Vol. 11, No. 1-2-3, Spring 2000 ISSN 1084-2810

SMARANDACHE NOTIONS JOURNAL

$/+_4\backslash$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	0	5	6	7	4	9	10	11	8	13	14	15	12
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	0	1	2	7	4	5	6	11	8	9	10	15	12	13	14
4	4	5	6	7	8	9	10	11	12	13	14	15	0	1	2	3
5	5	6	7	4	9	10	11	8	13	14	15	12	1	2	3	0
6	6	7	4	5	10	11	8	9	14	15	12	13	2	3	0	1
7	7	4	5	6	11	8	9	10	15	12	13	14	3	0	1	2
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	10	11	8	13	14	15	12	1	2	3	0	5	6	7	4
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	8	9	10	15	12	13	14	3	0	1	2	7	4	5	6
12	12	13	14	15	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	15	12	1	2	3	0	5	6	7	4	9	10	11	8
14	14	15	12	13	2	3	0	1	6	7	4	5	10	11	8	9
15	15	12	13	14	3	0	1	2	7	4	5	6	11	8	9	10

Mathematical Association
of the
UNIVERSITY OF CRAIOVA

The table on the first cover represents a Smarandache Group under the operation /$+_4$\ [Talukdar, 19].

This journal is an extension of the "Smarandache Function Journal", ISBN 1053-4792, Vol. 1-6, 1990-1995.

SNJ is yearly published (in the Spring - and sometimes in the Fall) in a 250-350 pages volume, and 800-1000 copies.

SNJ is a referred journal.

SNJ is reviewed, indexed, citated by the following journals: "Zentralblatt Für Mathematik" (Germany), "Referativnyi Zhurnal" and "Matematika" (Academia Nauk, Russia), "Mathematical Reviews" (USA), "Computing Review" (USA), "Libertas Mathematica" (USA), "Octogon" (Romania), "Indian Science Abstracts" (India), "Ulrich's International Periodicals Directory" (USA), "Gale Directory of Publications and Broadcast Media" (USA), "Historia Mathematica" (USA), "Journal of Recreational Mathematics" (USA), "The Mathematical Gazette" (U.K.), "Abstracts of Papers Presented to the American Mathematical Society" (USA), "Personal Computer World" (U.K.), "Mathematical Spectrum" (U.K.), "Bulletin of Pure and Applied Sciences" (India), Institute for Scientific Information (PA, USA), "Library of Congress Subject Headings" (USA).

SNJ is catalogued by the Library of Congress, Washington, D. C., at Call# QA246 .S63.

St.-Petersburg, Russia.

***Vera W. de Spinadel*, Centro de Matematica y Diseno MAyDI, Facultad de Arquitectura, Diseno y Urbanismo, Universidad de Buenos Aires, Argentina.**

***Homer B. Tilton*, Department of Mathematics, Phisics, and Astronomy, Pima Community College, Tucson, Arizona, USA.**

INDICATIONS TO AUTHORS

Authors of papers concerning any of Smarandache type functions, numbers, sequences, integer algorithms, paradoxes, experimental non-euclidean geometries, algebraic structures, neutrosophic probability/statistics/set/logic, quantum physics, etc. are encouraged to submit manuscripts to the Editors:

V. Seleacu & I. Bălăcenoiu
Department of Mathematics
University of Craiova, Romania.

The submitted manuscripts may be in the format of remarks, conjectures, solved/unsolved or open new proposed problems, notes, articles, miscellaneous, etc. They must be original work and camera ready [typewritten/computerized, format: 8.5 x 11 inches (≈ 21,6 x 28 cm)]. They are not returned, hence we advise the authors to keep a copy.

The title of the paper should be writting with capital letters. The author's name has to apply in the middle of the line, near the title. References should be mentioned in the text by a number in square brackets and should be listed alphabetically. Current address followed by e-mail address should apply at the end of the paper, after the references.

The paper should have at the beginning an up to a half-page abstract, followed by the key words.

All manuscripts are subject to anonymous review by two or more independent sources.

Every letter will be answered.

The authors will receive free copies of the journal.

Papers in electronic form are accepted. They can be e-mailed in Microsoft Word 97 (or lower) for Windows 98, WordPerfect 6.0 (or lower) for Windows 95, HTML, or text files.

Books on Smarandache notions can be ordered from:

Bell & Howell Information and Learning
(former University of Microfilm International)
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, Michigan 48106-1346, USA
Tel.: 1-800-521-0600 ext. 4806.

In year 2000 our journal celebrates 10 years of existence.

D - Form of SMARANDACHE GROUPOID

Dwiraj Talukdar
Head Dept . of Mathematics
NALBARI COLLEGE
Nalbari : Assam : India

Abstract :

The set of p different equivalence classes is $Zp = \{ [0], [1], [2], \text{------} [k] \text{-----} [p-1] \}$. For convenience, we have omitted the brackets and written k in place of [k]. Thus

$$Zp = \{ 0, 1, 2, \text{------} k \text{------} p-1 \}$$

The elements of Zp can be written uniquely as m - adic numbers. If $r = (a_{n-1}a_{n-2} \text{-----} a_1a_0)_m$ and $s = (b_{n-1}b_{n-2} \text{------} b_1b_0)_m$ be any two elements of Zp, then $r\Delta s$ is defined as $(|a_{n-1} - b_{n-1}|\,|a_{n-2} - b_{n-2}| \text{------} |a_1 - b_1|\,|a_0 - b_0|)_m$ then (Zp, Δ) is a groupoid known as SMARANDACHE GROUPOID . If we define a binary relation $r \equiv s \Leftrightarrow r \Delta C(r) = s \Delta C(s)$, where C(r) and C(s) are the complements of r and s respectively, then we see that this relation is equivalence relation and partitions Zp into some equivalence classes. The equivalence class $D_{sup(Zp)} = \{r \in Zp : r \Delta C(r) = Sup(Zp)\}$ is defined as D - form. Properties of SMARANDACHE GROUPOID and D - form are discussed here.

Key Words : SMARANDACHE GROUPOID, complement element and D - form.

1. Introduction :

Let m be a positive integer greater than one. Then every positive integer r can be written uniquely in the form $r = a_{n-1}m^{n-1} + a_{n-2}m^{n-2} + \text{----} + a_1m + a_0$ where $n \geq 0$, a_i is an integer, $0 \leq a_i < m$, m is called the base of r, which is denoted by $(a_{n-1}a_{n-2} \text{----} a_1a_0)_m$. The absolute difference of two integers $r = (a_{n-1}a_{n-2} \text{-----} a_1a_0)_m$ and $s = (b_{n-1}b_{n-2} \text{-----} b_1b_0)_m$ denoted by $r \Delta s$ and defined as

$$\begin{aligned} r \Delta s &= (|a_{n-1} - b_{n-1}|\,|a_{n-2} - b_{n-2}| \text{-----} |a_1 - b_1|\,|a_0 - b_0|)_m \\ &= (c_{n-1}c_{n-2} \text{-------} c_1c_0)_m, \quad \text{where} \quad c_i = |a_i - b_i| \quad \text{for} \quad i = 0, 1, 2 \text{-----} n-1. \end{aligned}$$

In this operation, $r \Delta s$ is not necessarily equal to $|r - s|$ i.e. absolute difference of r and s.

If the equivalence classes of Zp are expressed as m - adic numbers, then Zp with binary operation Δ is a groupoid, which contains some non-trivial groups. This groupoid is defined as SMARANDACHE GROUPOID . Some properties of this groupoid are established here.

2. Preliminaries :

We recall the following definitions and properties to introduce SMARANDACHE GROUPOID.

Definition 2.1 (2)

Let p be a fixed integer greater than one. If a and b are integers such that a-b is divisible by p, then a is congruent to b modulo p and indicate this by writting $a \equiv b \pmod p$. This congruence relation is an equivalence relation on the set of all integers.

The set of p different equivalence classes is $\mathbb{Z}p = \{ 0, 1, 2, 3, ------ p-1 \}$

Proposition 2.2 (1)

If $a \equiv b \pmod p$ and $c \equiv d \pmod p$

Then i) $a +_p c = b +_p d$

ii) $a \times_p c = b \times_p d$

Proposition 2.3 (2)

Let m be a positive integer greater than one. Then every integer r can be written uniquely in the form

$$r = a_{n-1}m^{n-1} + a_{n-2}m^{n-2} + ----- + a_1 m + a_0$$

$$= \sum_{i=0}^{n-1} a_i m^i \quad \text{for} \quad i = 0, 1, 2, -----, m-1 ;$$

Where $n \geq 0$, a_i is an integer $0 \leq a_i < m$. Here m is called the base of r, which is denoted by $(a_{n-1}a_{n-2} \cdots \quad \cdots \quad a_1a_0)_m$.

Proposition 2.4

If $r = (a_{n-1}a_{n-2} \cdots \quad \cdots \quad a_1a_0)_m$ and $s = (b_{n-1}b_{n-2} \cdots \quad \cdots \quad b_1b_0)_m$, then

i) $r = s$ if and only if $a_i = b_i$ for $i = 0, 1, 2, -----, n-1$.

ii) $r < s$ if and only if $(a_{n-1}a_{n-2} \cdots \quad \cdots \quad a_1a_0)_m < (b_{n-1}b_{n-2} \cdots \quad \cdots \quad b_1b_0)_m$

iii) $r > s$ if and only if $(a_{n-1}a_{n-2} \cdots \quad \cdots \quad a_1a_0)_m > (b_{n-1}b_{n-2} \cdots \quad \cdots \quad b_1b_0)_m$

3. Smarandache groupoid :

Definition 3.1

Let $r = (a_{n-1}a_{n-2} \cdots \quad a_i \quad \cdots \quad a_1a_0)_m$ and $s = (b_{n-1}b_{n-2} \cdots \quad b_i \quad \cdots \quad b_1b_0)_m$, then the absolute difference denoted by Δ of r and s is defined as

$r \Delta s = (c_{n-1}c_{n-2} --- \quad c_i \quad --- c_1c_0)_m$, where $c_i = |a_i - b_i|$ for $i = 0, 1, 2 ----- n-1$.

Here, $r \Delta s$ is not necessarily equal to $|r - s|$. For example

$5 = (101)_2$ and $6 = (110)_2$ and $5 \Delta 6 = (011)_2 = 3$ but $|5 - 6| = 1$.

In this paper, we shall consider $5 \Delta 6 = 3$, not $5 \Delta 6 = 1$.

Definition 3.2

Let $(\mathbb{Z}p, +p)$ be a commulative group of order $p = m^n$. If the elements of $\mathbb{Z}p$ are expressed as m - adic numbers as shown below :

$$
\begin{aligned}
0 &= (00 \quad \dots \quad \dots \quad 00)_m \\
1 &= (00 \quad \dots \quad \dots \quad 01)_m \\
2 &= (00 \quad \dots \quad \dots \quad 02)_m \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \\
m-1 &= (00 \quad \dots \quad \dots \quad 0\ m\text{-}1)_m \\
m &= (00 \quad \dots \quad \dots \quad 1\ 0)_m \\
m+1 &= (00 \quad \dots \quad \dots \quad 1\ 1)_m \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \\
m^2-1 &= (00 \quad \dots \quad \dots \quad m\text{-}1\ m\text{-}1)_m \\
m^2 &= (00 \quad \dots \quad \dots \quad 1\ 0\ 0)_m \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \\
m^n-1 &= (m\text{-}1\ m\text{-}1 \quad \dots \quad \dots \quad m\text{-}1\ m\text{-}1)_m
\end{aligned}
$$

The set $\mathbb{Z}p$ is closed under binary operation Δ. Thus $(\mathbb{Z}p, \Delta)$ is a groupoid. The elements $(00 \quad \dots \quad \dots \quad 00)_m$ and $(m\text{-}1\ m\text{-}1 \quad \dots \quad \dots \quad m\text{-}1\ m\text{-}1)_m$ are called infimum and supremum of $\mathbb{Z}p$.

The set H_1 of the elements noted below :

$$
\begin{aligned}
0 &= (00 \quad \dots \quad \dots \quad 00)_m \\
1 &= (00 \quad \dots \quad \dots \quad 01)_m \\
m &= (00 \quad \dots \quad \dots \quad 1\ 0)_m \\
m+1 &= (00 \quad \dots \quad \dots \quad 1\ 1)_m \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots
\end{aligned}
$$

$$\frac{m^{n-1}-m}{m-1} = (0\ 1 \quad \dots \quad \dots \quad 1\ 0)_m = \alpha \text{ (say)}$$

$$\frac{m^{n-1}-1}{m-1} = (0\ 1 \quad \dots \quad \dots \quad 1\ 1)_m = \beta \text{ (say)}$$

$$\frac{m^{n}-m}{m-1} = (1\ 1 \quad \dots \quad \dots \quad 1\ 0)_m = \gamma \text{ (say)}$$

$$\frac{m^{n}-1}{m-1} = (1\ 1 \quad \dots \quad \dots \quad 1\ 1)_m = \delta \text{ (say)}$$

is a proper subset of $\mathbb{Z}p$.

(H_1, Δ) is a group of order 2^n and its group table is as follows :

Δ	0	1	m	m+1	...	...	α	β	γ	δ
0	0	1	m	m+1	...	...	α	β	γ	δ
1	1	0	m+1	m	...	...	β	α	δ	γ
m	m	m+1	0	1	...	...	γ	δ	α	β
m+1	m+1	m	1	0	...	...	δ	γ	β	α
....	...	...	...	...	...	...	...	...	...	...
....	...	...	...	...	...	...	...	...	...	...
α	α	β	γ	δ	...	...	0	1	m	m+1
β	β	α	δ	γ	...	...	1	0	m+1	m
γ	γ	δ	α	β	...	...	m	m+1	0	1
δ	δ	γ	β	α	...	...	m+1	m	1	0

Table - 1

Similarly the proper sub-sets

$H_2 = \{ 0, 2, 2m, 2(m+1) \quad \dots \quad \dots \quad 2\alpha, 2\beta, 2\gamma, 2\delta \}$

$H_3 = \{ 0, 3, 3m, 3(m+1) \quad \dots \quad \dots \quad 3\alpha, 3\beta, 3\gamma, 3\delta \}$

... ...

$H_{m-1} = \{ 0, m-1, m(m-1), m^2-1 \quad \dots \quad \dots \quad (m-1)\alpha, (m-1)\beta, (m-1)\gamma, (m-1)\delta \}$

are groups of order 2^n under the operation absolute difference. So the groupoid $(\mathbf{Z}p, \Delta)$ contains mainly the groups (H_1, Δ), (H_2, Δ), (H_3, Δ) .... (H_{m-1}, Δ) and this groupoid is defined as SMARANDACHE GROUPOID . Here we use S.Gd. in place of SMARANDACHE GROUPOID.

Remarks 3.2

i) Let $(\mathbf{Z}p, +p)$ be a commutative group of order p, where $m^{n-1} < p < m^n$, then $(\mathbf{Z}p, \Delta)$ is not groupoid.

For example $(\mathbf{Z}_5, +5)$ is a commutative group of order 5, where $2^2 < p < 2^3$.

Here $\mathbf{Z}_5 = \{ 0, 1, 2, 3, 4 \}$ and

$0 = (0\ 0\ 0)_2$	$4 = (1\ 0\ 0)_2$
$1 = (0\ 0\ 1)_2$	$5 = (1\ 0\ 1)_2$
$2 = (0\ 1\ 0)_2$	$6 = (1\ 1\ 0)_2$
$3 = (0\ 1\ 1)_2$	$7 = (1\ 1\ 1)_2$

A composition table of $\mathbb{Z}_5$ is given below :

Δ	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	2	5
2	2	3	0	1	6
3	3	2	1	0	7
4	4	5	6	7	0

Table - 2

Hence $\mathbb{Z}_5$ is not closed under the operation Δ, i.e. $(\mathbb{Z}_5, \Delta)$ is not a groupoid.

ii) S. Gd. is not necessarily associative.

Let $1 = (\,00 \;\ldots\; \ldots\; 01)_m$

$2 = (\,00 \;\ldots\; \ldots\; 02)_m$ and

$3 = (\,00 \;\ldots\; \ldots\; 03)_m$ be three elements of $(\mathbb{Z}p, \Delta)$, then

$(1 \,\Delta\, 2) \,\Delta\, 3 = 2$ and

$1 \,\Delta\, (2 \,\Delta\, 3) = 0$

i.e. $(1 \,\Delta\, 2) \,\Delta\, 3 \neq 1 \,\Delta\, (2 \,\Delta\, 3)$.

iii) S. Gd. is commutative.

iv) S. Gd. has identity element $0 = (\,00 \;\ldots\; \ldots\; 0)_m$

v) Each element of S. Gd. is self inverse i.e. $\forall\, p \in \mathbb{Z}p, \quad p \,\Delta\, p = 0$.

Proposition 3.3

If (H, Δ) and (K, Δ) be two groups of order 2^n contained in S. Gd. $(\mathbb{Z}p, \Delta)$, then H is isomorphic to K.

Proof is obvious.

4. Complement element in S. Gd. $(\mathbb{Z}p, \Delta)$.

Definition 4.1

Let $(\mathbb{Z}p, \Delta)$ be a S. Gd., then the complement of any element $p \in \mathbb{Z}p$ is equal to $p \,\Delta\, \mathrm{Sup}(\mathbb{Z}p) = p \,\Delta\, m^n\text{-}1$ i.e. $C(p) = m^n\text{-}1 \,\Delta\, p$. This function is known as complement function and it satisfies the following properties.

i) $C(0) = m^n - 1$

ii) $C(m^n - 1) = 0$

ii) $C(C(p)) = p \quad \forall \quad p \in \mathbb{Z}p$

iv) If $p \leq q$ then $C(p) \geq C(q)$.

Definition 4.2

An element p of a S. Gd. $\mathbb{Z}p$ is said to be self complement if $p \Delta \sup(\mathbb{Z}p) = p$ i.e. $C(p) = p$.

If m is an odd integer greater than one, then $\frac{m^n - 1}{2}$ is the self complement of $(\mathbb{Z}p, \Delta)$.

If m is an even integer, then there exists no self complement in $(\mathbb{Z}p, \Delta)$.

Remarks 4.3

i) The complement of an element belonging to a S. Gd. is unique.

ii) The S. Gd. is closed under complement operation.

5. A binary relation in S. Gd.

Definition 5.1

Let $(\mathbb{Z}p, \Delta)$ be a S. Gd. An element p of $\mathbb{Z}p$ is said to be related to $q \in \mathbb{Z}p$ iff $p \Delta C(p) = q \Delta C(q)$ and written as $p \equiv q \Leftrightarrow p \Delta C(p) = q \Delta C(q)$.

Proposition 5.2

For the elements p and q of S. Gd. $(\mathbb{Z}p, \Delta)$, $p \equiv q \Leftrightarrow C(p) \equiv C(q)$.

