Smarandache functions of the first kind are the functions  $S_n : \mathbb{N}^* \to \mathbb{N}^*$  defined as follows:

(i) if  $n = u^i (u = 1 \text{ or } u = p)$ , prime number) then  $S_n(a)$  is the smallest positive integer k with the property that k! is a multiple of  $n^a$ .

(ii) if 
$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$$
 then  $S_n(a) = \max_{1 \le j \le t} S_{p_j^{\alpha_j}}(a)$ .

If n = p then  $S_n$  is the function  $S_p$  defined by F. Smarandache in [15]  $(S_p(a))$  is the smallest positive integer k such that k! is divisible by  $p^n$ .

The Smarandache function of the second kind  $S^k: \mathbb{N}^* \to \mathbb{N}^*$  are defined by  $S^k(n) = S_n(k), k \in \mathbb{N}^*$ .

For k=1, the function  $S^k$  is the Smarandache function, with the modification that S(1)=1.

If (a): 
$$1 = a_1, a_2, \ldots, a_n, \ldots$$

(b): 
$$1 = b_1, b_2, \ldots, b_n, \ldots$$

are two sequences with the property that

$$a_{kn} = a_k a_n$$
 ;  $b_{kn} = b_k b_n$ 

Let  $f_a^b: \mathbf{N}^* \mapsto \mathbf{N}^*$  be the function defined by  $f_a^b(n) = S_{a_n}(b_n)$ ,  $(S_{a_n}$  is the Smarandache function of the first kind).

It is easy to see that:

(i) if  $a_n = 1$  and  $b_n = n$  for every  $n \in \mathbb{N}^*$ , then  $f_a^b = S_1$ .

(ii) if 
$$a_n = n$$
 and  $b_n = 1$  for every  $n \in \mathbf{N}^*$ , then  $f_a^b = S^1$ .

The Smarandache functions the third kind are functions  $S_a^b = f_a^b$  in the case that the sequences (a) and (b) are different from those concerned in the situations (i) and (ii) from above.

In [4] it is proved that

$$S_n(a+b) \le S_n(a) + S_n(b) \le S_n(a)S_n(b) \text{ for } n > 1$$

$$\max \left\{ S^k(a), S^k(b) \right\} \le S^k(ab) \le S^k(a) + S^k(b) \text{ for every } a, b \in \mathbb{N}^*$$

$$\max \left\{ f_a^b(k), f_a^b(n) \right\} \le f_a^b(kn) \le b_n f_a^b(k) + b_k f_a^b(n)$$

so, for  $a_n = b_n = n$  it results

$$\max \{S_k(k), S_n(n)\} \le S_{kn}(kn) \le nS_k(k) + kS_n(n) \text{ for every } k, n \in \mathbb{N}^*.$$

This relation is equivalent with the following relation written by means of the Smarandache function:

$$\max\left\{S(k^k),S(n^n)\right\} \leq S\left((kn)^{kn}\right) \leq nS(k^k) + kS(n^n)$$

In [5] it is presents an other generalization of the Smarandache function. Let  $\mathcal{M} = \{S_m(n)|n, m \in \mathbb{N}^*\}$ , let  $A, B \in \mathcal{P}(\mathbb{N}^*) \setminus \emptyset$  and  $a = \min A$ ,  $b = \min B$ ,  $a^* = \max A$ ,  $b^* = \max B$ . The set I is the set of the functions  $I_A^B : \mathbb{N}^* \mapsto \mathcal{M}$  with

$$I_A^B(n) = \begin{cases} S_a(b) &, & \text{if } n < \max\{a,b\} \\ S_{a_k}(b_k) &, & \text{if } \max\{a,b\} \le n \le \max\{a^*,b^*\} \\ & & \text{where} \\ & a_k = \max_i \left\{a_i \in A \middle| a_i \le n\right\} \\ & b_k = \max_j \left\{b_j \in B \middle| b_j \le n\right\} \\ S_{a^*}(b^*) &, & \text{if } n > \max\{a^*,b^*\} \end{cases}$$

Let the rule  $\top: I \times I \mapsto I$ ,  $I_A^B \top I_C^D = I_{A \cup C}^{B \cup D}$  and the partial order relation  $\rho \subset I \times I$ ,  $I_A^B \rho I_C^D \Leftrightarrow A \subset C$  and  $B \subset D$ .