Proof : By definition

$$\begin{aligned} p \equiv q &\Leftrightarrow p \Delta C(p) = q \Delta C(q). \\ &\Leftrightarrow C(p) \Delta p = C(q) \Delta q \\ &\Leftrightarrow C(p) \Delta C(C(p)) = C(q) \Delta C(C(q)) \\ &\Leftrightarrow C(p) \equiv C(q) \end{aligned}$$

Proposition 5.3

Let $(\mathbb{Z}p, \Delta)$ be a S. Gd., then a binary relation $p \equiv q \Leftrightarrow p \Delta C(p) = q \Delta C(q)$ for $p, q \in \mathbb{Z}p$, is an equivalence relation.

Proof : Let $(\mathbb{Z}p, \Delta)$ be a S. Gd. and for any two elements p and q of $\mathbb{Z}p$, let us define a binary relation $p \equiv q \Leftrightarrow p \Delta C(p) = q \Delta C(q)$.

This relation is

i) reflexive for if p is an arbitrary element of $\mathbb{Z}p$, we get $p \Delta C(p) = p \Delta C(p)$ for all $p \in \mathbb{Z}p$. Hence $p \equiv p \Leftrightarrow p \Delta C(p) = p \Delta C(p) \quad \forall \quad p \in \mathbb{Z}p$.

ii) Symmetric, for if p and q are any elements of $\mathbb{Z}p$ such that

$p \equiv q$, then
$$\begin{aligned} p \equiv q &\Leftrightarrow p \Delta C(p) = q \Delta C(q) \\ &\Leftrightarrow q \Delta C(q) = p \Delta C(p) \\ &\Leftrightarrow q \equiv p \end{aligned}$$

iii) transitive, for p, q, r are any three elements of $\mathbb{Z}p$ such that

$p \cong q$ and $q \cong r$, then

$p \cong q \Leftrightarrow p \Delta C(p) = q \Delta C(q)$ and

$q \cong r \Leftrightarrow q \Delta C(q) = r \Delta C(r)$.

Thus $p \Delta C(p) = r \Delta C(r) \Leftrightarrow p \cong r$

Hence $p \cong q$ and $q \cong r$ implies $p \cong r$

6. D - Form of S. Gd.

Let $(\mathbb{Z}p, \Delta)$ be a S. Gd. of order m^n. Then the equivalence relation referred in the proposition 5.3 partitions $\mathbb{Z}p$ into mutually disjoint classes.

Definition 6.1

If r be any element of S. Gd. $(\mathbb{Z}p, \Delta)$ such that $r \Delta C(r) = x$, then the equivalence class generated by x is denoted by Dx and defined by

$$Dx = \{ r \in \mathbb{Z}p : r \Delta C(r) = x \}$$

The equivalence class generated by sup($\mathbb{Z}$p) and defined by

$D_{\sup(\mathbb{Z}p)} = \{ r \in \mathbb{Z}p : r \Delta C(r) = \sup(\mathbb{Z}p) \}$ is called the D - form of $(\mathbb{Z}p, \Delta)$.

Example 6.2

Let $(\mathbb{Z}_9, +9)$ be a commutative group, then $\mathbb{Z}_9 = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8 \}$. If the elements of $\mathbb{Z}_9$ are written as 3-adic numbers, then

$\mathbb{Z}_9 = \{(00)_3, (01)_3, (02)_3, (10)_3, (11)_3, (12)_3, (20)_3, (21)_3, (22)_3\}$ and

$(\mathbb{Z}_9, \Delta)$ is a S. Gd. of order $3^2 = 9$. Its composition table is as follows :

Δ	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	0	1	4	3	4	7	6	7
2	2	1	0	5	4	3	8	7	6
3	3	4	5	0	1	2	3	4	5
4	4	3	4	1	0	1	4	3	4
5	5	4	3	2	1	0	5	4	3
6	6	7	8	3	4	5	0	1	2
7	7	6	7	4	3	4	1	0	1
8	8	7	6	5	4	3	2	1	0

Table - 3

Here $0 \Delta C(0) = 0 \Delta 8 = 8$

$1 \Delta C(1) = 1 \Delta 7 = 6$

$2 \Delta C(2) = 2 \Delta 6 = 8$

$3 \Delta C(3) = 3 \Delta 5 = 2$

$4 \Delta C(4) = 4 \Delta 4 = 0$

$5 \Delta C(5) = 5 \Delta 3 = 2$

$6 \Delta C(6) = 6 \Delta 2 = 8$

$7 \Delta C(7) = 7 \Delta 1 = 6$

$8 \Delta C(8) = 8 \Delta 0 = 8$

Hence $D_8 = \{ 0, 2, 6, 8 \} = \{(00)_3, (02)_3, (20)_3, (22)_3 \}$

$D_6 = \{ 1, 7 \}$

$D_2 = \{ 3, 5 \}$

$D_0 = \{ 4 \}$

The self complement element of $(\mathbb{Z}_9, \Delta)$ is 4 and D- form of this S. Gd. is $\{0, 2, 6, 8\} = D_8$

Here $\mathbb{Z}_9 = D_0 \cup D_2 \cup D_6 \cup D_8$.

Proposition 6.3

Any two equivalence classes in a S. Gd. $(\mathbb{Z}p, \Delta)$ are either disjoint or identical.

Proof is obvious.

Proposition 6.4

Every S. Gd. $(\mathbb{Z}p, \Delta)$ is equal to the union of its equivalence classes.

Proof is obvious.

Proposition 6.5

Every D- form of a S. Gd. $(\mathbb{Z}p, \Delta)$ is a commutative group.

Proof : Let $(\mathbb{Z}p, \Delta)$ be a S. Gd. of order $P = m^n$. The elements of D- form of this groupoid are as follows.

$$
\begin{array}{lllll}
0 & = (00 & \dots & \dots & 00)_m \\
m-1 & = (00 & \dots & \dots & 0\ m\text{-}1)_m \\
m^2-m & = (00 & \dots & \dots & m\text{-}1\ 0)_m \\
m^2-1 & = (00 & \dots & \dots & m\text{-}1\ m\text{-}1)_m \\
\dots & \dots & \dots & \dots & \\
\dots & \dots & \dots & \dots & \\
m^{n-1}-m & = (0\ m\text{-}1 & \dots & \dots & m\text{-}1\ 0)_m \\
m^{n-1}-1 & = (0\ m\text{-}1 & \dots & \dots & m\text{-}1\ m\text{-}1)_m \\
m^n-m & = (m\text{-}1\ m\text{-}1 & \dots & \dots & m\text{-}1\ 0)_m \\
m^n-1 & = (m\text{-}1\ m\text{-}1 & \dots & \dots & m\text{-}1\ m\text{-}1)_m
\end{array}
$$

$\therefore D_{m^n-1} = \{ 0, m-1, m^2-m, m^2-1, \text{-------}, m^{n-1}-m, m^{n-1}-1, m^n-m, m^n-1 \}$

Here (D_{m^n-1}, Δ) is a commutative group and its table is given below :

Δ	0	m-1	m^2-m	m^2-1	...	m^{n-1}-m	m^{n-1}-1	m^n-m	m^n-1
0	0	m-1	m^2-m	m^2-1	...	m^{n-1}-m	m^{n-1}-1	m^n-m	m^n-1
m-1	m-1	0	m^2-1	m^2-m	...	m^{n-1}-1	m^{n-1}-m	m^n-1	m^n-m
m^2-m	m^2-m	m^2-1	0	m-1	...	m^n-m	m^n-1	m^{n-1}-m	m^{n-1}-1
m^2-1	m^2-1	m^2-m	m-1	0	...	m^n-1	m^n-m	m^{n-1}-1	m^{n-1}-m
- - -		- - -			...	- - -		- - -	
m^{n-1}-m	m^{n-1}-m	m^{n-1}-1	m^n-m	m^n-1	...	0	m-1	m^2-m	m^2-1
m^{n-1}-1	m^{n-1}-1	m^{n-1}-m	m^n-1	m^n-m	...	m-1	0	m^2-1	m^2-m
m^n-m	m^n-m	m^n-1	m^{n-1}-m	m^{n-1}-1	...	m^2-m	m^2-1	0	m-1
m^n-1	m^n-1	m^n-m	m^{n-1}-1	m^{n-1}-m	...	m^2-1	m^2-m	m-1	0

Table - 4

Remarks 6.6

Let $(\mathbf{Z}p, \Delta)$ be a S. Gd. of order m^n.

The equivalence relation $p \equiv q \Leftrightarrow p \,\Delta\, C(p) = q \,\Delta\, C(q)$ partitions $\mathbf{Z}p$ into some equivalence classes.

i) If m is odd integer, then the number of elements belonging to the equivalence classes are not equal. In the example 6.2, the number of elements belonging to the equivalence classes D_0, D_2, D_6, D_8 are not equal due to m = 3.

ii) If m is even integer, then the number of elements belonging to the equivalence classes are equal.

For example, $\mathbf{Z}_{16} = \{ 0, 1, 2, \text{-------}, 15 \}$ be a commutative group. If the elements of $\mathbf{Z}_{16}$ are expressed as 4- adic numbers, then $(\mathbf{Z}_{16}, \Delta)$ is a S. Gd. The composition table of $(\mathbf{Z}_{16}, \Delta)$ is given below :

Δ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	1	2	5	4	5	6	9	8	9	10	13	12	13	14
2	2	1	0	1	6	5	4	5	10	9	8	9	14	13	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	4	5	6	7	8	9	10	11
5	5	4	5	6	1	0	1	2	5	4	5	6	9	8	9	10
6	6	5	4	5	2	1	0	1	10	5	6	7	8	9	10	11
7	7	6	5	4	3	2	1	0	7	6	5	4	11	10	9	8
8	8	9	10	11	4	5	6	7	0	1	2	3	4	5	6	7
9	9	8	9	10	5	4	5	6	1	0	1	2	5	4	5	6
10	10	9	8	9	6	5	4	5	2	1	0	1	6	5	4	5
11	11	10	9	8	7	6	5	4	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	13	14	9	8	9	10	5	4	5	6	1	0	1	2
14	14	13	12	13	10	9	8	9	6	5	4	5	2	1	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Table - 5

Here $0 \Delta C(0) = 15 = 15 \Delta C(15)$

$1 \Delta C(1) = 13 = 14 \Delta C(14)$

$2 \Delta C(2) = 13 = 13 \Delta C(13)$

$3 \Delta C(3) = 15 = 12 \Delta C(12)$

$4 \Delta C(4) = 7 = 11 \Delta C(11)$

$5 \Delta C(5) = 5 = 10 \Delta C(10)$

$6 \Delta C(6) = 5 = 9 \Delta C(9)$

$7 \Delta C(7) = 7 = 8 \Delta C(8)$

Hence $D_{15} = \{ 0, 3, 12, 15 \}$, $D_{13} = \{ 1, 2, 13, 14 \}$

$D_7 = \{ 4, 8, 7, 11 \}$, $D_5 = \{ 5, 6, 9, 10 \}$

The number of elements of the equivalence classes are equal due to m = 4, which is even integer.

Acknowledgement :

I wish to express my gratitude to Prof. Sashi Sarma, Nalbari and Sjt. Panchanan Sarma, Bidyapur, Nalbari for their encouragement in preparing this paper.

References :

1. David M. Burton - Elementary number theory
   2nd edition, University book stall
   New Delhi (1994).

2. Mc Coy, N.H.- Introduction to Modern Algebra
   Boston Allyu and Bacon INC (1965)

3. Talukdar, D & Das N.R.- Measuring associativity in a groupoid of natural numbers
   The Mathematical Gazette
   Vol. 80. No.- 488 (1996), 401 - 404

4. Talukdar, D - Some Aspects of inexact groupoids
   J. Assam Science Society
   37(2) (1996), 83 - 91

5. Talukdar, D - A Klein 2^n - group, a generalization of Klein 4 group
   GUMA Bulletin
   Vol. 1 (1994), 69 - 79

6. Hall, M - The theory of groups
   Macmillan Co. 1959.

------ ***** ------

The notions of the SMARANDACHE GROUP and the SMARANDACHE BOOLEAN RING

D. Talukdar.
Deptt. of Mathematics
Nalbari College, Nalbari
Assam, India

The notions of the Snmarandache group and the Smarandache Boolean ring are introduced here with the help of group action and ring action i.e. module respectively. The centre of the Smarandache groupoid is determined. These are very important for the study of Algebraic structures.

1. The centre of the Smarandache groupoid :

Definition 1.1

An element a of the smarandache groupoid ($\mathbb{Z}p$, Δ) is said to be conjugate to b if there exists r in $\mathbb{Z}p$ such that $a = r \Delta b \Delta r$.

Definition 1.2

An element a of the smarandache groupoid ($\mathbb{Z}p$, Δ) is called a self conjugate element of $\mathbb{Z}p$ if $a = r \Delta a \Delta r$ for all $r \in \mathbb{Z}p$.

Definition 1.3

The set $\mathbb{Z}p^*$ of all self conjugate elements of ($\mathbb{Z}p$, Δ) is called the centre of $\mathbb{Z}p$ i.e.

$$\mathbb{Z}p^* = \{ a \in \mathbb{Z}p : a = r \Delta a \Delta r \ \forall\, r \in \mathbb{Z}p \}.$$

Definition 1.4

Let $(\mathbb{Z}_9, +_9)$ be a commutative group, then $\mathbb{Z}_9=\{ 0, 1, 2, 3, 4, 5, 6, 7, 8 \}$. If the elements of $\mathbb{Z}_9$ are written as 3-adic numbers, then

$$\mathbb{Z}_9 = \{ (00)_3, (01)_3, (02)_3, (10)_3, (11)_3, (12)_3, (20)_3, (21)_3, (22)_3 \}$$

and ($\mathbb{Z}_9$, Δ) is a smarandache groupoid of order 9. Conjugacy relations among the elements of $\mathbb{Z}_9$ are determined as follows :

$0 \Delta 0 \Delta 0 = 0$	$0 \Delta 1 \Delta 0 = 1$	$0 \Delta 2 \Delta 0 = 2$	$0 \Delta 3 \Delta 0 = 3$	$0 \Delta 4 \Delta 0 = 4$
$1 \Delta 0 \Delta 1 = 0$	$1 \Delta 1 \Delta 1 = 1$	$1 \Delta 2 \Delta 1 = 0$	$1 \Delta 3 \Delta 1 = 3$	$1 \Delta 4 \Delta 1 = 4$
$2 \Delta 0 \Delta 2 = 0$	$2 \Delta 1 \Delta 2 = 1$	$2 \Delta 2 \Delta 2 = 2$	$2 \Delta 3 \Delta 2 = 3$	$2 \Delta 4 \Delta 2 = 4$
$3 \Delta 0 \Delta 3 = 0$	$3 \Delta 1 \Delta 3 = 1$	$3 \Delta 2 \Delta 3 = 2$	$3 \Delta 3 \Delta 3 = 3$	$3 \Delta 4 \Delta 3 = 4$
$4 \Delta 0 \Delta 4 = 0$	$4 \Delta 1 \Delta 4 = 1$	$4 \Delta 2 \Delta 4 = 0$	$4 \Delta 3 \Delta 4 = 3$	$4 \Delta 4 \Delta 4 = 4$
$5 \Delta 0 \Delta 5 = 0$	$5 \Delta 1 \Delta 5 = 1$	$5 \Delta 2 \Delta 5 = 2$	$5 \Delta 3 \Delta 5 = 3$	$5 \Delta 4 \Delta 5 = 4$
$6 \Delta 0 \Delta 6 = 0$	$6 \Delta 1 \Delta 6 = 1$	$6 \Delta 2 \Delta 6 = 2$	$6 \Delta 3 \Delta 6 = 3$	$6 \Delta 4 \Delta 6 = 4$
$7 \Delta 0 \Delta 7 = 0$	$7 \Delta 1 \Delta 7 = 1$	$7 \Delta 2 \Delta 7 = 0$	$7 \Delta 3 \Delta 7 = 3$	$7 \Delta 4 \Delta 7 = 4$
$8 \Delta 0 \Delta 8 = 0$	$8 \Delta 1 \Delta 8 = 1$	$8 \Delta 2 \Delta 8 = 2$	$8 \Delta 3 \Delta 8 = 3$	$8 \Delta 4 \Delta 8 = 4$
$0 \Delta 5 \Delta 0 = 5$	$0 \Delta 6 \Delta 0 = 6$	$0 \Delta 7 \Delta 0 = 7$	$0 \Delta 8 \Delta 0 = 8$	
$1 \Delta 5 \Delta 1 = 3$	$1 \Delta 6 \Delta 1 = 6$	$1 \Delta 7 \Delta 1 = 7$	$1 \Delta 8 \Delta 1 = 6$	
$2 \Delta 5 \Delta 2 = 5$	$2 \Delta 6 \Delta 2 = 6$	$2 \Delta 7 \Delta 2 = 7$	$2 \Delta 8 \Delta 2 = 8$	
$3 \Delta 5 \Delta 3 = 5$	$3 \Delta 6 \Delta 3 = 0$	$3 \Delta 7 \Delta 3 = 1$	$3 \Delta 8 \Delta 3 = 2$	
$4 \Delta 5 \Delta 4 = 3$	$4 \Delta 6 \Delta 4 = 0$	$4 \Delta 7 \Delta 4 = 1$	$4 \Delta 8 \Delta 4 = 0$	
$5 \Delta 5 \Delta 5 = 5$	$5 \Delta 6 \Delta 5 = 0$	$5 \Delta 7 \Delta 5 = 1$	$5 \Delta 8 \Delta 5 = 2$	
$6 \Delta 5 \Delta 6 = 5$	$6 \Delta 6 \Delta 6 = 6$	$6 \Delta 7 \Delta 6 = 7$	$6 \Delta 8 \Delta 6 = 8$	
$7 \Delta 5 \Delta 7 = 3$	$7 \Delta 6 \Delta 7 = 6$	$7 \Delta 7 \Delta 7 = 7$	$7 \Delta 8 \Delta 7 = 6$	
$8 \Delta 5 \Delta 8 = 5$	$8 \Delta 6 \Delta 8 = 6$	$8 \Delta 7 \Delta 8 = 7$	$8 \Delta 8 \Delta 8 = 8$	

Table - 1

Here $\mathbf{Z}_9^* = \{0, 1, 3, 4\}$, the set of all self conjugate elements of $\mathbf{Z}_9$ is called the centre of $(\mathbf{Z}_9, \Delta)$. Again $(\mathbf{Z}_9^*, \Delta)$ is an abelian group.

D-form of the Smarandache groupoid $(\mathbf{Z}_9, \Delta)$ is given by $D_8 = \{0, 2, 6, 8\}$. Again (D_8, Δ) is an abelian group. The group table (2) and group table (3) are given below.

Δ	0	1	3	4
0	0	1	3	4
1	1	0	4	3
3	3	4	0	1
4	4	3	1	0

Table - 2

Δ	0	2	6	8
0	0	2	6	8
2	2	0	8	6
6	6	8	0	2
8	8	6	2	0

Table - 3

From table (2) and table (3); it is clear that $(\mathbf{Z}_9^*, \Delta)$ and (D_8, Δ) are isomorphic to each other.

Definition 1.3

The groups $(\mathbb{Z}p^*, \Delta)$ and $(D_{\sup(\mathbb{Z}p)}, \Delta)$ of the Smarandache groupoid $(\mathbb{Z}p, \Delta)$ are isomorphic to each other.

Proof is obvious.

2. The Smarandache group :

To introduce the Smarandache group, we have to explain group action on a set. Let A be a group and B a set. An action of A on B is a map, $B \times A \to B$ written $(b, p) \to b \Delta P$ such that

i) for every $p, q \in A$ and $b \in B$, we have

$((b, p), q) = ((b \Delta p) \Delta q) = (b \Delta p) \Delta q = b \Delta (p \Delta q)$

and ii) for every $b \in B$, we have $(b, 0) = b \Delta 0 = b$

where 0 denotes the identity element of the group A.

If a group A has an action on B, we say that B is a A-set or A-space. Here in this paper we shall use B(A) in place of "B is a A - space".

Note :

If B is a proper subgroup of A, then we get a map $A \times B \to A$ defined by $(a, b) \to a \Delta b \in A$. This is a group action of B on A. Then we say that A is a B - set or B - space i.e. A(B) is a B - space. In this paper, by proper subgroup, we mean a group contained in A, different from the trivial groups.

Definition 2.1

The smarandache group is defined to be a group A such that A(B) is a B - space, where B is a proper subgroup of A

Examples 2.2

i) The D - form of $(\mathbb{Z}p, \Delta)$ defined by

$D_{\sup(\mathbb{Z}p)} = \{ r \in \mathbb{Z}p : r \Delta C(r) = \mathrm{Sup}(\mathbb{Z}p)\} = A$ is a Smarandache group. If B is a proper subgroup of A, then the action of B on A is the map, $A \times B \to A$ defined by $(a, q) = a \Delta q$ for all $a \in A$ and $q \in B$. This action is a B - action i.e. A(B) is a B - space.

ii) The centre of $(\mathbb{Z}p, \Delta)$ defined by

$\mathbb{Z}p^* = \{ a \in \mathbb{Z}p : a = r \Delta a \Delta r \quad \forall r \in \mathbb{Z}p\} = A$ is a Smarandache group. If B be a proper subgroup of A, then the action of B on A is the map, $A \times B \to A$ defined by $(a, p) = a \Delta p$ for all $a \in A$ and for all $p \in B$. This action is a B - space i.e. A(B) is a B - space.

iii) The Addition modulo m of two integers $r = (a_{n-1}a_{n-2}$ ------------------ $a_1a_0)_m$ and $s = (b_{n-1}b_{n-2}$ --------- $b_1b_0)_m$ denoted by $/+_m\backslash$ and defined as

$$\begin{aligned} r /+_m\backslash s &= (a_{n-1}a_{n-2} \;\cdots\;\cdots\; a_1a_0)_m /+_m\backslash (b_{n-1}b_{n-2} \;\cdots\;\cdots\; b_1b_0)_m \\ &= (a_{n-1}/+_m\backslash b_{n-1}\; a_{n-2}/+_m\backslash b_{n-2} \;\cdots\;\cdots\; a_1/+_m\backslash b_1\; a_0/+_m\backslash b_0)_m \\ &= (c_{n-1}\; c_{n-2} \;\cdots\;\cdots\; c_1c_0)_m, \quad \text{where } c_i = a_i/+_m\backslash b_i \quad \text{for } i = 0, 1, 2, \ldots n-1. \end{aligned}$$

The group ($\mathbf{Z}p$, $/+_{m}\backslash$) is a Smarandache group. The group ($\mathbf{Z}p$, $/+_{p}\backslash$) contains a proper subgroup (H = { 0, 1, 2, 3, ... p-1}, $/+_{m}\backslash$)

Then the action of H on $\mathbf{Z}p$ is the map $\mathbf{Z}p \times H \rightarrow \mathbf{Z}p$ defined by (a, r) = $a/+_{m}\backslash r$ for all $a \in \mathbf{Z}p$ and for all $r \in H$. This action is a H - space i.e. $\mathbf{Z}p(H)$ is a H - space.

iv) The set $\mathbf{Z}_{16}$ = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 } can be written as

$\mathbf{Z}_{16} = \{(00)_4, (01)_4, (02)_4, (03)_4, (10)_4, (11)_4, (12)_4, (13)_4, (20)_4, (21)_4, (22)_4, (23)_4, (30)_4, (31)_4, (32)_4, (33)_4\}$

is a smarandache group under the operation $/+_{4}\backslash$ and its group table is as follows:

$/+_{4}\backslash$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	0	5	6	7	4	9	10	11	8	13	14	15	12
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	0	1	2	7	4	5	6	11	8	9	10	15	12	13	14
4	4	5	6	7	8	9	10	11	12	13	14	15	0	1	2	3
5	5	6	7	4	9	10	11	8	13	14	15	12	1	2	3	0
6	6	7	4	5	10	11	8	9	14	15	12	13	2	3	0	1
7	7	4	5	6	11	8	9	10	15	12	13	14	3	0	1	2
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	10	11	8	13	14	15	12	1	2	3	0	5	6	7	4
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	8	9	10	15	12	13	14	3	0	1	2	7	4	5	6
12	12	13	14	15	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	15	12	1	2	3	0	5	6	7	4	9	10	11	8
14	14	15	12	13	2	3	0	1	6	7	4	5	10	11	8	9
15	15	12	13	14	3	0	1	2	7	4	5	6	11	8	9	10

Table - 4

From above group table, it is clear that (H = { 0, 1, 2, 3, 8, 9, 10, 11 }, $/+_{4}\backslash$) is a subgroup of ($\mathbf{Z}_{16}$, $/+_{4}\backslash$). Then the action of H on $\mathbf{Z}_{16}$ is the map $\mathbf{Z}_{16} \times H \rightarrow \mathbf{Z}_{16}$ defined by (a, r) = $a/+_{4}\backslash r$ for all $a \in \mathbf{Z}_{16}$ and for all $r \in H$. This action is a H - space i.e. $\mathbf{Z}_{16}(H)$ is a H - space.

Here (K = { 0, 1, 2, 3 }, $/+_{4}\backslash$) be a subgroup of (H, $/+_{4}\backslash$). Then the action of K on H is the map, $H \times K \rightarrow H$ defined by (b, p) = $b/+_{4}\backslash p$ for all $b \in H$ and for all $p \in K$. This action is a K - space i.e. H(K) is a K - space. Hence H is a Smarandache group contained in the Smarandache group $\mathbf{Z}_{16}$. So it is called the Smarandache sub-group.

3. The Smarandache sub-group :

Definition 3.1

The smarandache sub-group is defined to be a smarandache group B which is a proper subset of smarandache group A (with respect to the same induced operation).

4. Smarandache quotient group :

Let (H, Δ) be smarandache subgroup of the Smarandache group (B, Δ), then the quotient group $^{B}/_{H} = V$ is defined as smarandache quotient group such that V(K) is a K - space, where K is a proper subgroup of V i.e. the group action of K on V is a map $V \times K \rightarrow V$, defined by

$((H \Delta a), H \Delta p) = (H \Delta a) \Delta (H \Delta p)$ for all $H \Delta a \in V$ and $H \Delta p \in K$

5. Smarandache Boolean ring :

Definition 5.1

The intersection of two integers $r = (a_{n-1}a_{n-2} \quad \ldots \quad \ldots \quad a_1a_0)_m$ and $s = (b_{n-1}b_{n-2} \quad \ldots \quad \ldots \quad b_1b_0)_m$ denoted by $r \cap s$ and defined as

$$r \cap s = (a_{n-1} \cap b_{n-1}\, a_{n-2} \cap b_{n-2} \quad \ldots \quad \ldots \quad a_1 \cap b_1\, a_0 \cap b_0)_m$$
$$= (c_{n-1}c_{n-2} \quad \ldots \quad \ldots \quad c_1c_0)_m$$

where $c_i = a_i \cap b_i = \min (a_i, b_i)$ for $i = 0, 1, 2, \quad \ldots \quad \ldots \quad , n-1$

If the equivalence classes of are expressed as m - adic numbers, then with binary operation $\cap$ is a groupoid, which contains some non trivial groups. This groupoid is smarandache groupoid. Here $(\mathbf{Z}p^*, \Delta, \cap)$ and $(\mathbf{Z}_{\sup(\mathbf{Z}p)}, \Delta, \cap)$ are Boolean ring.

Definition 5.2

The smarandache Boolean ring is defined to be a Boolean ring A such that the Abelian group (A, Δ) has both left and right B - module, where B is a non trivial sub-ring of A.

From above, we mean an Abelian group (A, Δ) together with a map, $B \times A \rightarrow A$, written $(b, p) = b \cap p \in A$ such that for every $b, c \in B$ and $p, q \in A$, we have

i) $b \cap (p \Delta q) = (b \cap p) \Delta (b \cap q)$

ii) $(b \Delta c) \cap p = (b \cap p) \Delta (c \cap p)$ iii) $(b \cap c) \cap p = b \cap (c \cap p)$

Again from the map, $A \times B \rightarrow A$, written $(p, b) = p \cap b \in A$ such that for every $p, q \in A$ and $b, c \in B$, we get

i) $(p \Delta q) \cap b = (p \cap b) \Delta (q \cap b)$

ii) $p \cap (b \Delta c) = (p \cap b) \Delta (p \cap c)$ iii) $p \cap (b \cap c) = (p \cap b) \cap c$

Definition 5.3

The smarandache Boolean sub-ring is defined to be a smrandache Boolean ring B which is a proper subset of a Smarandache Boolean ring A. (with respect to the same induced operation).

Definition 5.4

The smarandache Boolean ideal is defined to be an ideal B of Smarandache Boolean ring A such that the Abelian group (C, Δ) has both left and right B - module, where C is a proper subset of B. From above we mean an Abelian group (C, Δ) together with a map, $C \times B \to C$ written $(c, p) = C \cap P \in C$ such that this mapping satisfies all the postulates of both left and right B - module.

Examples 5.5

Let $(\mathbf{Z}_{256}, +_{256})$ be an Abelian group, then $\mathbf{Z}_{256} = \{ 0, 1, 2, \ldots \ldots \ldots 255 \}$. If the elements $\mathbf{Z}_{256}$ of are written as 4 - adic numbers, then

$\mathbf{Z}_{256} = \{ (0000)_4, (0001)_4, (0002)_4, (0003)_4, (0010)_4, \ldots \ldots \ldots \ldots, (3333)_4 \}$and $(\mathbf{Z}_{256}, \Delta)$ is a smarandache groupoid of order 256. The centre of $(\mathbf{Z}_{256}, \Delta)$ is

$\mathbf{Z}^*_{256} = \{ 0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, 81, 84, 85 \}$. Here $(\mathbf{Z}^*_{256}, \Delta, \cap)$ is a Smarandache Boolean ring and its composition tables are given below :

Δ	0	1	4	5	16	17	20	21	64	65	68	69	80	81	84	85
0	0	1	4	5	16	17	20	21	64	65	68	69	80	81	84	85
1	1	0	5	4	17	16	21	20	65	64	69	68	81	80	85	84
4	4	5	0	1	20	21	16	17	68	69	64	65	84	85	80	81
5	5	4	1	0	21	20	17	16	69	68	65	64	85	84	81	80
16	16	17	20	21	0	1	4	5	80	81	84	85	64	65	68	69
17	17	16	21	20	1	0	5	4	81	80	85	84	65	64	69	68
20	20	21	16	17	4	5	0	1	84	85	80	81	68	69	64	65
21	21	20	17	16	5	4	1	0	85	84	81	80	69	68	65	64
64	64	65	68	69	80	81	84	85	0	1	4	5	16	17	20	21
65	65	64	69	68	81	80	85	84	1	0	5	4	17	16	21	20
68	68	69	64	65	84	85	80	81	4	5	0	1	20	21	16	17
69	69	68	65	64	85	84	81	80	5	4	1	0	21	20	17	16
80	80	81	84	85	64	65	68	69	16	17	20	21	0	1	4	5
81	81	80	85	84	65	64	69	68	17	16	21	20	1	0	5	4
84	84	85	80	81	68	69	64	65	20	21	16	17	4	5	0	1
85	85	84	81	80	69	68	65	64	21	20	17	16	5	4	1	0

Table - 5

∩	0	1	4	5	16	17	20	21	64	65	68	69	80	81	84	85
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
4	0	0	4	4	0	0	4	4	0	0	4	4	0	0	4	4
5	0	1	4	5	0	1	4	5	0	1	4	5	0	1	4	5
16	0	0	0	0	16	16	16	16	0	0	0	0	16	16	16	16
17	0	1	0	1	16	17	16	17	0	1	0	1	16	17	16	17
20	0	0	4	4	16	16	20	20	0	0	4	4	16	16	20	20
21	0	1	4	5	16	17	20	21	0	1	4	5	16	17	20	21
64	0	0	0	0	0	0	0	0	64	64	64	64	64	64	64	64
65	0	1	0	1	0	1	0	1	64	65	64	65	64	65	64	65
68	0	0	4	4	0	0	4	4	64	64	68	68	64	64	68	68
69	0	1	4	5	0	1	4	5	64	65	68	69	64	65	68	69
80	0	0	0	0	16	16	16	16	64	64	64	64	80	80	80	80
81	0	1	0	1	16	17	16	17	64	65	64	65	80	81	80	81
84	0	0	4	4	16	16	20	20	64	64	68	68	80	80	84	84
85	0	1	4	5	16	17	20	21	64	65	68	69	80	81	84	85

Table - 6

Some smarandache Boolean sub-rings of $(\mathbb{Z}^{\bullet}_{256}, \Delta, \cap)$ are given below :

i) $H_1 = \{ 0, 1, 4, 5, 16, 17, 20, 21 \}$
$H_2 = \{ 0, 1, 4, 5, 64, 65, 68, 69 \}$
$H_3 = \{ 0, 1, 4, 5, 80, 81, 84, 85 \}$
$H_4 = \{ 0, 5, 16, 21, 64, 69, 80, 85 \}$
$H_5 = \{ 0, 1, 16, 17, 64, 65, 80, 81 \}$
$H_6 = \{ 0, 1, 4, 5 \}$
$H_7 = \{ 0, 1, 16, 17 \}$
$H_8 = \{ 0, 1, 64, 65 \}$
$H_9 = \{ 0, 1, 80, 81 \}$ etc.

Here smarandache Boolean subrings H_1, H_2, H_6, H_7, H_8 are ideals of $(\mathbb{Z}^{\bullet}_{256}, \Delta)$.

References :

1. David M. Burton - Elementary number theory
2nd edition, University book stall
New Delhi (1994).

2. Mc Coy, N.H - Introduction to Modern Algebra
Boston Allyu and Bacon INC (1965)

3. Talukdar, D & Das N.R.- Measuring associativity in a groupoid of natural numbers
The Mathematical Gazette
Vol. 80. No.- 488 (1996), 401 - 404

4. Talukdar, D - Some Aspects of inexact groupoids
J. Assam Science Society
37(2) (1996), 83 - 91

5. Talukdar, D - A Klein 2^n - group, a generalization of Klein 4 group
GUMA Bulletin
Vol. 1 (1994), 69 - 79

6. Hall, M - The theory of groups
Macmillan Co. 1959.

7. Talukdar, D - Klein 2^n – group Action on a set of Fuzzy Subsets. The Journal of Fuzzy Mathematics. Vol. 6, 1998.

FROM BOLYAI'S GEOMETRY TO SMARANDACHE ANTI-GEOMETRY

Angela Vasiu and Nicolaie Oprea*

Mathematics Department, "Babeş-Bolyai" University Cluj, Romania
*Mathematics Department, "Nord University" Baia Mare, Romania

---

ABSTRACT

It is considered the notion of absolute Geometry in its evolution, from the first Non-euclidean Geometry of Lobacewski, Bolyai and Gauss till that of Smarandache Anti-Geometry.

Key words: Euclidean Geometry, Non-euclidean Geometry, Hilbert's axioms and incidence structures deduced from them, Smarandache Geometries, Hjelmslev-Barbilian structures

Any theory or deductive system has two distinguishable parts:

1. the specifical part
2. the logical part.

When we formalize one, or both of parts of a theory, we obtain next classification for axiomatical theories:

1. nonformalized,
2. semiformalized, or
3. formalized axiomatical theory.

When J. Bolyai in [3] in 1831, and N. Lobacewski in [12], in 1826 began the studies about non-euclidean Geometries the formalization in mathematics was not yet introduced. Their contributions should be considered as more important as at that moment the formalized system of axioms of Geometry of D. Hilbert was not given.

The way we can establish the metamathematical analyse of a theory are two:

1. sintactical (that is directly)
2. semantical, by the interpretations and models.

J. Bolyai and N. Lobacevski worked only sintactically and not semantically in the study of the metamathematical analyse of their theory.

On the first way, non-contradiction of their Geometry given in [3] and [12] could not be proved, because in a such a way they could not convince that the set of correct affirmations of their theory were exhausted and that has excluded the possibility to meet a proposition p correct constructed such that p and $\neg p$ to be verified in their Geometry. Later were given semantically the proofs that their non-euclidean Geometry is consistent, and so is non-contradictory, by the models of Berltrami in 1868, after that of Cayley-Klein in 1871 and of H. Poincaré in 1882.

In Bolyai-Lobacewski's absolute Geometry from a point A to a line a, A is not incident with a, there exists a parallel. In this absolute Geometry we have only two possibilities:

1. the Euclidean Geometry
2. the Hyperbolical Geometry.

The elliptical Geometry with none parallel through a point to a line, are excluded from this absolute Geometry. Also this absolute Geometry contained only continuous Geometry.

In 1903 in [6], D. Hilbert proved that the hyperbolical plane geometry can be introduced without to use the tridimensional space, and that is possible to renounce to the axioms of continuity. This is an important moment for research in Geometry because from that moment the notion of absolute Geometry changes its meaning and begins to be different considered to different moments and to different authors. The absolute plane of Bolyai becomes a particular case of the absolute plane in recently researches of Geometry.

From 1889, when D. Hilbert in [5] gave a formalized system of axioms for absolute Geometry, appeared more directions of investigation in Geometry. The incidence structures are largely used and so are introduced a great variety of affine and projective planes and affine and projective spaces.