It is easy to see that  $(I, T, \rho)$  is a semilattice.

The elements  $u, v \in I$  are  $\rho$ -strictly preceded by w if:

- (i)  $w \rho u$  and  $w \rho v$
- (ii)  $\forall x \in I \setminus \{w\}$  so that  $x \rho u$  and  $x \rho v \Rightarrow x \rho w$ .

Let  $I^{\#}=\{(u,v)\in I\times I|u,v \text{ are }\rho\text{-strictly preceded }\}$ , the rule  $\bot:I^{\#}\mapsto I$ ,  $I_A^B\bot I_C^D=I_{A\cap C}^{B\cap D}$  and the order partial relation r,  $I_A^BrI_C^D\Leftrightarrow I_C^D\rho I_A^B$ . Then the structure  $(I^{\#},\bot,r)$  is called the return of semilattice  $(I,\top,\rho)$ .

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### OTHER SMARANDACHE TYPE FUNCTIONS

# by J. Castillo I40 & Window Rock Rd. Lupton, Box 199, AZ 86508, USA

- 1) Let  $f: N \longrightarrow N$  be a strictly increasing function and x an element in N. Then:
  - a) Inferior Smarandache f-Part of x,

ISf(x) is the smallest k such that  $f(k) \le x \le f(k+1)$ .

b) Superior Smarandache f-Part of x,

SSf(x) is the smallest k such that  $f(k) < x \le f(k+1)$ .

#### Particular Cases:

- a) Inferior Smarandache Prime Part:
  For any positive real number n one defines ISp(n) as the largest prime number less than or equal to n.
  The first values of this function are (Smarandache[6] and
  - Sloane[5]):
  - 2,3,3,5,5,7,7,7,7,11,11,13,13,13,13,17,17,19,19,19,19,23,23.
- b) Superior Smarandache Prime Part: For any positive real number n one defines SSp(n) as the smallest prime number greater than or equal to n. The first values of this function are (Smarandache[6] and Sloane[5]):
  - 2,2,2,3,5,5,7,7,11,11,11,11,13,13,17,17,17,17,19,19,23,23,23.
- c) Inferior Smarandache Square Part: For any positive real number n one defines ISs(n) as the largest square less than or equal to n. The first values of this function are (Smarandache[6] and Sloane[5]):
- 0,1,1,1,4,4,4,4,4,9,9,9,9,9,9,9,16,16,16,16,16,16,16,16,16,25,25.
  b) Superior Smarandache Square Part:
   For any positive real number n one defines SSs(n) as the smallest square greater than or equal to n.
   The first values of this function are (Smarandache[6] and Sloane[5]):
- 0,1,4,4,4,9,9,9,9,16,16,16,16,16,16,16,25,25,25,25,25,25,25,25,25,36.

  - e) Superior Smarandache Cube Part:
    For any positive real number n one defines SSs(n) as the smallest cube greater than or equal to n.
    The first values of this function are (Smarandache[6] and Sloane[5]):

- - f) Inferior Smarandache Factorial Part: For any positive real number n one defines ISf(n) as the largest factorial less than or equal to n. The first values of this function are (Smarandache[6] and Sloane[5]):

- 2) Let g: A ---> A be a strictly increasing function, and let "~" be a given internal law on A. Then we say that f: A ---> A is smarandachely complementary with respect to the function g and the internal law "~" if: f(x) is the smallest k such that there exists a z in A so that x~k = g(z).