The great importance of geometrical transformations for geometrical problems was put by F. Klein in "The programme from Erlangen" in 1872, when he began to consider the Geometry as the study of invariant properties to a group of transformations. From that moment the system of axioms of many Geometries are based on the notions of theory of groups. This group is given as an abstract group, and geometrical structure is a consequence of structure of group. This fact was possible, after that it was proved that the geometry can be transposed in the group of its automorphisms generated by axial symmetries. A such a system of axioms is more simple than a classical one, it is easily adopted to the special qualities of non-euclidean Geometries. Compared with a such a system of axioms, the system of axioms of D. Hilbert is more complicated.

As it is the calcullus in a field for Analytical Geometry a method of work, as the calcullus in the group generated by the axial symmetries becomes a method for proofs in Geometry, after J. Hjelmslev in [7] introduced it. In [16] Thompsen proved that this can become an efficient method of demonstration for the theorems of Euclidean Geometry. This is an attrative method because the hypothesis and conclusions of a theorem can be written simply as relations of group.

The first system of axioms of absolute plane geometry formulated in theory of groups was given by A. Schmidt in [13], and after that F. Bachman in [1]. From that date this method is largely used in Geometry as in [9], [10], [11], [17] and many others works.

In 1954 after E. Sperner gives a group proof of theorem of Desargues for a large classes of Geometries, in absolute geometry are included new-types of geometry, as geometry with centre, with perpendicular nuclei [10], and many others.

The classical Geometries are extended, because are not made hypothesis of order, of continuity or mobility [1], [9], [11], [17].

In 1967 H. Wolf in [21], includes near Euclidean, elliptical, and hyperbolical Geometry, also Minkowski's Geometry.

The geometries constructed over a field of characteristic 2 are included later, by a more general system of axioms of R. Lingenberg in [11].

Another generalization of incidence structures was that in which it were considered geometrical structures to which the line incident with two different points is not unique. A such a theory is consistent and as a model for it we have the Geometry over a ring. Such structures were introduced by J. Hjelmslev in [8] and D. Barbilian in [2]. A new direction of study in Geometry begins from this moment, in which we have also some results.

The researches of absolute Geometry have a natural continuity, the notion of absolute Geometry is a notion in evolution in the modern literature of speciality. This help us to understand better the life, the transformations in the life, and finally this could bring us more wisdom and increasing degree of understanding of human condition, and such to answer to the deep desire of their creatores: that the mathematics to become also a force of life.

Such Florentin Smarandache even in 1969 said that it is natural to consider a new Geometry denying not only one axiom from the axioms of D.H. Hilbert from [5] but more or even all of them, what he did in 1985 in [14] and in 1997 in [15].

So he introduced so called "Smarandache Anti-Geometry". It seems strange but it is natural. We should remember that when J. Bolyai the genial discoverer of first non-euclidean Geometry was deeply implied in his great work even the great Gauss said that the people are not prepared to receive a new Geometry, a such a new theory. And that was the truth. J. Bolyai suffered very much at that time seing that he can not be understood, but he was convinced that not only in Mathematics, but in the whole history of thinking his conception represents a crucial point. Besides the value of his discoveries in Mathematics, J. Bolyai must be discovered and then, inevitable loved, as a great thinker preoccupied of the problems of harmonious integration in the life. As we showed in [18], [19] J. Bolyai always was thirsty of harmony and with a stoical wisdom he supported his ideas until the end of his life, a life full of misunderstanding. In spite of all what he met as nonunderstanding he continued to believe in what he created and he felt them to be true.

As J. Bolyai, N. Lobacevski was not understood during his life and his work was not recognized at that time. Their contributions today have to be appreciate even more as at their time the formalized theories has not been introduced.

Beyond the mathematical contribution their works represent an opening meditation of human condition which have not been enough exploited. Feeling the potential of this opening in the understanding of the human complexity we suggested it as a direction of research and to try to imply, we all scientists, to get an amelioration of the human condition as in [18], [19], [20] we did. This research can be done by the utilisation of mathematical ideas and theories to the construction of a model of self-knowledge. Have we ever put the question which are the axioms which stay at the base of the existence? As any theory the human existence should have some axioms, propositions, theorems, conjectures, false affirmations etc. The consequences of false affirmations in our behaviour can be clearly observed: the pollution of the mind, of the nature, ecological perturbations etc.

We all can realize that the elimination or diminuation of false affirmations about the existence and man, could bring harmony and peace. Taking in consideration the profoundness and credibility of scientists we can hope more and more from us paing attention to this noble work. The incredible technical progress and discoveries of the science have a correspondent in the science of selfknowledge.

The Anti-Geometry introduced by Florentin Smarandache in [14], [15] would correspond to the understanding of the degradation of human condition. Even this "Anti-Geometry" could be a model for this kind of "inner Geometry", in the sense that the degree of degradation represents the different levels of negation of our inner possibilities, of our natural qualities.

References

[1] Bachmann, F., *Aufbau der Geometrie aus dem Spiegelungsbegriff*, Berlin-Göttingen-Heidelberg, 1959.

[2] Barbilian, D., *Zur Axiomatik der projecktiven Ringgeometrien I, II*, Uber Deutsch. Math. Verein 50(1940), 179-229 and 51(1941), 34-76.

[3] Bolyai, J., *Appendix*, Ed. Acad. 1950, Romania.

[4] Hessenberg, G., *Begründung der elliptischen Geometrie*. Math. Annalen 61(1905), 173-184.

[5] Hilbert, D., *Grundlagen der Geometrie*, 1. Aufl. Berlin 1899, 7. Aufl. Berlin 1930.

[6] Hilbert, D., *Neue Begründung der Bolyai-Lobatschefskyscher Geometrie*, Math. Annalen 57(1903), 137-150, Printed as addendum II in [5].

[7] Hjelmslev, J., *Neue Begründung der ebenen Geometrie*, Math. Annalen 64(1907), 449-474.

[8] Hjelmslev, J., *Einleintung in die allgemeine Kongruenzlehre*, Danske V d. Selsk., mat.-fys. Medd. 8, Nr.11(1929), 10, Nr.1(1929); 19, nr.12(1942), 22 nr.6, (1945); 25, Nr.10(1949).

[9] Karzel, H., *Ein Axiomensystem der absoluten Geometrie*, Arch. Math. 6(1955), pp.66-76.

[10] Karzel, H., *Spiegelung geometrien mit echtem zentrum*, Arch. Math. 9(1958), pp.140-146.

[11] Lingenberg. R., *Über Gruppen mit einen invarianten System involutorischer Erzeugender, in dem der allgemeine Satz von den drei Spiegelungen gilt I, II, III, IV*, Math. Ann. 137(1959), pp.14-26, 83-106; 132(1961), pp.184-224; 158(1965), pp.297-325.

[12] Lobacewski. N., *Cercetări geometrice cu privire la teoria liniilor paralele*, 1952, Ed. Acad., Romania.

[13] Schmidt. A., *Die Dualität von Inzidenz und Senkrechtstehen in der absoluten Geometrie*, Math. Ann. 118(1943), pp.609-625.

[14] Smarandache. Florentin (1985). *Paradoxist Mathematics* (Lecture), Bloomsburg University. Math. Dept, PA, U.S.A.

[15] Smarandache. Florentin (1997), *Collected Papers II*. pp.5-28, University of Kishinev.

[16] Thomsen, G., *Grundlagen der elementar-geometrie in gruppen algebricher Behandlung*, Hamburg, Math. Eizelschrisften 15(1933).

[17] Vasiu. Angela. *Grupuri (G, S) cu sistem de generatori involutivi I, II*, Studia Univ. Babeş-Bolyai. Cluj. 3(1973), pp.12-20, 1(1974), pp.33-36.

[18] Vasiu Adrian. Vasiu Angela, *Bolyai invites us to more wisdom. to our awakening*, Proceedings of Symp. in Geometry "Babeş-Bolyai" Univ. Nr.2. 1993, pp.209-216.

[19] Vasiu Angela. Vasiu Adrian, *The Geometry a support of a formative teaching*, Dokia Publishing House Ltd, ISBN 973-9074-46-4, Cluj, Romania, 1998.

[20] Vasiu Adrian. Vasiu Angela, *The foundation of the natural integration in the life*, Dokia Publishing House Ltd, ISBN 973-9074-57-X. Cluj, Romania, 1998.

[21] Wolff, H., *Minkowskische und absolute Geometrie I, II*, Math. Ann. 171(1967).

Smarandache's new geometries a provocation for an ammelioration of human condition

Angela Vasiu

Mathematics Department, "Babeş-Bolyai" University Cluj, Romania

ABSTRACT

Are remarked the new Geometries of Smarandache and it is given a relationship and an application of Smarandache Paradoxist Geometry to the ammelioration of human condition by a better understanding of ourselves and of others.

Key words: Non-euclidean Geometry, Bolyai/Lobacewski/Gauss and Riemann Geometry, Smarandache Paradoxist Geometry

In [2], [3], Florentin Smarandache introduced a new type of Geometry. In this Smarandacheian space it is proposed to be considered the theory deduced from the Absolute Geometry of Bolyai and Lobacewski in which the axiom of parallel it is accepted for some pairs of points and lines and it is denied for others. This new Geometry generalizes and unites in the same time: Euclid, Bolyai/Lobacewski/Gauss and Riemann Geometries.

If the first Non-euclidean Geometry introduced by Lobacewski, Bolyai and Gauss surprised the world, such that Gauss said that the people were not prepared to receive a new theory, now we know and accept many kinds of new Geometries. Even in 1969 Florentin Smarandache had put the problem to study a new Geometry in which the parallel from a point to a line to be unique only for some pairs of points and lines and for others: none or more, even infinitely many parallels could be drawn through some points to a line.

Are nowadays people surprise for such new ideas and new Geometries? Certainly not! After then the formalized theories were introduced in Mathematics, a lot of new Geometries could be accepted and semantically to be proved to be non-contradictory by the models created for them as in [1].

In [4] we introduced a new notion for understanding the great diversity of human condition, that of "inner Geometry". Conformly with this notion we differ so much after the degree of manifestation of our inner possibilities, and from here, after our own blockade of them. To be able to understand and to improve our interhuman relationships these new types of Geometries could help in at least two directions. For a hand, we are in different type of "inner Geometry" from a moment to another moment, and for the other hand: from a person to other one, this "inner Geometry" could be different. In this acceptation we can treat each other with more wisdom, we can find an explanation of so exposed human condition, to be more conscious about the greatness of self knowledge and to imply more in the ammelioration of the existence as a theory in which we want to be with more concilliation. Smarandache's Geometries could be considered in this way, as an important reflection about human condition and his Paradoxist Geometry to find a new model in the theory of existence.

References

[1] Sandy P. Chimienti and Mihaly Bencze, *Smarandache Anti-Geometry*, Bulletin of Pure and Applied Sciences Vol.17E, No.1, 1998, p.103-114.

[2] Smarandache, Florentin, *Paradoxist Mathematics* (Lecture), Bloomsburg University, Math. Dept, PA, U.S.A, 1985.

[3] Smarandache, Florentin, *Collected Papers*, Vol.II, University of Kishinev, Press Kishinev, p.5-28, 1997.

[4] Vasiu Adrian, Vasiu Angela, *The foundation of the natural integration in the life*, Dokia Publishing House Ltd, 1998.

THE AVERAGE OF THE ERDOS FUNCTION

Sabin Tabirca* Tatiana Tabirca*

**Transilvania University of Brasov, Computer Science Department

The aim of this article is to establish the complexity order of the Erdos function average. This will be studied based on some recent results about the Smarandache function.

1. INTRODUCTION

The main results used in this paper are reviewed in the following. These deal with the main properties of the Smarandache and Erdos functions.

The Smarandache function [Smarandache, 1980] is $S: N^* \to N$ defined by

$$S(n) = \min\{k \in N | k! = Mn\} \ (\forall n \in N^*). \tag{1}$$

The function $P : N^* \to N$ defined by

$$P(n) = \min\{ p \in N \mid n = Mp \wedge p \text{ is prim}\} (\forall n \in N^* \setminus \{1\}),\ P(1) = 0 \tag{2}$$

is named classically the Erdos function. Both functions satisfy the same main properties:

$$(\forall a, b \in N^*)\ (a,b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\},\ P(a \cdot b) = \max\{P(a), P(b)\}. \tag{3}$$

$$(\forall a \in N^*)\ P(a) \le S(a) \le a \text{ and the equalities occur } iif \text{ a is prim.} \tag{4}$$

Erdos [1991] found that these two functions have the same values for all most of the natural numbers $\lim\limits_{n\to\infty} \frac{\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right|}{n} = 0$. This important result was extended by Ford [1999] to

$$\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right| = n \cdot e^{-\left(\sqrt{2} + a_n\right)\sqrt{\ln n \cdot \ln \ln n}},\ \text{where } \lim_{n\to\infty} a_n = 0. \tag{5}$$

Obviously, both functions are neither increasing nor decreasing functions. In this situation, many researchers have tried to study properties concerning their average. Many results that have been published so far deal with complexity orders of the average.

Let us denote $E(f(n)) = \frac{1}{n} \cdot \sum_{i=1}^{n} f(n)$ the average of function $f : N^* \to R$. The average *E(S(n))* was intensively studied by Tabirca [1997, 1998] and Luca [1999]. Tabirca [1998] proved that $(\forall\ n > c_p) E(S(n)) \le a_p \cdot n + b_p$, where $\lim\limits_{p\to\infty} a_p = \lim\limits_{p\to\infty} b_p = 0$. This means that the order *O(n)* is not properly chosen for $E(S(n))$. Tabirca [1998] conjectured that the average $E(S(n))$ satisfies the equation $E(S(n)) \leq \frac{n}{\ln n}$. Finally and the most important, Luca [1999] proposed the equation

$$\frac{1}{2} \cdot \left[\pi(n) - \pi(\sqrt{n})\right] < E(S(n)) < \pi(n) + \frac{5}{2} \cdot \ln \ln n + \frac{1}{n} + \frac{31}{5} \tag{7}$$

where $\pi(x)$ denotes the number of prim numbers less than or equal to x. Thus, the complexity order for the average $E(S(n))$ is indeed $O\left(\frac{n}{\log n}\right)$.

2. THE COMPLEXITY ORDER FOR THE ERDOS FUNCTION

Some of the above results are used to find the complexity order of $E(P(n))$. Based on the well-known formula $\lim_{n \to \infty} \frac{\pi(n)}{\frac{n}{\ln n}} = 1$, Equation (7) gives

$$\frac{1}{2} \leq \liminf_{n \to \infty} \frac{E(S(n))}{\frac{n}{\ln n}} \leq \limsup_{n \to \infty} \frac{E(S(n))}{\frac{n}{\ln n}} \leq 1. \tag{8}$$

Theorem 1.

$$\liminf_{n \to \infty} \frac{E(S(n))}{\frac{n}{\ln n}} = \liminf_{n \to \infty} \frac{E(P(n))}{\frac{n}{\ln n}}, \ \limsup_{n \to \infty} \frac{E(S(n))}{\frac{n}{\ln n}} = \limsup_{n \to \infty} \frac{E(P(n))}{\frac{n}{\ln n}} \tag{9}$$

Proof Let us denote $A = \{i = \overline{1,n} \mid S(i) > P(i)\}$ the set of the numbers that do not satisfy the equation $S(i)=P(i)$. The cardinal of this set is $|A| = n \cdot e^{-\left(\sqrt{2}+a_n\right)\sqrt{\ln n \cdot \ln \ln n}}$, where $\lim_{n \to \infty} a_n = 0$.

The proof is started from the following equation

$$\left|E(S(n)) - E(P(n))\right| = \frac{1}{n} \cdot \left|\sum_{i=1}^{n} S(i) - \sum_{i=1}^{n} P(i)\right| = \frac{1}{n} \cdot \left[\sum_{i \in A} (S(i) - P(i))\right] \leq \sum_{i \in A} \frac{S(i)}{n}. \tag{10}$$

Because we have $\left(\forall\, i = \overline{1,n}\right) S(i) \leq n$, Equation (10) gives

$$\left|E(S(n)) - E(P(n))\right| \leq |A| = n \cdot e^{-\left(\sqrt{2}+a_n\right)\sqrt{\ln n \cdot \ln \ln n}} \text{ and}$$

$$\left|\frac{E(S(n))}{\frac{n}{\ln n}} - \frac{E(P(n))}{\frac{n}{\ln n}}\right| \leq \ln n \cdot e^{-\left(\sqrt{2}+a_n\right)\sqrt{\ln n \cdot \ln \ln n}}. \tag{11}$$

Because $\lim_{n\to\infty} a_n = 0$, the equation $\lim_{n\to\infty} \ln n \cdot e^{-\left(\sqrt{2}+a_n\right)\sqrt{\ln n \cdot \ln\ln n}} = 0$ is found true, thus

$$\liminf_{n\to\infty} \frac{E(S(n))}{\frac{n}{\ln n}} = \liminf_{n\to\infty} \frac{E(P(n))}{\frac{n}{\ln n}}, \ \limsup_{n\to\infty} \frac{E(S(n))}{\frac{n}{\ln n}} = \limsup_{n\to\infty} \frac{E(P(n))}{\frac{n}{\ln n}} \text{ holds.} \quad \blacklozenge$$

Theorem 2 is obtained as a direct consequence of Theorem1.

Theorem 2

$$E(P(n)) = O(\frac{n}{\ln n})$$

Proof The equation $\frac{1}{2} \le \liminf_{n\to\infty} \frac{E(P(n))}{\frac{n}{\ln n}} \le \limsup_{n\to\infty} \frac{E(P(n))}{\frac{n}{\ln n}} \le 1$ is found true applying Theorem 1. From that, there is a natural number N_1 such that that

$$\left(\forall\, n \ge N_1\right)\left(\frac{1}{2} - \varepsilon\right) \cdot \frac{n}{\ln n} \le E(P(n)) \le \left(1+\varepsilon\right) \cdot \frac{n}{\ln n}. \tag{12}$$

Therefore, the equation $E(P(n)) = O(\frac{n}{\ln n})$ holds. ♦

The right question that comes from (12) is the following "**Is the equation** $E(P(n)) \le \frac{n}{\ln n}$ **true?**". This has been investigated for all the natural numbers less than 1000000 and it has been found true. Equation (7) can be adapt to the average *E(P(n))* but obviously the inequality that is found is not an answer to the question. Therefore, we may conjecture the following: **The equation** $E(P(n)) \le \frac{n}{\ln n}$ **holds for all *n*>1.**

References

Erdos, P. (1991) *Problem 6674*, Amer. Math. Monthly. **98**, 965.

Ford, K. (1999) The normal behaviours of the Smarandache function, *Smarandache Notions Journal,* Vol. **10,** No.1-3, 81-86.

Luca, F. (1999) The average Smarandache function, Personal communication to S. Tabirca. [will appear in *Smarandache Notion Journal*].

Tabirca, S. and Tabirca, T. (1997) Some upper bounds for Smarandache's function, *Smarandache Notions Journal*, **8**, 205-211.

Tabirca, S. and Tabirca, T. (1998) Two new functions in number theory and some upper bounds for Smarandache's function, *Smarandache Notions Journal*, **9**, No. 1-2, 82-91.

ON THE CONVERGENCE OF THE ERDOS HARMONIC SERIES

Tatiana Tabirca* Sabin Tabirca*

*Transilvania University of Brasov, Computer Science Department

The purpose of this article is to study the convergence of a few series with the Erdos function. The work is based on results concerning the convergence of some series with the Smarandache function.

1. INTRODUCTION

The results used in this article are presented briefly in the following. These concern the relationship between the Smarandache and the Erdos functions and the convergence of some series. These two functions are important function in Number Theory. They are defined as follows:

- The Smarandache function [Smarandache, 1980] is $S: N^* \to N$ defined by

$$S(n) = \min\{k \in N | k! = Mn\} \ (\forall n \in N^*). \tag{1}$$

- The Erdos function is $P: N^* \to N$ defined by

$$P(n) = \min\{p \in N \mid n = Mp \wedge p \text{ is prim}\} (\forall n \in N^* \setminus \{1\}), P(1) = 0. \tag{2}$$

The main properties of them are:

$$(\forall a, b \in N^*) \ (a,b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\}, P(a \cdot b) = \max\{P(a), P(b)\}. \tag{3}$$

$$(\forall a \in N^*) \ P(a) \le S(a) \le a \text{ and the equalities occur } iif \text{ a is prim.} \tag{4}$$

Erdos [1991] found the relationship between these two functions that is given by

$$\lim_{n \to \infty} \frac{\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right|}{n} = 0. \tag{5}$$

This important result was extended by Ford [1999] to

$$\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right| = n \cdot e^{-\left(\sqrt{2}+a_n\right)\sqrt{\ln n \cdot \ln \ln n}}, \text{ where } \lim_{n \to \infty} a_n = 0. \tag{6}$$

Equations (5-6) are very important because allow us to translate convergence properties on the Smarandache function to convergence properties on the Erdos function. This translation represents the main technique that is used to obtain the convergence of some series with the function P.

2. THE ERDOS HARMONIC SERIES

The Erdos harmonic series can be defined by $\sum_{n\geq 2}\frac{1}{P^a(n)}$. This is one of the important series with the Erdos function and its convergence is studied starting from the convergence of the Smarandache harmonic series $\sum_{n\geq 2}\frac{1}{S^a(n)}$. Some results concerning series with the function S are reviewed briefly in the following:

- If $(x_n)_{n>0}$ is an increasing sequence such that $\lim_{n\to\infty} x_n = \infty$, then the series $\sum_{n>1}\frac{x_{n+1}-x_n}{S(x_n)}$ is divergent. [Cojocaru, 1997].
  (7)
- The series $\sum_{n\geq 2}\frac{1}{S^2(n)}$ is divergent. [Tabirca, 1998]
  (8)
- The series $\sum_{n\geq 2}\frac{1}{S^a(n)}$ is divergent for all a>0. [Luca, 1999] (9)

These above results are translated to the similar properties on the Erdos function.

Theorem 1. If $(x_n)_{n>0}$ is an increasing sequence such that $\lim_{n\to\infty} x_n = \infty$, then the series $\sum_{n>1}\frac{x_{n+1}-x_n}{P(x_n)}$ is divergent.

Proof The proof is obvious based on the equation $P(x_n)\leq S(x_n)$. Therefore, the equation $\frac{x_{n+1}-x_n}{P(x_n)}\geq\frac{x_{n+1}-x_n}{S(x_n)}$ and the divergence of the series $\sum_{n>1}\frac{x_{n+1}-x_n}{S(x_n)}$ give that the series $\sum_{n>1}\frac{x_{n+1}-x_n}{P(x_n)}$ is divergent.

♦

A direct consequence of Theorem 1 is the divergence of the series $\sum_{n>1}\frac{1}{P(a\cdot n+b)}$, where $a,b>0$ are positive numbers. This gives that $\sum_{n\geq 2}\frac{1}{P(n)}$ is divergent and moreover that $\sum_{n\geq 2}\frac{1}{P^a(n)}$ is divergent for all $a<1$.

Theorem 2. The series $\sum_{n\geq 2}\frac{1}{P^a(n)}$ is divergent for all $a>1$.

Proof The proof studies two cases.

Case 1. $a\geq\frac{1}{2}$.

In this case, the proof is made by using the divergence of $\sum_{n\geq 2}\frac{1}{S^a(n)}$.

Denote $A=\left\{i=\overline{2,n}\mid S(i)=P(i)\right\}$ and $B=\left\{i=\overline{2,n}\mid S(i)>P(i)\right\}$ a partition of the set $\left\{i=\overline{1,n}\right\}$. We start from the following simple transformation

$$\sum_{i=2}^{n}\frac{1}{P^a(i)}=\sum_{i=2}^{n}\frac{1}{S^a(i)}+\sum_{i\in B}\left[\frac{1}{P^a(i)}-\frac{1}{S^a(i)}\right]=\sum_{i=2}^{n}\frac{1}{S^a(i)}+\sum_{i\in B}\frac{S^a(i)-P^a(i)}{P^a(i)\cdot S^a(i)}. \tag{10}$$

An $i\in B$ satisfies $S^a(i)-P^a(i)\geq 1$ and $P(i)<S(i)\leq n$ thus, (10) becomes

$$\sum_{i=2}^{n}\frac{1}{P^a(i)}\geq\sum_{i=2}^{n}\frac{1}{S^a(i)}+\sum_{i\in B}\frac{1}{n^{2\cdot a}}=\sum_{i=2}^{n}\frac{1}{S^a(i)}+\frac{1}{n^{2\cdot a}}\cdot|B|. \tag{11}$$

The series $\sum_{n\geq 2}\frac{1}{P^a(n)}$ is divergent because the series $\sum_{n\geq 2}\frac{1}{S^a(n)}$ is divergent and

$$\lim_{n\to\infty}\frac{|B|}{n^{2\cdot a}}=\lim_{n\to\infty}\frac{n\cdot e^{-(\sqrt{2}+a_n)\cdot\sqrt{\ln n\cdot\ln\ln n}}}{n^{2\cdot a}}=\lim_{n\to\infty}\frac{1}{n^{2\cdot a-1}\cdot e^{(\sqrt{2}+a_n)\cdot\sqrt{\ln n\cdot\ln\ln n}}}=0.$$

Case 2. $\frac{1}{2}>a>1$.

The first case gives that the series $\sum_{n\geq 2}\frac{1}{P^{\frac{1}{2}}(n)}$ is divergent.

Based on $P^{\frac{1}{2}}(n) > P^{a}(n)$, the inequality $\sum_{i=2}^{n} \frac{1}{P^{a}(i)} > \sum_{i=2}^{n} \frac{1}{P^{\frac{1}{2}}(i)}$ is found. Thus, the series $\sum_{n \geq 2} \frac{1}{S^{a}(n)}$ is divergent. ♦

The technique that has been applied to the proof of Theorem 2 can be used in the both ways. Theorem 2 started from a property of the Smarandache function and found a property of the Erdos function. Opposite, Finch [1999] found the property $\lim_{n \to \infty} \frac{\sum_{i=2}^{n} \frac{\ln S(i)}{\ln i}}{n} = \lambda$ based on the similar property $\lim_{n \to \infty} \frac{\sum_{i=2}^{n} \frac{\ln P(i)}{\ln i}}{n} = \lambda$, where λ=0.6243299 is the Golomb-Dickman constant. Obviously, many other properties can be proved using this technique. Moreover, Equations (5-6) gives a very interesting fact - **"the Smarandache and Erdos function may have the same behavior on the convergence problems."**

References

Cojocaru, I. And Cojocaru, S. (1997) On a function in Number Theory, *Smarandache Notions Journal*, **8**, No. 1-2-3, 164-169

Erdos, P. (1991) *Problem 6674*, Amer. Math. Monthly. **98**, 965.

Ford, K. (1999) The normal behaviours of the Smarandache function, *Smarandache Notions Journal*, Vol. **10**, No.1-2-3, 81-86.

Luca, F. (1999) On the divergence of the Smarandache harmonic series, *Smarandache Notions Journal*, **10**, No. 1-2-3, 116-118.

Tabirca, S. and Tabirca, T. (1998) The convergence of the Smarandache harmonic series, *Smarandache Notions Journal*, **9**, No. 1-2, 27-35.

Smarandache, F. (1980) A Function in number theory, *Analele Univ. Timisoara*, **XVIII**.

An experimental evidence on the validity of third Smarandache conjecture on primes

Felice Russo
Micron Technology Italy
Avezzano (Aq) Italy

Abstract

In this note we report the results regarding the check of the third Smarandache conjecture on primes [1],[2] for $p_n \leq 2^{25}$ and $2 \leq k \leq 10$. In the range analysed the conjecture is true. Moreover, according to experimental data obtained, it seems likely that the conjecture is true for all primes and for all positive values of k..

Introduction

In [1] and [2] the following function has been defined:

$$C(n,k) = p_{n+1}^{\frac{1}{k}} - p_n^{\frac{1}{k}}$$

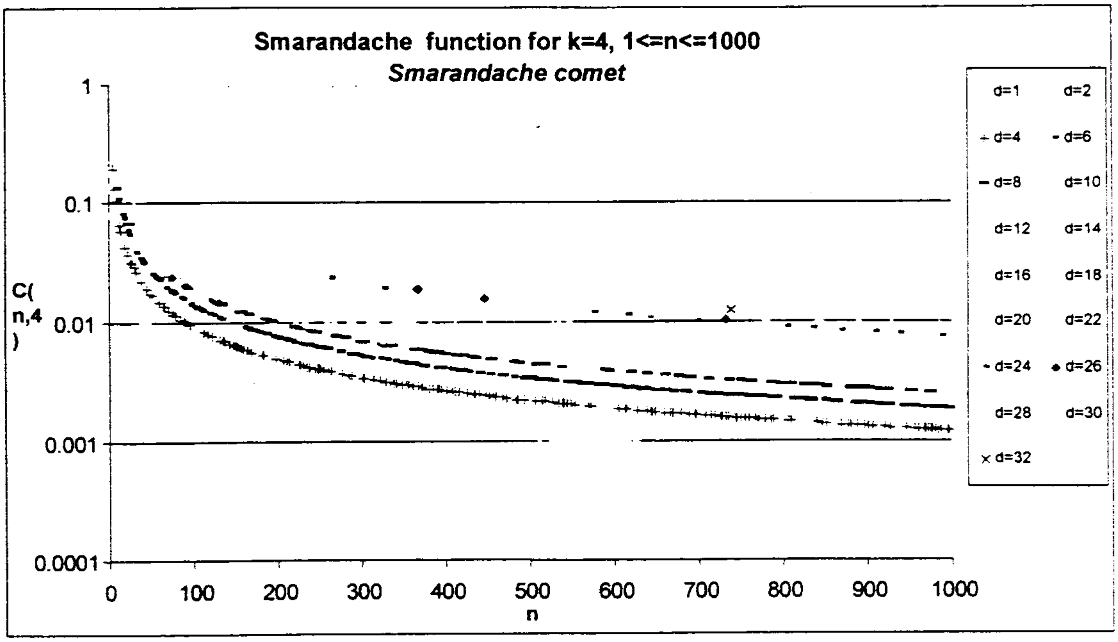
where p_n is the n-th prime and k is a positive integer. Moreover in the above mentioned papers the following conjecture has been formulated by F. Smarandache:

$$C(n,k) < \frac{2}{k} \text{ for } k \geq 2$$

This conjecture is the generalization of the Andrica conjecture (k=2) [3] that has not yet been proven. This third Smarandache conjecture has been tested for $p_n \leq 2^{25}$, $2 \leq k \leq 10$ and in this note the result of this search is reported. The computer code has been written utilizing the Ubasic software package.

Experimental Results

In the following graph the Smarandache function for k=4 and n<1000 is reported. As we can see the value of C(k,n) is modulated by the prime's gap indicated by $d_n = p_{n+1} - p_n$.
We call this graph the Smarandache "comet".



In the following table, instead, we report:

- the largest value Max_C(n,k) of Smarandache function for $2 \le k \le 10$ and $p_n \le 2^{25}$
- the difference $\Delta(k)$ between 2/k and Max_C(n,k)
- the value of p_n that maximize C(n,k)
- the value of 2/k

k	2	3	4	5	6	7	8	9	10
Max_C(n,k)	0.67087	0.31105	0.19458	0.13962	0.10821	0.08857	0.07564	0.06598	0.05850
Δ	0.32913	0.35562	0.30542	0.26038	0.22512	0.19715	0.17436	0.15624	0.14150
p_n	7	7	7	7	7	3	3	3	3
2/k	1	0.666..	0.5	0.4	0.333..	0.2857..	0.25	0.222..	0.20

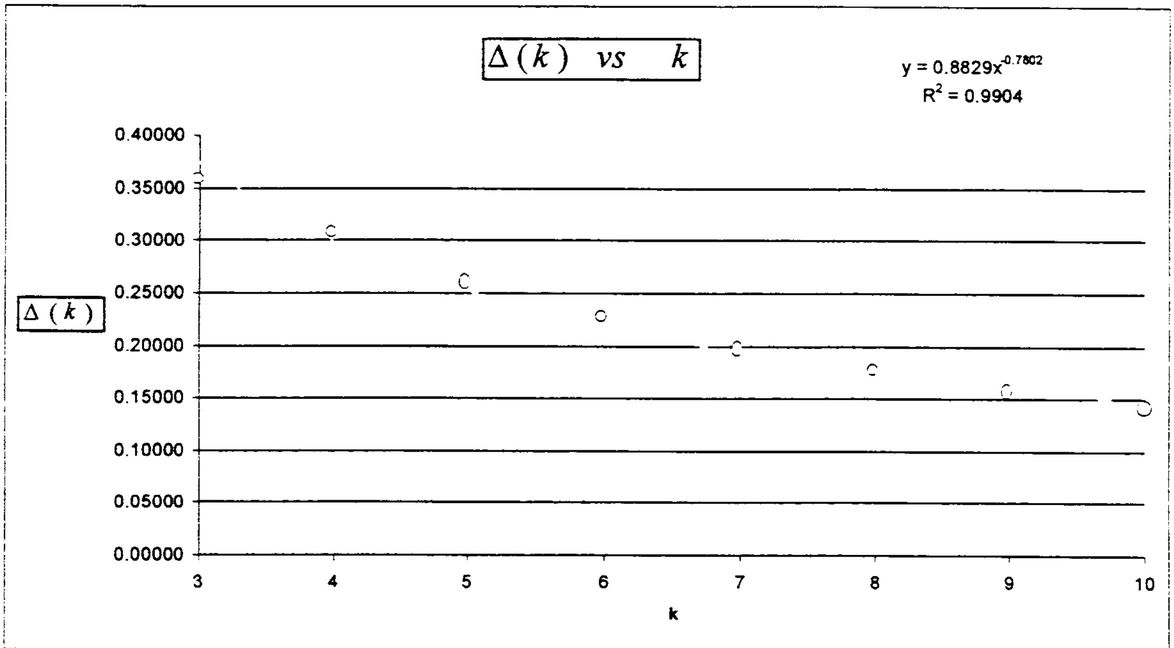
According to previous data the third Smarandache conjecture is verified in the range of k and p_n analysed due to the fact that Δ is always positive.
Moreover since the Smarandache C(n,k) function falls asymptotically as n increases it is likely that the estimated maximum is valid also for $p_n > 2^{25}$.

We can also analyse the behaviour of difference $\Delta(k)$ versus the k parameter that in the following graph is showed with white dots. We have estimated an interpolating function:

$$\Delta(k) \approx 0.88 \cdot \frac{1}{k^{0.78}} \quad \text{for } k>2$$

with a very good R^2 value (see the continuous curve). This result reinforces the validity of the third Smarandache conjecture since:

$$\Delta(k) \rightarrow 0 \quad \textit{for } k \rightarrow \infty$$



New Question

According to previous experimental data can we reformulate the third Smarandache conjecture with a tighter limit as showed below?

$$C(n,k) < \frac{2}{k^{2 \cdot a_0}}$$

where $k \geq 2$ and a_0 is the Smarandache constant [4],[1].

References:

[1] See http://www.gallup.unm.edu/~smarandache/ConjPrim.txt

[2] Smarandache, Florentin, "Conjectures which Generalize Andrica's Conjecture", Octogon, Braşov, Vol. 7, No. 1, 1999, 173-176.

[3] Weisstein, Eric W., "Andrica's conjecture" in CRC Concise Encyclopedia of Mathematics, CRC Press, Florida, 1998.

[4] Sloane, Neil, Sequence A038458 ("Smarandache Constant" = 0.567148130202017714646846875...) in <An On-Line Version of the Encyclopedia of Integer Sequences>, http://www.research.att.com/cgi-bin/access.cgi/as/njas/sequences/ eishis.cgi.

Some results about four Smarandache U-product sequences

Felice Russo
Micron Technology Italy
Avezzano (Aq) – Italy

Abstract

In this paper four Smarandache product sequences have been studied: Smarandache Square product sequence, Smarandache Cubic product sequence, Smarandache Factorial product sequence and Smarandache Palprime product sequence. In particular the number of primes, the convergence value for Smarandache Series, Smarandache Continued Fractions, Smarandache Infinite product of the mentioned sequences has been calculated utilizing the Ubasic software package. Moreover for the first time the notion of Smarandache Continued Radicals has been introduced. One conjecture about the number of primes contained in these sequences and new questions are posed too.

Introduction

In [1] Iacobescu describes the so called Smarandache U-product sequence.
Let u_n $n \geq 1$, be a positive integer sequence. Then a U-sequence is defined as follows:

$$U_n = 1 + u_1 \cdot u_2 \cdot \ldots \cdot u_n$$

In this paper differently from [1], we will call this sequence a U-sequence of the first kind because we will introduce for the first time a U-sequence of the second kind defined as follows:

$$U_n = \left|1 - u_1 \cdot u_2 \cdot \cdots \cdot u_n\right|$$

In this paper we will discuss about the "Square product", "Cubic product", "Factorial product" and "Primorial product" sequences. In particular we will analyze the question posed by Iacobescu in [1] on the number of primes contained in those sequences. We will also analyze the convergence values of the Smarandache Series [2], Infinite product [3], Simple Continued Fractions [4] of the four sequences. Moreover for the first time we will introduce the notion of Smarandache Continued Radicals and we will analyse the convergence of sequences reported above.