#### Particular Cases:

- a) Smarandache Square Complementary Function:
   f: N ---> N, f(x) = the smallest k such that xk is a
   perfect square.
   The first values of this function are (Smarandache[6] and
   Sloane[5]):
   1,2,3,1,5,6,7,2,1,10,11,3,14,15,1,17,2,19,5,21,22,23,6,1,26,3,7.
- b) Smarandache Cubic Complementary Function:
   f: N ---> N, f(x) = the smallest k such that xk is a
   perfect cube.
   The first values of this function are (Smarandache[6] and
   Sloane[5]):
   1,4,9,2,25,36,49,1,3,100,121,18,169,196,225,4,289,12,361,50.
  More generally:
- c) Smarandache m-power Complementary Function: f: N ---> N, f(x) = the smallest k such that xk is a perfect m-power.
- d) Smarandache Prime Complementary Function:
   f: N ---> N, f(x) = the smallest k such that x+k is a prime.
   The first values of this function are (Smarandache[6] and Sloane[5]):
   1,0,0,1,0,3,2,1,0,1,0,3,2,1,0,1,0,3,2,1,0,5,4,3,2,1,0,1,0,5.

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# SURVEY ON THE RESEARCH OF SMARANDACHE NOTIONS

# by M. L. Perez, editor

The American CRC Press, Boca Raton, Florida, published, in December 1998, a 2000 pages "CRC Concise Encyclopedia of Mathematics", by Eric W. Weisstein, ISBN 0-8493-9640-9, internationally distributed.

Among the entries included in this prestigious encyclopedia there also are the following:

- "Smarandache functions"
- [i.e., Pseudosmarandache Function (p. 1459), Smarandache Ceil Function (p. 1659), Smarandache Function (p. 1660) the most known, Smarandache-Kurepa Function (p. 1661), Smarandache Near-to-Primordial (p. 1661)], Smarandache-Wagstaff Function (p. 1663)]
- "Smarandache sequences" [41 such sequences are listed (pp. 1661-1663), in addition of 7 other Smarandache concatenated sequences (pp. 310-311))
- "Smarandache constants" [11 such constants are listed (pp. 1659-1660)]
  - "Smarandache paradox" (p. 1661).

Five large pages from the above encyclopedia are dedicated to these notions.

Other contributors to the Smarandache Notions are cited as well in this wonderful mathematical treasure: C. Ashbacher, A. Begay, M. Bencze, J. Brown, E. Burton, I. Cojocaru, S. Cojocaru, J. Castillo, C. Dumitrescu, Steven Finch, E. Hamel, F. Iacobescu, H. Ibstedt, K. Kashihara, H. Marimutha, M. Mudge, I. M. Radu, J. Sandor, V. Seleacu, N. J. A. Sloane, S. Smith, Ralf W. Stephan, L. Tutescu, David W. Wilson, E. W. Weisstein, etc.

Professor Eric W. Weisstein from the University of Virginia has extended more results on Smarandache sequences, such as:
- The Smarandache Concatenated Odd Sequence:

- 1, 13, 135, 1357, 13579, 1357911, 135791113, 13579111315, ... (Sloane's A019519) contains another prime term:
- SCOS(2570) = 13579111315...51375139, which has 9725 digits! This is the largest consecutive odd number sequence prime ever found.

Conjecture 1: There is a finite number of primes in this sequence.
- The Smarandache Concatenated Prime Sequence:

- 2, 23, 235, 2357, 235711, 23571113, 2357111317, ...
- (Sloane's A019518) is prime for terms 1, 2, 4, 128, 174, 342, 435, 1429, ... (Sloane's A046035) with no other less than 1960.
- Conjecture 2: There is a finite number of primes in this sequence. The Smarandache Concatenated Square Sequence:
- 1, 14, 149, 14916, 1491625, 149162536, 14916253649, ...
- (Sloane's A019521) contains a prime only 149 (the third term) in the first 1828 terms.
- Conjecture 3: There is only a prime in this sequence.

- The Smarandache Concatenated Cubic Sequence: 1, 18, 1827, 182764, 182764125, 182764125216, ... (Sloane's A019522) contains no prime in the first 1356 terms. Conjecture 4: There is no prime in this sequence.