Sequences details

o ***Smarandache square product sequence of the first and second kind.***

In this case the sequence u_n is given by:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144........

that is the square of n. The first 20 terms of the sequence U_n $(1 \le n \le 20)$ both the first and second kind are reported in the table below:

Smarandache Square product sequence (first kind)	Smarandache Square product sequence (second kind)
2	0
5	3
37	35
577	575
14401	14399
518401	518399
25401601	25401599
1625702401	1625702399
131681894401	131681894399
13168189440001	13168189439999
1593350922240001	1593350922239999
229442532802560001	229442532802559999
38775788043632640001	38775788043632639999
7600054456551997440001	7600054456551997439999
1710012252724199424000001	1710012252724199423999999
437763136697395052544000001	437763136697395052543999999
126513546505547170185216000001	126513546505547170185215999999
40990389067797283140009984000001	40990389067797283140009983999999
14797530453474819213543604224000001	14797530453474819213543604223999999
5919012181389927685417441689600000001	5919012181389927685417441689599999999

o ***Smarandache cubic product sequence of the first and second kind.***

In this case the sequence u_n is given by:

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728......

that is the cube of n. Here the first 17 terms for the sequence U_n of the first and second kind.

Smarandache Cubic product sequence (first kind)	Smarandache Cubic product sequence (second kind)
2	0
9	7
217	215
13825	13823
1728001	1727999
373248001	373247999
128024064001	128024063999
65548320768001	65548320767999
47784725839872001	47784725839871999
47784725839872000001	47784725839871999999
63601470092869632000001	63601470092869631999999
109903340320478724096000001	109903340320478724095999999
241457638684091756838912000001	241457638684091756838911999999
662559760549147780765974528000001	662559760549147780765974527999999
2236139191853373760085164032000000001	2236139191853373760085164031999999999
9159226129831418921308831875072000000001	9159226129831418921308831875071999999999
44999277975861761160390291002228736000000001	44999277975861761160390291002228735999999999

o Smarandache factorial product sequence of the first and second kind.

In this case the sequence u_n is given by:

1, 2, 6, 24. 120, 720. 5040, 40320, 362880…….

that is the factorial of n. The first 13 terms of the U_n sequence of the first and second kind follow.

Smarandache Factorial product sequence (first kind)	Smarandache Factorial product sequence (second kind)
2	0
3	1
13	11
289	287
34561	34559
24883201	24883199
125411328001	125411327999
5056584744960001	5056584744959999
1834933472251084800001	1834933472251084799999
6658606584104736522240000001	6658606584104736522239999999
265790267296391946810949632000000001	265790267296391946810949631999999999
1273139632993994167495597712474112000000000001	1273139632993994167495597712474111999999999999
79278669759579679560737708640087148855296000000000001	79278669759579679560737708640087148855295999999999999

o *Smarandache primorial product sequence of the first and second kind.*

In this case the sequence u_n is given by:

2, 3, 5, 7, 11, 101, 121,131, 151, 181, 191, 313, 353, 353, 373......

that is the sequence of palindromic primes. Below the first 17 terms of the U_n sequence of the first and second kind.

Smarandache Palprime product sequence (first kind)	Smarandache Palprime product sequence (second kind)
3	1
7	5
31	29
211	209
2311	2309
233311	233309
28230511	28230509
3698196811	3698196809
558427718311	558427718309
101075417014111	101075417014109
19305404649695011	19305404649695009
6042591655354538131	6042591655354538129
2133034854340151959891	2133034854340151959889
795622000668876681038971	795622000668876681038969
304723222625617976883792551 1	3047232226256179768837925509
22153378548824269194517184577 1	221533785488242691945171845769
16770107561459971780249508724789 1	167701075614599717802495087247889

Results

For all above sequences the following qestions have been studied:

1. How many terms are prime?
2. Is the Smarandache Series convergent?
3. Is the Smarandache Infinite product convergent?
4. Is the Smarandache Simple Continued Fractions convergent?
5. Is the Smarandache Continued Radicals convergent?

For this purpose the software package Ubasic Rev. 9 has been utilized. In particular for the item n. 1, a strong pseudoprime test code has been written [5]. Moreover, as already mentioned above, the item 5 has been introduced for the first time; a Smarandache Continued Radicals is defined as follows:

$$\sqrt{a(1)+\sqrt{a(2)+\sqrt{a(3)+\sqrt{a(4)+\ldots\ldots}}}}$$

where a(n) is the nth term of a Smarandache sequence. Here below a summary table of the obtained results:

	# Primes	SS_cv	SIP_cv	SSCF_cv	SCR_cv
Square 1st kind	12/456=0.026	0.7288315379…..	0	2.1989247812….	2.3666079803….
Square 2nd kind	1/463=0.0021	∞	∞	0.3301888340….	1.8143775546….
Cubic 1st kind	@	0.6157923201…..	0	2.1110542477….	2.6904314681….
Cubic 2nd kind	@	∞	∞	0.1427622842….	2.2446613806….
Factorial 1st kind	5/70=0.071	0.9137455924…..	0	2.3250021620….	2.2332152218….
Factorial 2nd kind	2/66=0.033	∞	∞	0.9166908563….	1.6117607295….
Palprime 1st kind	10/363=0.027	0.5136249121…..	0	3.1422019345….	2.5932060878….
Palprime 2nd kind	9/363=0.024	1.2397048573…..	0	1.1986303614….	2.1032632883….

Legend:

\# primes (Number of primes/number of sequence terms checked)
SS_cv (Smarandache Series convergence value)
SIP_cv (Smarandache Infinite Product convergence value)
SSCF_cv (Smarandache Simple Continued Fractions convergence value)
SCR _cv (Smarandache Continued Radicals convergence value)
@ (This sequence contain only one prime as proved by M. Le and K. Wu [6])

About the items 2,3,4 and 5 according to these results the answer is: yes, all the analyzed sequences converge except the Smarandache Series and the Smarandache Infinite product for the square product (2^{nd} kind), cubic product (2^{nd} kind) and factorial product (2^{nd} kind). In particular notice the nice result obtained with the convergence of Smarandache Simple Continued Fractions of Smarandache palprime product sequence of the first kind.
The value of convergence is roughly π with the first two decimal digits correct.

$$\pi \approx 3+\cfrac{1}{7+\cfrac{1}{31+\cfrac{1}{211+\cfrac{1}{2311+\cfrac{1}{233311+\cfrac{1}{28230511+\cdots}}}}}}$$

Analogously for the cubic product sequence of the second kind the simple continued fraction converge roughly to $\pi-3$, while for the factorial product sequence of the second kind the continued radical converge roughly (two first decimal digits correct) to the golden ratio ϕ, that is:

$$\pi \approx 3+\left[0+\cfrac{1}{7+\cfrac{1}{215+\cfrac{1}{13823+\cfrac{1}{1727999+\cfrac{1}{373247999+\cdots}}}}}\right]$$

$$\phi \approx \sqrt{0+\sqrt{1+\sqrt{11+\sqrt{287+\sqrt{34559+\cdots}}}}}$$

About the item 1, the following table reports the values of n in the sequence that generate a strong pseudoprime number and its digit's number.

	n	d
Square 1st kind	1/2/3/4/5/9/10/11/1324/65/76	1/1/2/3/5/12/14/16/20/48/182/223
Square 2nd kind	2	1
Cubic 1st kind	1	1
Cubic 2nd kind	2	1
Factorial 1st kind	1/2/3/7/14	1/1/2/125/65
Factroial 2nd kind	3/7	2/12..
Palprime 1st kind	1/2/3/4/5/7/10/19/57/234	1/1/2/3/4/8/15/39/198/1208
Palprime 2nd kind	2/3/4/5/7/10/19/57/234	1/2/3/4/8/15/39/198/1208

Please note that the primes in the sequence of palprime of the first and second kind generate pairs of twin primes. The first ones follow:

(3.5) (5.7) (29.31) (209,211) (2309,2311) (28230509,28230511) (101075417014109,101075417014111) ....

Due to the fact that the percentage of primes found is very small and that according to Prime Number Theorem, the probability that a randomly chosen number of size n is prime decreases as 1/d (where d is the number of digits of n) we are enough confident to pose the following conjecture:

- *The number of primes contained in the Smarandache Square product sequence (1st and 2nd kind), Smarandache Factorial product sequence (1st and 2nd kind) and Smarandache Palprime product sequence (1st and 2nd kind) is finite.*

New Questions

- Is there any Smarandache sequence whose SS, SIP, SSCF and SCR converge to some known mathematical constants?

- Are all the estimated convergence values irrational or trascendental?

- Is there for each prime inside the Smarandache Palprime product sequence of the second kind the correspondent twin prime in the Smarandache Palprime product sequence of the first kind?

- Are there any two Smarandache sequences a(n) and b(n) whose Smarandache Infinite Product ratio converge to some value k different from zero?

$$\lim_{n \to \infty} \frac{\prod_n \frac{1}{a(n)}}{\prod_n \frac{1}{b(n)}} = k$$

- Is there any Smarandache sequence a(n) such that:

$$\lim_{n \to \infty} e^{\sum_n \frac{1}{a(n)}} \cong \pi$$

- For the four sequences of first kind a(n), study:

$$\lim_{n \to \infty} \sum_n \frac{a(n)}{R(a(n))}$$

where R(a(n)) is the reverse of a(n). (For example if a(n)=17 then R(a(n))=71 and so on).

References

[1] F. Iacobescu, Smarandache partition type and other sequences, Bull. Pure Appl. Sci. Sec. E 16(1997), No. 2, 237-240.
[2] C. Ashbacher, Smarandache Series convergence, to appear
[3] See http://www.gallup.unm.edu/~smarandache/product .txt
[4] C. Zhong, On Smarandache Continued fractions, Smarandache Notions Journal, Vol. 9, No. 1-2, 1998, 40-42
[5] D.M. Bressoud, Factorization and primality testing, Springer Verlag, 1989, p. 77
[6] M. Le and K. Wu , The primes in the Smarandache Power product Sequence, Smarandache Notions Journal, Vol. 9, No. 1-2, 1998, 97-97

On an unsolved question about the Smarandache Square-Partial-Digital Subsequence

Felice Russo
Micron Technology Italy
Avezzano (Aq) – Italy

Abstract

In this note we report the solution of an unsolved question on Smarandache Square-Partial-Digital Subsequence. We have found it by extesive computer search.
Some new questions about palindromic numbers and prime numbers in SSPDS are posed too.

Introduction

The Smarandache Square-Partial-Digital Subsequence (SSPDS) is the sequence of square integers which admits a partition for which each segment is a square integer [1],[2],[3].
The first terms of the sequence follow:

49, 144, 169, 361, 441, 1225, 1369, 1444, 1681, 1936, 3249, 4225, 4900, 11449, 12544, 14641, 15625, 16900 ...

or

7, 12, 13, 19, 21, 35, 37, 38, 41, 44, 57, 65, 70, 107, 112, 121, 125, 130, 190, 191, 204, 205, 209, 212, 223, 253 ...

reporting the value of n^2 that can be partitioned into two or more numbers that are also squares (A048653) [5].
Differently from the sequences reported in [1], [2] and [3] the proposed ones don't contain terms that admit 0 as partition. In fact as reported in [4] we don't consider the number zero a perfect square.
So, for example, the term 256036 and the term 506 respectively, are not reported in the above sequences because the partion 256/0/36 contains the number zero.
L. Widmer explored some properties of S SPDS's and posed the following question [2]:

Is there a sequence of three or more consecutive integers whose squares are in SPDS?

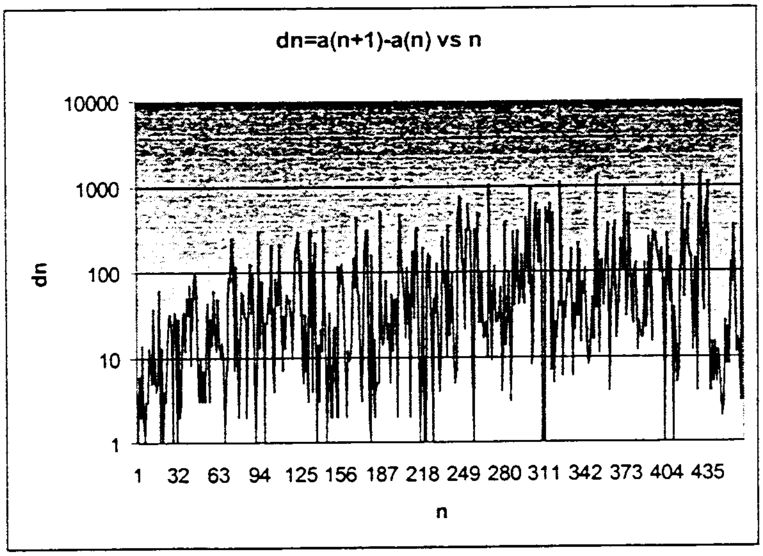
This note gives an answer to this question.

Results

A computer code has been written in Ubasic Rev. 9.
After about three week of work only a solution for three consecutive integers has been found. Those consecutive integers are: 12225, 12226,12227.

n	n^2	Partition
12225	149450625	1, 4, 9, 4, 50625
12226	149475076	1, 4, 9, 4, 75076
12227	149499529	1, 4, 9, 4, 9, 9, 529

No other three consecutive integers or more have been found for terms in SSPDS up to about 3.3E+9. Below a graph of distance dn between the terms of sequence A048653 versus n is given; in particular dn=a(n+1)-a(n) where n is the n-th term of the sequence.



According to the previous results we are enough confident to offer the following conjecture:

- ***There are no four consecutive integers whose squares are in SSPDS.***

New Questions

Starting with the sequence (A048646), reported above, the following sequence can be created [5] (A048653):

7, 13, 19, 37, 41, 107, 191, 223, 379, 487, 1093, 1201, 1301, 1907, 3019, 3371, 5081, 9041, 9721, 9907......

that we can call " Smarandache Prime-Square-Partial-Digital-Subsequence " because all the squares of these primes can be partitioned into two or more numbers that are also squares.
By looking this sequence the following questions can be posed:

1. *Are there other palindromic primes in this sequence beyond the palprime 191?*

2. *Is there at least one plandromic prime in this sequence which square is a palindromic square?*

3. *Are there in this sequence other two or more consecutive primes beyond 37 and 41?*

If we look now at the terms of the sequence A048653 we discover that two of them are very interesting:

121 and 212

Both numbers are palindromes and their squares are in SSPDS and palindromes too. In fact 121^2=14641 can be partitioned as: 1,4,64,4 and 212^2=44944 can be partitioned in five squares that are also palindromes: 4, 4, 9, 4, 4. These are the only terms found by our computer search. So the following question arises:

1. How many other SSPDS palindromes do exist ?

References

[1] Sylvester Smith, "A Set of Conjectures on Smarandache Sequences", Bulletin of Pure and Applied Sciences, (Bombay, India), Vol. 15 E (No. 1), 1996, pp. 101-107.
[2] L.Widmer, Construction of Elements of the Smarandache Square-Partial-Digital Sequence, Smarandache Notions Journal, Vol. 8, No. 1-2-3, 1997, 145-146.
[3] C. Dumitrescu and V. Seleacu, Some notions and questions in Number Theory, Erhus University Press, Glendale, Arizona, 1994
[4] E. W. Weisstein, CRC Concise Encyclopedia of Mathematics, CRC Press 1999, p. 1708
[5] N. Sloane, On-line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences

On A Conjecture By Russo

Charles Ashbacher
Charles Ashbacher Technologies
Hiawatha, Iowa USA
e-mail 71603.522@compuserve.com

The Smarandache Square-Partial-Digital Subsequence(SSPDS) is the sequence of square integers which can be partitioned so that each element of the partition is a perfect square[1][2][3]. For example, 3249 is in SSPDS since 3249 can be partitioned into $324 = 18^2$ and $9 = 3^2$.

The first terms of the sequence are:

49, 144, 169, 361, 441, 1225, 1369, 1444, 1681, 1936, 3249, 4225, 4900, 11449, 12544, 14641, ...

where the square roots are

7, 12, 13, 19, 21, 35, 37, 38, 41, 44, 57, 65, 70, 107, 112, 121, ...

this sequence is assigned the identification code A048653[4].

L. Widmer examined this sequence and posed the following question[2]:

Is there a sequence of three or more consecutive integers whose squares are in SPDS?

For the purposes of this examination, we will assume that 0 is not a perfect square. For example, the number 90 will not be considered as a number that can be partitioned into two perfect squares. Furthermore, elements of the partition are not allowed to have leading zeros. For example, 101 cannot be partitioned into perfect squares.

Russo[5] considered this question and concluded that the only additional solution to the Widmer question up to 3.3E+9 was

n	n^2	Partition
12225	149450625	1,4,9,4,50625
12226	149475076	1,4,9,4,75076
12227	149499529	1,4,9,4,9,9,529

and made the following conjecture:

There are no four consecutive integers whose squares are in SSPDS.

The purpose of this short paper is to present several additional solutions to the Widmer question as well as a counterexample to the Russo conjecture.

A computer program was written in the language Delphi Ver. 4 and run for all numbers n, where $n \leq 100{,}000{,}000$ and the following ten additional solutions were found

n	n^2	Partition
376779	141962414841	1, 4, 1, 9, 6241, 4, 841
376780	141963168400	1, 4, 196, 3168400
376781	141963921961	1, 4, 196392196,1

n	n^2	Partition
974379	949414435641	9, 4, 9, 4, 1, 4, 4356, 4, 1
974380	949416384400	9,4,9,4,16,384400
974381	949418333161	9, 4, 9, 4, 1833316, 1

n	n^2	Partition
999055	998110893025	9, 9, 81, 1089, 3025
999056	998112891136	9, 9, 81, 1, 289, 1, 1, 36
999057	998114889249	9, 9, 81, 1, 4, 889249

n	n^2	Partition
999056	998112891136	9, 9, 81, 1, 289, 1, 1, 36
999057	998114889249	9, 9, 81, 1, 4, 889249
999058	998116887364	9, 9, 81, 16, 887364

n	n^2	Partition
2000341	4001364116281	400, 1, 36, 4, 116281
2000342	4001368116964	400, 1, 36, 81, 16, 9, 64
2000343	4001372117649	400,1,3721, 1764,9

n	n^2	Partition
2063955	4259910242025	4, 25, 9, 9, 1024, 2025
2063956	4259914369936	4, 25, 9, 9, 1, 4, 36, 9, 9, 36
2063957	4259918497849	4, 25, 9, 9, 1849, 784, 9

n	n^2	Partition
2083941	4342810091481	43428100, 9, 1, 4, 81
2083942	4342814259364	434281, 4, 25, 9, 36, 4
2083943	4342818427249	434281, 842724, 9

n	n^2	Partition
4700204	22091917641616	2209, 1, 9, 1764, 16, 16
4700205	22091927042025	2209, 1, 9, 2704, 2025
4700206	22091936442436	2209, 1, 9, 36, 4, 42436

n	n^2	Partition
5500374	30254114139876	3025, 4, 1, 1, 4, 139876
5500375	30254125140625	3025, 4, 1, 25, 140625
5500376	30254136141376	3025, 4, 1, 36, 141376

n	n^2	Partition
80001024	6400163841048576	6400, 16384, 1048576
80001025	6400164001050625	6400, 1, 6400, 1050625
80001026	6400164161052676	6400, 1, 64, 16, 1052676

n	n^2	Partition
92000649	8464119416421201	8464, 1, 1, 9, 4, 16, 421201
92000650	8464119600422500	8464, 1, 19600, 4, 22500
92000651	8464119784423801	8464, 1, 1, 9, 784, 423801

Pay particular attention to the four consecutive numbers 999055, 999056, 999057 and 999058. These four numbers are a counterexample to the conjecture by Russo.

Given the frequency of three consecutive integers whose squares are in SSPDS, the following conjecture is made:

There are an infinite number of three consecutive integer sequences whose squares are in SSPDS.

In terms of larger sequences, the following conjecture also appears to be a safe one:

There is an upper limit to the length of consecutive integer sequences whose squares are in SSPDS.

We close with an unsolved question:

What is the length of the largest sequence of consecutive integers whose squares are in SSPDS?

References

1] Sylvester Smith, "A Set of Conjectures on Smarandache Sequences", **Bulletin of Pure and Applied Sciences**, (Bombay, India), Vol. 15 E (No. 1), 1996, pp. 101-107.

[2] L.Widmer, "Construction of Elements of the Smarandache Square-Partial-Digital Sequence", **Smarandache Notions Journal**, Vol. 8, No. 1-2-3, 1997, 145-146.

[3] C. Dumitrescu and V. Seleacu, **Some notions and questions in Number Theory**, Erhus University Press, Glendale, Arizona, 1994.

[4] N. Sloane, "On-line Encyclopedia of Integer Sequences", *http://www.research.att.com/~njas/sequences.*

[5] F. Russo, "On An Unsolved Question About the Smarandache Square-Partial-Digital Subsequence" *http://www.gallup.unm.edu/~smarandache/russo1.htm.*

A functional recurrence to obtain the prime numbers using the Smarandache prime function.

Sebastián Martín Ruiz. Avda de Regla, 43. Chipiona 11550Cádiz Spain.

Theorem: We are considering the function:

For n ≥ 2 , integer:

$$\mathrm{F}(n) = n + 1 + \sum_{m=n+1}^{2n} \prod_{i=n+1}^{m} \left[-E\left[-\frac{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-2}{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-1} \right] \right]$$

one has: $p_{k+1} = \mathrm{F}(p_k)$ for all $k \geq 1$ where $\{p_k\}_{k\geq 1}$ are the prime numbers and E(x) is the greatest integer less than or equal to x.

Observe that the knowledge of p_{k+1} only depends on knowledge of p_k and the knowledge of the fore primes is unnecessary.

Observe that this is a functional recurrence strictly closed too.

Proof:

Suppose that we have found a function $G(i)$ with the following property:

$$G(i) = \begin{cases} 1 & \textit{if } i \textit{ is compound} \\ 0 & \textit{if } i \textit{ is prime} \end{cases}$$

This function is called Smarandache Prime Function (Reference)

Consider the following product:

$$\prod_{i=p_k+1}^{m} G(i)$$

If $p_k < m < p_{k+1}$ $\prod_{i=p_k+1}^{m} G(i) = 1$ since $i : p_k + 1 \leq i \leq m$ are all compounds.

If $m \geq p_{k+1}$ $\prod_{i=p_k+1}^{m} G(i) = 0$ since the $G(p_{k+1}) = 0$ factor is in the product.

Here is the sum:

$$\sum_{m=p_k+1}^{2p_k} \prod_{i=p_k+1}^{m} G(i) = \sum_{m=p_k+1}^{p_{k+1}-1} \prod_{i=p_k+1}^{m} G(i) + \sum_{m=p_{k+1}}^{2p_k} \prod_{i=p_k+1}^{m} G(i) = \sum_{m=p_k+1}^{p_{k+1}-1} 1 =$$

$$= p_{k+1} - 1 - (p_k + 1) + 1 = p_{k+1} - p_k - 1$$

The second sum is zero since all products have the factor $G(p_{k+1}) = 0$.

Therefore we have the following relation of recurrence:

$$p_{k+1} = p_k + 1 + \sum_{m=p_k+1}^{2p_k} \prod_{i=p_k+1}^{m} G(i)$$

Let's now see that we can find $G(i)$ with the asked property.
Considerer:

$$(1) \quad E\left(\tfrac{i}{j}\right) - E\left(\tfrac{i-1}{j}\right) = \begin{cases} 1 & si\ j \mid i \\ 0 & si\ j \nmid i \end{cases} \qquad j = 1, 2, \ldots, i \qquad i \geq 1$$

We shall deduce this later.

We deduce of this relation:

$$d(i) = \sum_{j=1}^{i} \left(E\left(\tfrac{i}{j}\right) - E\left(\tfrac{i-1}{j}\right) \right)$$ where $d(i)$ is the number of divisors of i.

If i is prime $d(i) = 2$ therefore:

$$-E\left[-\tfrac{d(i)-2}{d(i)-1}\right] = 0$$

If i is compound $d(i) > 2$ therefore:

$$0 < \tfrac{d(i)-2}{d(i)-1} < 1 \Rightarrow -E\left[-\tfrac{d(i)-2}{d(i)-1}\right] = 1$$

Therefore we have obtained the function $G(i)$ which is:

$$G(i) = -E\left[-\frac{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-2}{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-1} \right] \qquad i \geq 2 \text{ integer}$$

To finish the demonstration of the theorem it is necessary to prove (1)

If $j = 1 \quad j \mid i \quad E\left(\frac{i}{j}\right) - E\left(\frac{i-1}{j}\right) = i - (i-1) = 1$

If $j > 1$

$$i = jE(\tfrac{i}{j}) + r \quad 0 \leq r < j$$
$$i - 1 = jE(\tfrac{i-1}{j}) + s \quad 0 \leq s < j$$

If $j \mid i \Rightarrow r = 0 \Rightarrow jE(\frac{i}{j}) = jE(\frac{i-1}{j}) + s + 1 \Rightarrow \left.\begin{matrix} j \mid s+1 \\ s+1 \leq j \end{matrix}\right\} \Rightarrow j = s+1$

$\Rightarrow jE(\frac{i}{j}) = jE(\frac{i-1}{j}) + j \Rightarrow E(\frac{i}{j}) = E(\frac{i-1}{j}) + 1$

If $j \nmid i \Rightarrow r > 0 \Rightarrow 0 = j(E(\frac{i}{j}) - E(\frac{i-1}{j})) + (r-s) + 1 \Rightarrow j \mid r - s + 1$

Therefore $r - s + 1 = 0$ or $r - s + 1 = j$

If $s \neq 0 \Rightarrow r - s < j - 1 \Rightarrow r - s + 1 = 0 \Rightarrow E(\frac{i}{j}) = E(\frac{i-1}{j})$

If $s = 0 \Rightarrow j \mid i - 1 \Rightarrow E(\frac{i}{j}) = E(\frac{i-1}{j} + \frac{1}{j}) = \frac{i-1}{j} = E(\frac{i-1}{j})$

With this, the theorem is already proved.

Reference:

[1] E. Burton, "Smarandache Prime and Coprime Functions",
http://www.gallup.unm.edu/~smarandache/primfnct.txt
[2] F. Smarandache, "Collected Papers", Vol. II, 200 p., p. 137,
Kishinev University Press, Kishinev, 1997.

The general term of the prime number sequence and the Smarandache prime function.

Sebastián Martín Ruiz. Avda de Regla, 43 Chipiona 11550 Cádiz Spain.

Let 's consider the function $d(i)$ = number of divisors of the positive integer number i. We have found the following expression for this function:

$$d(i) = \sum_{k=1}^{i} E\left(\tfrac{i}{k}\right) - E\left(\tfrac{i-1}{k}\right)$$

We proved this expression in the article "A functional recurrence to obtain the prime numbers using the Smarandache Prime Function".

We deduce that the folowing function:

$$G(i) = -E\left[-\tfrac{d(i)-2}{i}\right]$$

This function is called the Smarandache Prime Function (Reference)
It takes the next values:

$$G(i) = \begin{cases} 0 \ \ if \quad i \ \ is \ \ prime \\ 1 \ \ if \ i \ \ is \ \ compound \end{cases}$$

Let is consider now $\pi(n)$ =number of prime numbers smaller or equal than n.
It is simple to prove that:

$$\pi(n) = \sum_{i=2}^{n}(1 - G(i))$$

Let is have too:

$$\begin{array}{l} If \ \ 1 \le k \le p_n - 1 \ \Rightarrow \ E\left(\frac{\pi(k)}{n}\right) = 0 \\ If \ \ C_n \ge k \ge p_n \ \Rightarrow \ E\left(\frac{\pi(k)}{n}\right) = 1 \end{array}$$

We will see what conditions have to carry C_n.

Therefore we have te following expression for p_n n-th prime number:

$$p_n = 1 + \sum_{k=1}^{C_n}\left(1 - E\left(\tfrac{\pi(k)}{n}\right)\right)$$

If we obtain C_n that only depends on n, this expression will be the general term of the prime numbers sequence, since π is in function with G and G does with $d(i)$ that is expressed in function with i too. Therefore the expression only depends on n.

E[x]=The highest integer equal or less than n

Let is consider $C_n = 2(E(n \log n) + 1)$
Since $p_n \sim n \log n$ from of a certain n_0 it will be true that

$$(1) \quad p_n \leq 2(E(n \log n) + 1)$$

If n_0 it is not too big, we can prove that the inequality is true for smaller or equal values than n_0.

It is necessary to that:

$$E\left[\frac{\pi(2(E(n \log n)+1))}{n}\right] = 1$$

If we check the inequality:

$$(2) \quad \pi(2(E(n \log n) + 1)) < 2n$$

We will obtain that:

$$\frac{\pi(C_n)}{n} < 2 \Rightarrow E\left[\frac{\pi(C_n)}{n}\right] \leq 1 \quad ; \quad C_n \geq p_n \Rightarrow E\left[\frac{\pi(C_n)}{n}\right] = 1$$

We can experimentaly check this last inequality saying that it checks for a lot of values and the difference tends to increase, wich makes to think that it is true for all n.

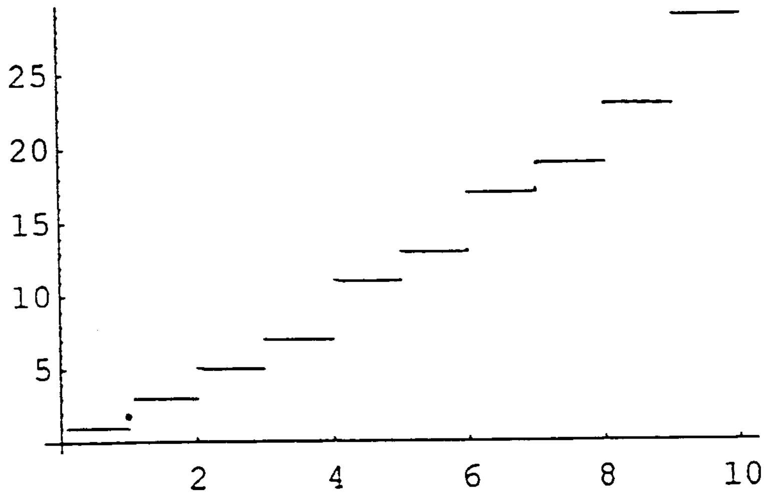
Therefore if we prove that the next inequalities are true:

$$(1) \quad p_n \leq 2(E(n \log n) + 1)$$
$$(2) \quad \pi(2(E(n \log n) + 1)) < 2n$$

which seems to be very probable; we will have that the general term of the prime numbers sequence is:

$$p_n = 1 + \sum_{k=1}^{2(E(n \log n)+1)} \left[1 - E\left[\frac{\sum_{j=2}^{k} \left[1 + E\left[-\frac{\sum_{s=1}^{j} (E(j/s) - E((j-1)/s)) - 2}{j} \right] \right]}{n} \right] \right]$$

If now we consider the general term defined in the same way but for all real number greater than zero the following grafic is obtained:



Let is observe that if $0<x<1$ P(x)=1 si x=1 P(x)=2 and for $n-1<x\leq n$ P(x)= p_n

Reference:

[1] E. Burton, "Smarandache Prime and Coprime Functions" Http://www.gallup.unm.edu/~Smarandache/primfnct.txt

[2] F. Smarandache, "Collected Papers", Vol. II, 200 p.,p.137, Kishinev University Press.

Expressions of the Smarandache Coprime Function

Sebastian Martin Ruiz, Avda de Regla, 43 Chipiona, 11550 Cadiz, Spain

Smarandache Coprime Function is defined this way:

$$C_k(n_1, n_2, \cdots, n_k) = \begin{cases} 0 & if \quad n_1, n_2, \cdots, n_k \; are\ coprime\ numbers \\ 1 & otherwise \end{cases}$$

We see two expressions of the Smarandache Coprime Function for k=2.
EXPRESSION 1:

$$C_2(n_1, n_2) = -E\left[-\frac{n_1 n_2 - lcm(n_1, n_2)}{n_1 n_2}\right]$$

$E(x)$ = the biggest integer number smaller or equal than x.
If n_1, n_2 are coprime numbers:

$$lcm(n_1, n_2) = n_1 n_2 \quad \Rightarrow C_2(n_1, n_2) = -E\left[\frac{0}{n_1 n_2}\right] = 0$$

If n_1, n_2 aren't coprime numbers:

$$lcm(n_1, n_2) < n_1 n_2 \quad \Rightarrow 0 < \frac{n_1 n_2 - lcm(n_1, n_2)}{n_1 n_2} < 1 \Rightarrow C_2(n_1, n_2) = 1$$

EXPRESSION 2:

$$C_2(n_1,n_2) = 1 - E\left[- \frac{\displaystyle\prod_{\substack{d \mid n_1 \\ d>1}} \prod_{\substack{d' \mid n_2 \\ d'>1}} \mid d-d' \mid}{\displaystyle\prod_{d|n_1} \prod_{d|n_2} (d+d')} \right]$$

If n_1, n_2 are coprime numbers then $d \neq d' \quad \forall\, d, d' \neq 1$

$$\Rightarrow 0 < \frac{\displaystyle\prod_{\substack{d \mid n_1 \\ d>1}} \prod_{\substack{d' \mid n_2 \\ d'>1}} \mid d-d' \mid}{\displaystyle\prod_{d|n_1} \prod_{d|n_2} (d+d')} < 1 \Rightarrow C_2(n_1,n_2) = 0$$

If n_1, n_2 aren't coprime numbers $\exists d = d' \quad d > 1, d' > 1 \Rightarrow C_2(n_1,n_2) = 1$
Smarandache coprime function for $k \geq 2$.

$$C_k(n_1,n_2,\cdots,n_k) = -E\left[\frac{1}{GCD(n_1,n_2,\cdots,n_k)} - 1\right]$$

If $n_1, n_2, \cdots, n_k$ are coprime numbers:

$$GCD(n_1,n_2,\cdots,n_k) = 1 \Rightarrow C_k(n_1,n_2,\cdots,n_k) = 0$$

If $n_1, n_2, \cdots, n_k$ aren't coprime numbers: $GCD(n_1,n_2,\cdots,n_k) > 1$

$$0 < \frac{1}{GCD} < 1 \Rightarrow -E\left[\frac{1}{GCD} - 1\right] = 1 = C_k(n_1,n_2,\cdots,n_k)$$

References:

1. E. Burton, "Smarandache Prime and Coprime Functions"
2. F. Smarandache, "Collected Papers", Vol. II, 200 p.,p. 137, Kishinev University Press.

NOTE ON THE DIOPHANTINE EQUATION $2x^2 - 3y^2 = p$

Mladen V. Vassilev - Missana and Krassimir T. Atanassov

The solving of the Diophantine equation

$$2x^2 - 3y^2 = 5 \tag{1}$$

i.e.,

$$2x^2 - 3y^2 - 5 = 0$$

was put as an open Problem 78 by F. Smarandache in [1]. Below this problem is solved completely. Also, we consider here the Diophantive equation

$$l^2 - 6m^2 = -5, \tag{2}$$

i.e.,

$$l^2 - 6m^2 + 5 = 0$$

and the Pellian equation

$$u^2 - 6v^2 = 1, \tag{3}$$

i.e.,

$$u^2 - 6v^2 - 1 = 0.$$

Here we use variables x and y only for equation (1) and l, m for equation (2).

We will need the following denotations and definitions:

$$\mathcal{N} = \{1, 2, 3, ...\};$$

if

$$F(t, w) = 0$$

is an Diophantive equation, then:

(a_1) we use the denotation $< t, w >$ if and only if (or briefly: iff) t and w are integers which satisfy this equation.

(a_2) we use the denotation $< t, w > \in \mathcal{N}^2$ iff t and w are positive integers;

$K(t, w)$ denotes the set of all $< t, w >$;

$K^o(t, w)$ denotes the set of all $< t, w > \in \mathcal{N}^2$;

$K'(t, w) = K^o(t, w) - \{< 2, 1 >\}$.

LEMMA 1: If $< t, w > \in \mathcal{N}^2$ and $< x, y > \neq < 2, 1 >$, then there exists $< l, m >$, such that $< l, m > \in \mathcal{N}^2$ and the equalities

$$x = l + 3m \text{ and } y = l + 2m \tag{4}$$

hold.

LEMMA 2: Let $< l, m > \in \mathcal{N}^2$. If x and y are given by (1), then x and y satisfy (4) and $< x, y > \in \mathcal{N}^2$.

We shall note that lemmas 1 and 2 show that the map $\varphi : K^0(l,m) \rightarrow K'(x,y)$ given by (4) is a bijection.

Proof of Lemma 1: Let $< x, y > \in \mathcal{N}^2$ be chosen arbitrarily, but $< x, y > \neq < 2, 1 >$. Then $y \geq 2$ and $x > y$. Therefore,

$$x = y + m \tag{5}$$

and m is a positive integer. Subtracting (5) into (1), we obtain

$$y^2 - 4my + 5 - 2m^2 = 0. \tag{6}$$

Hence

$$y = y_{1,2} = 2m \pm \sqrt{6m^2 - 5}. \tag{7}$$

For $m = 1$ (7) yields only

$$y = y_1 = 3.$$

indeed

$$1 = y = y_2 < 2$$

contradicts to $y \geq 2$.