David W. Wilson (wilson@cabletron.com) proved that - The Smarandache Permutation Sequence: 1342, 135642, 13579108642, 135791112108642, 13578642, 1357911131412108642, ... has no perfect power in its terms. Proof: Their last digits should be: either 2 for exponents of the form 4k+1, either 8 for exponents of the form 4k+3, where  $k \ge 0$ . 12 is not a perfect power. All remaining elements are congruent to 2 (mod 4), and are therefore not a perfect power, either. QED. The Smarandache Binarv Sieve (Item http;//www.gallup.unm.edu/~smarandache/SNAOINT.txt): 1,3,5,9,11,13,17,21,25,27,29,33,35,37,43,49,51,53,57,59,65,67,69, 73,75,77,81,85,89,91,97,101,107,109,113,115,117,121,123,129,131,1 33,137,139,145,149,... (Starting to count on the natural numbers set at any step from 1:

- delete every 2-nd numbers
- delete, from the remaining ones, every 4-th numbers ... and so on: delete, from the remaining ones, every  $2^k$ -th numbers,  $k = 1, 2, 3, \ldots$ )
  Conjectures:
- a) There are an infinity of primes that belong to this sequence;
- b) There are an infinity of numbers of this sequence which are composite.

The second conjecture has been proved true by David W. Wilson: One way to see this is to note that any sequence with positive density over the positive integers contains an infinitude of composites (the density of this sequence is 1/2 \* 3/4 \* 7/8 \* 15/16 \* 31/32 \* ... = 0.28878809508660242127... > 0.)

Another way to see this is to note that this sequence contains all numbers of the form  $(4^k-1)/3$  for  $k \ge 3$ , which are all composite.

Also, in the "Bulletin of Pure and Applied Sciences", Delhi, India, Vol. 17E, No. 1, 1998 (pp. 103-114, 115-116, 117-118, 123-124) four articles present the "Smarandache noneuclidean geometries".

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## BOOK REVIEWS

Logic As Algebra, by Paul Halmos and Steven Givant, The Mathematical Association of America, Washington, D. C., 1998. 152 pp., \$27.00(paper). ISBN 0-88385327-2.

It can be strongly argued that logic is the most ancient of all the mathematical sub-disciplines. When mathematics as we know it was being created so many years ago, it was necessary for the concepts of rigid analytical reasoning to be developed. Of the three earliest areas, geometry was born out of the necessity of accurately measuring land plots and large buildings and number theory was required for sophisticated counting techniques. Logic, the third area, had no "practical" godfather, other than being the foundation for rigorous reasoning in the other two. In the intervening years, so many additional areas of mathematics have been developed, with logic and logical reasoning continuing to be the fundamental building block of them all. Therefore, every mathematician should have some exposure to logic, with the simple history lesson automatically being included. This short, but excellent book fills that niche.

The title accurately sets the theme for the entire book. Algebra is nothing more than a precise notation in combination with a rigorous set of rules of behavior. When logic is approached in that way, it becomes much easier to understand and apply. This is especially necessary in the modern world where computing is so ubiquitous. Many areas of mathematics are incorporated into the computer science major, but none is more widely used than logic. Written at a level that can be comprehended by anyone in either a computer science or mathematics major, it can be used as a textbook in any course targeted at these audiences.

The topics covered are standard although the algebraic approach makes it unique. One simple chapter subheading, 'Language As An Algebra', succinctly describes the theme. Propositional calculus, Boolean algebra, lattices and predicate calculus are the main areas examined. While the treatment is short, it is thorough, providing all necessary details for a sound foundation in the subject. While the word "readable" is sometimes overused in describing books, it can be used here without hesitation.

Sometimes neglected as an area of study in their curricula, logic is an essential part of all mathematics and computer training, whether formal or informal. The authors use a relatively small number of pages to present an extensive amount of knowledge in an easily understandable way. I strongly recommend this book.

### Reviewed by

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In Polya's Footsteps: Miscellaneous Problems and Essays, by Ross Honsberger, The Mathematical Association of America, Washington, D. C., 1997. 328 pp., \$28.95(paper). ISBN 0-88385-326-4.

The greatest scientist of all time was quoted as saying that the reason that he saw further than others was that he stood on the shoulders of giants. As the title of this book suggests, there is another route, namely walking the same path as others. Given our individual differences and how we vary from day to day, even the most beaten of paths can present differing appearances. When walking through a forest, some days you may see the moss, other days the ground cover and then on others we pay particular attention to the leaves. In this collection of problems, Ross

Honsberger proves once again that he is the best at picking the high quality, sturdy building material from the large, stable, yet uninspiring stack of wood.

This is a collection of problems to build on. Many of the them were taken from those proposed and rejected from mathematics competitions, both national and international. Given the quality of these problems, those that were accepted in favor of them must have indeed been gems. It is fortunate that **Crux Mathematicorum**, a journal of the Canadian Mathematical Society, publishes problems of this type so that the rest of us may enjoy them. The range of topics is extensive, with very detailed proofs of all problems. The most striking aspect of many of them is that the approach used in the proof is "non-obvious." Which is the mathematical term for ,"now, how did they ever think of that?" Which is what makes them so charming and emphasizes how exciting mathematics is. There used to be a television game show where contestants competed by claiming that they could name a song in the fewest notes. If there was a similar contest concerning the elegance and directness of proofs, some of those in this book would provide stiff competition.

Classic works of art or music always provide enjoyment, even after many repetitions. High quality, elegant proofs of mathematical problems do the same thing to those willing to experience them. This is one book that will allow you to do that.

Reviewed by

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Computer Analisis of Number Sequences, by Henry Ibstedt, American Research Press, Lupton, Az., 1998. 87 pp., \$9.95 (paper), ISBN 1-879585-59-6.

Playing with numbers is one activity that all mathematicians enjoy. It is considered a pleasurable occupational hazard. Finding "new" properties of numbers is a joy that cannot be accurately described, only experienced. In this book, the author presents and to some extent explores a set of problems in recreational mathematics. Nearly all of the problems originated in the mind of Florentin Smarandache, the creator of innumerable problems in many areas of mathematics. While many are somewhat contrived, they are all fun to read and think through.

For exemple, there are the three sequences of numbers formed by the repeated concatenation of the elements of a set of integers

```
Smarandache Odd Sequence (SOS):
1, 13, 135, 1357, 13579, 1357911, 135791113, ...

Smarandache Even Sequence (SES):
2, 24, 246, 2468, 246810, 24681012, ...

Smarandache Prime Sequence (SPS):
23, 235, 2357, 235711, 23571113, ...

where questions like the following are presented.
```

How many primes are there in the SOS and SPS sequences? How many perfect powers are there in the SES sequence?

Like the large Marsenne primes, the current largest Known prime in either of these sequences is an accurate barometer of the state of current factoring capability. As no less a mathemetician as Pal Erdos has noted, it will probably never be known if there is an infinite number of primes in either the SOS or SPS sequences. However, if someone ever resolves the issue, it will no doubt be headline news in the mathematics community. Any technique powerfull enough to resolve this issue will certainly be one that can be used elsewhere.

It is just an interesting collection of problems in recreational mathematics that can be worked on just for the joy of exploration. That alone makes it well worth reading.

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CRC Concise Encyclopedia of Mathematics, by Eric W. Weisstein, CRC Press, Boca Raton, FL, USA, 1998, 1969 pp., \$79.95 (alk. paper), ISBN 0-8493-9640-9.

The best ever published encyclopedia of mathematics. Also very accessible and well organized, with many cross-references.

From

http://www.amazon.com/exec/obidos/ISBN=0849396409/ericstreasuretroA/,

"The CRC Concise Encyclopedia of Mathematics is a compendium of mathematical definitions, formulas, figures, tabulations, and references. Its informal style makes it accessible to a broad spectrum of readers with a diverse range of mathematical backgrounds and interests. This fascinating, useful book draws connections to other areas of mathematics and science as well as demonstrates its actual implementation providing a highly readable, distinctive text diverging from the all-too-frequent specialized jargon and dry formal exposition.

Through its thousands of explicit examples, formulas, and derivations, The CRC Concise Encyclopedia of Mathematics gives the reader a flavor of the subject without getting lost in minutiae, stimulating his or her thirst for additional information and exploration.

This book serves as handbook, dictionary, and encyclopedia extensively cross-linked and cross-referenced, not only to other related entries, but also to web sites on the Internet. Standard mathematical references, combined with a few popular ones, are also given at the end of most entries, providing a resource for more reading and exploration. In The CRC Concise Encyclopedia of Mathematics, the most useful and interesting aspects of the topic are thoroughly discussed, enhancing technical definitions."

# CONTENTS

ARTICLES:	
Micha Fleuren, Smarandache Factors and Reverse Factors  Henry Ibstedt, Smarandache Continued Fractions	5 39
Constantin Dumitrescu, Carmen Rocsoreanu, Some Connections Between the Smarandache Function and the Fibonnaci Sequence.	50
Y. V. Chebrakov, V. V. Shmagin, Computative Paradoxes in	
Modern Data Analysis	61 81
Factorials I	87
Steven R. Finch,The Average Value of the SmarandacheFunctionSabin Tabirca,Smarandache Magic Squares	95 100
Sabin Tabircă, Tatiana Tabircă, Some Inequalities Concerning	100
Smarandache's Function	105
Perfect Numbers	114
Series	116
Function A Result Obtained Using Smarandache's	119
Function	123
Florian Luca, On A Series Involving S(1)·S(2)·S(n) Sebastián Martín Ruiz, A Congruence with Smarandache's	128
Function	
NOTES:	
Mihaly Bencze, A New Inequality for the Smarandache Function Maohua Le, A Formula for the Smarandache Function	140 142
Maohua Le, On Smarandache Divisor Product	146
Sequence	
Theorem	152 154
<b>Maohua Le</b> , Primes in the Sequences $\{n^n+1\}_{n=1}^{\infty}$ and $\{n^n+1\}_{n=1}^{\infty}$ .	156
Maohua Le, On the Smarandache Prime Additive Complement Sequence	158

	e, An Inequality for the Smarandache Function	
	On Smarandache Simple Functions	
	pacher, Maohua Le, On Smarandache Simple Continued	
Fractions		163
		165
		167
		169
Charles Ashb	bacher, Smarandache Series Converges	171
Maohua Le,	The Primes in the Smarandache Symmetric Sequence	172 174 176
	The Lower Bound for the Smarandache Counter	178
Maohua Le,	On Smarandache Pseudo-Primes of Second Kind	180
MATHEMA	ATICAL HISTORY:	
	It, Summary of the First International Conference	
_	the Type Notions in Number Theory (University of	
	gust 21-24, 1997)	181
Henry Ibsted	lt, Preambul to the First International Conference	
on Smarandad	che Type Notions in Number Theory (University of	
Craiova, Aug	gust 21-24, 1997)	182
Henry Ibsted	lt, The Smarandache Sequence Inventory	183
Henry Ibsted	lt, List of Smarandache Documents	191
I. Bălăcenoi	u, V. Seleacu, History of the Smarandache	
		192
J. Castillo,	Other Smarandache Type Functions	202
M. L. Perez,	Survey on the Research of Smarandache Notions	205
REVIEWS	S:	
	s	207

The "Smarandache Notions" is now online at:
http://www.gallup.unm.edu/~smarandache/
and selected papers are periodically added to our web site. The
authors are asked to send, together with their hard copy
manuscripts, a floppy disk with HTML or ASCHII files to be put in
the Internet as well.

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Mr. Pål Grønås from Norway successfully defended his Master Degree (Hovedfag) thesis in mathematics with the title "Grøbnergenererende mengder for idealer i en spesiell type algebraer -- Smarandache-funksjonen" (Norwegian) in 1998 under the supervision of Professor Dr. Øyvind Solberg.