Let $m > 1$. Then

$$2m - \sqrt{6m^2 - 5} < 0.$$

Therefore $y = y_2$ is impossible again. Thus we always have

$$y = y_1 = 2m + \sqrt{6m^2 - 5}. \tag{8}$$

Hence

$$y - 2m = \sqrt{6m^2 - 5}. \tag{9}$$

The left-hand side of (9) is a positive integer. Therefore, there exists a positive integer l such that

$$6m^2 - 5 = l^2.$$

Hence l and m satisfy (2) and $< l, m > \in \mathcal{N}^2$.

The equalities (4) hold because of (5) and (8). $\diamond$

Proof of Lemma 2: Let $< l, m > \in \mathcal{N}^2$. Then we check the equality

$$2(l + 3m)^2 - 3(l + 2m)^2 = 5,$$

under the assumption of validity of (2) and the lemma is proved. $\diamond$

Theorem 108 a, Theorem 109 and Theorem 110 from [2] imply the following

THEOREM 1: There exist sets $K_i(l,m)$ such that

$$K_i(l,m) \subset K(l,m) \quad (i = 1, 2),$$

$$K_1(l,m) \cap K_2(l,m) = \emptyset,$$

and $K(l,m)$ admits the representation

$$K(l,m) = K_1(l,m) \cup K_2(l,m).$$

The fundamental solution of $K_1(l,m)$ is $< -1, 1 >$ and the fundamental solution of $K_2(l,m)$ is $< 1, 1 >$.

Moreover, if $< u, v >$ runs $K(u,v)$, then:

(b_1) $< l, m >$ runs $K_1(l,m)$ iff the equality

$$l + m\sqrt{6} = (-1 + \sqrt{6})(u + v\sqrt{6}) \tag{10}$$

holds;

(b_2) $< l, m >$ runs $K_2(l,m)$ iff the equality

$$l + m\sqrt{6} = (1 + \sqrt{6})(u + v\sqrt{6}) \tag{11}$$

holds.

We must note that the fundamental solution of (3) is $< 5, 2 >$. Let u_n and v_n be given by

$$u_n + v_n\sqrt{6} = (5 + 2\sqrt{6})^n \quad (n \in \mathcal{N}. \tag{12}$$

Then u_n and v_n satisfy (11) and $< u_n, v_n > \in \mathcal{N}^2$. Moreover, if n runs $\mathcal{N}$, then $< u_n, v_n >$ runs $K^o(u,v)$.

Let the sets $K_i^o(l,m)$ $(i = 1,2)$ are introduced by

$$K_i^o(l,m) = K_i(l,m) \cap \mathcal{N}^2. \tag{13}$$

As a corollary from the above remark and Theorem 1 we obtain

THEOREM 2: The set $K^o(l,m)$ may be represented as

$$K^o(l,m) = K_1^o(l,m) \cup K_2^o(l,m), \tag{14}$$

where

$$K_1^o(l,m) \cap K_2^o(l,m) = \emptyset. \tag{15}$$

Moreover:

(c_1) If n runs $\mathcal{N}$ and the integers l_n and m_n are defined by

$$l_n + m_n\sqrt{6} = (-1 + \sqrt{6})(5 + 2\sqrt{6})^n, \tag{16}$$

then l_n and m_n satisfy (2) and $< l_n, m_n >$ runs $K_1^o(l,m)$;

(c_2) If n runs $\mathcal{N} \cup \{0\}$ and the integers l_n and m_n are defined by

$$l_n + m_n\sqrt{6} = (1 + \sqrt{6})(5 + 2\sqrt{6})^n, \tag{17}$$

then l_n and m_n satisfy (2) and $< l_n, m_n >$ runs $K_2^o(l,m)$.

Let φ be the above mentioned bijection. The sets $K_i'^o(x,y)$ $(i=1,2)$ are introduced by

$$K_i'^o(x,y) = \varphi(K_i^o(l,m)). \tag{18}$$

From Theorem 2, and especially from (14), (15), and (18) we obtain

THEOREM 3: The set $K'^o(x,y)$ may have the representation

$$K'^o(x,y) = K_1^o(x,y) \cup K_2^o(x,y), \tag{19}$$

where

$$K_1^o(x,y) \cap K_2^o(x,y) = \emptyset. \tag{20}$$

Moreover:

(d_1) If n runs $\mathcal{N}$ and the integers x_n and y_n are defined by

$$x_n = l_n + 3m_n \text{ and } y_n = l_n + 2m_n, \tag{21}$$

where l_n and m_n are introduced by (16), then x_n and y_n satisfy (1) and $< x_n, y_n >$ runs $K_1^o(x,y)$;

(d_2) If n runs $\mathcal{N} \cup \{0\}$ and the integers x_n and y_n are defined again by (21), but l_n and m_n now are introduced by (17), then x_n and y_n satisfy (1) and $< x_n, y_n >$ runs $K_2^o(x,y)$.

Theorem 3 completely solves F. Smarandache's Problem 78 from [1], because l_n and m_n could be expressed in explicit form using (16) or (17) as well.

*

* *

Below we shall introduce a generalization of Smarandache's problem 87 from [1].

If we have to consider the Diophantine equation

$$2x^2 - 3y^2 = p, \tag{22}$$

where $p \neq 2$ is a prime number, then using [2, Ch. VII, exercize 2] and the same method as in the case of (1), we obtain the following result.

THEOREM 4: (1) The necessary and sufficient condition for the solvability of (22) is:

$$p \equiv 5(mod24) \quad \text{or} \quad p \equiv 23(mod24) \qquad (23);$$

(2) If (23) is valid, then there exists exactly one solution $< x, y > \in \mathcal{N}^2$ of (22) such that the inequalities $x < \sqrt{\frac{3}{2}.p}; y < \sqrt{\frac{2}{3}.p}$ hold. Every other solution $< x, y > \in \mathcal{N}^2$ of (22) has the form:

$$x = l + 3m$$

$$y = l + 2m,$$

where $< l, m > \in \mathcal{N}^2$ is a solution of the Diophantine equation

$$l^2 - 6m^2 = -p.$$

The question how to solve the Diophantine equation, a special case of which is the above one, is considered in Theorem 110 from [2].

REFERENCES:

[1] F. Smarandache, Proposed Problems of Mathematics. Vol. 2, U.S.M., Chişinău, 1997.

[2] T. Nagell, Introduction to Number Theory. John Wiley & Sons, Inc., New York, 1950.

Mladen V. Vassilev - Missana
5, Victor Hugo Str., Sofia-1124, Bulgaria
MRL, P.O.Box 12, Sofia-1113, Bulgaria

Krassimir T. Atanassov
MRL, P.O.Box 12, Sofia-1113, Bulgaria
CLBME - BAS, Acad. G. Bonchev Str., Bl. 105, Sofia-1113
e-mail: krat@bgcict.acad.bg

REMARK ON THE 62-th SMARANDACHE'S PROBLEM

Hristo Aladjov and Krassimir Atanassov

In [1] Florian Smarandache formulated 105 unsolved problems.

The 62-th problem is the following:

Let $1 \leq a_1 < a_2 < \ldots$ be an infinite sequence of integers such that any three members do not constitute an arithmetic progression. Is it true that always

$$\sum_{n \geq 1} \frac{1}{a_n} \leq 2?$$

In [2-4] some counterexamples are given.

Easily it can be seen that the set of numbers $\{1, 2, 4, 5, 10\}$ does not contain three numbers which are members of an arithmetic progression. On the other hand

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{10} = 2\frac{1}{20} > 2.$$

Therefore, Smarandache's problem is not true in the present form, because the sum of the members of every one sequence with the above property and with first members 1, 2, 4, 5, 10 will be bigger than 2.

Some modifications of the above problem are discussed in [3,4].

We can construct the sequence which contains the minimal possible members, satisfying the Smarandache's property. The first 100 members of this sequence are:

$$1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41, 82, 83, 85, 86, 91, 92, 94, 95, 109, 110, 112,$$
$$113, 118, 119, 121, 122, 244, 245, 247, 248, 253, 254, 256, 257, 271, 272, 274, 275, 280, 281, 283,$$
$$284, 325, 326, 328, 329, 334, 335, 337, 338, 352, 353, 355, 356, 361, 362, 364, 365, 730, 731, 733,$$
$$734, 739, 740, 742, 743, 757, 758, 760, 761, 766, 767, 769, 770, 811, 812, 814, 815, 820, 821, 823,$$
$$824, 838, 839, 841, 842, 847, 848, 850, 851, 973, 974, 976, 977$$

In another paper the properties of this sequence will be discussed in details. Some of them are given in [3,4].

We must note that it was checked by a computer that the sum of the first 18567 members of the sequence (the 18567-th member is 4962316) is 3.00000013901583..., i.e. for this sequence

$$\sum_{n \geq 1} \frac{1}{a_n} > 3.$$

It can be easily seen that if the first member of the sequence satisfying the Smarandache's property is not 1, or if its second member is not 2, then

$$\sum_{n \geq 1} \frac{1}{a_n} < 3.$$

On the other hand, there are an infinite number of sequences for which

$$\sum_{n\geq 1} \frac{1}{a_n} > 2,$$

because, for example, all sequences (their number is, obviously, infinite) generated by the above one without only one of its members will satisfy the last inequality.

This number will be discussed in the next paper of ours, too.

Now we shall cite the following unsolved problem from [2]:

Given a sequence of integers $a_1 \leq a_2 \leq ... \leq a_k \leq ...$ where no three form an arithmetic progression, is there any bound on the sum

$$\sum_{n\geq 1} \frac{1}{a_n}?$$

From the above remark it follows that 3 is a bound of all sequences with the above property without the first sequence shown above. Some properties of this bound also will be discussed in the next paper of ours.

REFERENCES:

[1] F. Smarandache, Only problems, not solutions!. Xiquan Publ. House, Chicago, 1993.

[2] C. Ashbacher, Pluckings from the Tree of Smarandache Sequences and Functions. American Research Press, Lupton, 1998.

[3] K. Atanassov, On the 62-nd Smarandache's problem. Notes on Number Theory and Discrete Mathematics, Vol. 5 (1999), No. 3, 100-101.

[4] K. Atanassov, On Some of the Smarandache's Problems. American Research Press, Lupton, 1999.

Hristo Aladjov and Krassimir Atanassov
CLBME - Bulgarian Academy of Sciences
Acad. G. Bonchev Str., 105 Block,
Sofia-1113, Bulgaria
e-mails: {aladjov,krat}@bgcict.acad.bg

The Integral Values of $\log_{k^n} S(n^k)$

Maohua Le

Abstract: Let k, n be distinct positive integers. In this paper, we prove that $\log_{k^n} S(n^k)$ is never a positive integer.

Key words: Smarandache function, logarithm, integral value.

For any positive integer a, let S(a) denote the Smarandache function of a. In [2, Problem 22], Muller posed the following problem:

Problem: Is it possible to find two distinct positive integers k and n such that $\log_{k^n} S(n^k)$ is a positive integer?

In this paper, we completely solve the above problem as follows:

Theorem: For any distinct positive integers k and n, $\log_{k^n} S(n^k)$ is never a positive integer.

Proof: If $\log_{k^n} S(n^k)$ is a positive integer, then we have $k > 1$, $n > 1$ and

(1) $\log_{k^n} S(n^k) = m,$

where m is a positive integer. By (1), we get

(2) $S(n^k) = k^{nm}$.

By (1), we have

(3) $S(n^k) = S(n^{k-1} * n) \leq S(n^{k-1}) + S(n) \leq \ldots kS(n).$

Therefore, by (2) and (3), we get

(4) $k^{nm} \leq kS(n) \leq kn.$

If $k > n > 1$, then from (4) we obtain

(5) $k^2 \leq k^n \leq k^{nm} \leq kn \leq k(k-1) < k^2$

a contradiction. If $n > k > 1$, then we have

(6) $2^n \leq k^n \leq k^{nm} \leq kn \leq (n-1)n.$

It is impossible, since $n \geq 3$. Thus, the theorem is proved.

References

[1] M. H. Le, "An Inequality Concerning the Smarandache Function", **Smarandache Notions Journal**, 9(1998), 124-125.
[2] R. Muller, **Unsolved Problems Related to Smarandache Function**, Number Theory Publishing Company, 1997.

Maohua Le
Department of Mathematics
Zhanjiang Normal College
Zhanjiang, Guangdong
P.R. China

On the Functional Equation $S(n)^2 + S(n) = kn$

Rongi Chen and Maohua Le

Abstract

For any positive integer a, let S(a) denote the Smarandache function of a. In this paper, we prove that the equation $S(n)^2 + S(n) = kn$ has infinitely many positive integer solutions for every positive integer k. Moreover, the size of the number of solutions does not depend on the parity of k.

Key Words: Smarandache function, functional equation, number of solutions.

1. Introduction

Let N be the set of positive integers. For any positive integer a, let

(1) $S(a) = \min \{ r \mid r \in N, a \mid r! \}$.

Then S(a) is called the Smarandache function of a. Let k be a fixed positive integer. In this paper we deal with the equation

(2) $S(n)^2 + S(n) = kn, n \in N$.

For any positive integer x, let N(k,x) denote the number of solutions n with $n \leq x$, and let N(k) denote the number of all solutions n of (2). A computer search showed that $N(1, 10^4) = 23$, $N(2, 10^4) = 33$, $N(3, 10^4) = 20$, $N(4,10^4) = 24$, $N(5,10^4) = 11$ and $N(6, 10^4) = 26$. In [1] Ashbacher posed the following questions:

Question 1: Is $N(k) = \infty$ for k = 1, 2, 3, 4, 5 or 6?
Question 2: Is there a positive integer k for which N(k) = 0?
Question 3: Is there a largest positive integer for which N(k) > 0?
Question 4: Is there more solutions n when k is even than when k is odd?

In this paper, we completely solve the above-mentioned questions. In fact, we prove a general result as follows:

Theorem: The positive integer n is a solution of (2) if and only one of the following conditions is satisfied.

(i) n = 1 for k = 2.
(ii) n = 4 for k = 5.
(iii) $n = p(p+1)$ for k = 1, where p is a prime with p > 3.
(iv) $n = p(p+1)/k$ for k > 1, where p is a prime with $p \equiv -1 \pmod{k}$.

Corollary 1: As $x \to \infty$, we have

$$N(k,x) \sim 2\sqrt{(kx)} \,/\, (\varphi(k)\log(kx)).$$

Corollary 2: For any positive integers k_1 and k_2, we have

$$\frac{N(k_1)}{N(k_2)} = \frac{\varphi(k_2)}{\varphi(k_1)} \sqrt{(k_1/k_2)}$$

By our results, we observe that (2) has infinitely many solutions n for every positive integer k. Moreover, the size of N(k,x) does not depend on the parity of k.

2. Preliminaries

Lemma 1: For any positive integers u and v, we have $S(u) \le S(uv)$.

Proof: Let $a = S(u)$ and $b = S(uv)$. By (1), a and b are least positive integers satisfying $u \mid a!$ and $uv \mid b!$ respectively. So we have $a \le b$. The lemma is proved.

Lemma 2: For any positive integer u with $u > 1$, there exists a prime factor d such that $d \mid S(u)$.

Proof: Let $u = p_1^{t1} p_2^{t2} \ldots p_k^{tk}$ be the prime factorization of u. Then, by [2], we have

$$S(u) = \max\left(S(p_1^{t1}), S(p_2^{t2}), \ldots, S(p_k^{tk}) \right)$$

and $p_i \mid S(p_i^{ti})$ for $i = 1, 2, 3, \ldots, k$. This proves the lemma.

Lemma 3: For any positive integer u, we have

$$S(u) \begin{cases} = u, & \text{if } u = 1, 4 \text{ or } p, \text{ where p is a prime.} \\ \le u/2, & \text{otherwise.} \end{cases}$$

Proof: See [4].

Lemma 4: For any coprime positive integers, u and v, we have $S(uv) = \max(S(u), S(v))$.

Proof: Let $a = S(u)$, $b = S(v)$ and $c = S(uv)$. By (1), a, b and c are least positive integers satisfying $u \mid a!$, $v \mid b!$ and $uv \mid c!$ respectively. This implies that $c \ge \max(a,b)$.
If $a \ge b$, then we have $u \mid a!$ and $v \mid a!$. Since $\gcd(u,v) = 1$, we get $uv \mid a!$. So we have $a \ge c$. This implies that $c = a = \max(a,b)$. By the same method, we can prove that if $a \le b$, then $c = b = \max(a,b)$. The lemma is proved.

Lemma 5: For any positive number x, let $\Pi(x)$ denote the number of primes p with $p \le x$. As $x \to \infty$, we have $\Pi(x) \sim x/\log x$.

Proof: See [3].

Lemma 6: Let a,b be integers satisfying $a > 1$ and $\gcd(a,b) = 1$. For any positive number x, let $\Pi(x;a,b)$ denote the number of primes p such that $p \le x$ and $p \equiv b \pmod a$. As $x \to \infty$, we have $\Pi(x;a,b) \sim x/\varphi(a)\log x$, where $\varphi(a)$ is the Euler function of a.

Proof: See [5].

3. Proofs

Proof of Theorem: Clearly, if n satisfy (i) or (ii), then it is a solution of (2). If n satisfy (iii), then $n = p(p+1)$, where p is a prime with $p > 3$. Since $\gcd(p,p+1) = 1$, by Lemma 4, we get

(5) $$S(n) = S(p(p+1)) = \max(S(p), S(p+1)).$$

Further, since $p+1 \geq 6$ is not a prime, by Lemma 3, we get $S(p+1) \leq (p+1)/2 < p$. Hence, we see from (5) that $S(n) = S(p) = p$. It implies that $S(n)^2 + S(n) = p^2 + p = n$ and n is a solution of (2) for $k = 1$. By the same method, we can prove that if n satisfy the condition (iv), then it is a solution of (2) for $k > 1$. Thus, the sufficient condition of our theorem is proved.

We now prove the necessary condition. Let n be a solution of (2), and let $t = S(n)$. We get from (2) that

$$(6) \qquad t(t+1) = kn.$$

If $n = 1$ or 4, then $t = 1$ or 4, and n is a solution of (2) for $k = 2$ or 5. From below, we may assume that $n \neq 1$ or 4. Since $\gcd(t,t+1) = 1$, by Lemma 4, we get from (6) that

$$(7) \qquad S(kn) = S(t(t+1)) = \max(S(t),S(t+1)).$$

If $S(t) \leq S(t+1)$, then from (7) we get

$$(8) \qquad S(kn) = S(t+1).$$

By Lemma 1, we have $S(kn) \geq S(n) = t$. Hence, by (8) we obtain

$$(9) \qquad S(t+1) \geq t.$$

Since $n \neq 1$ or 4, by Lemma 3, we see from (9) that either $t = 3$ or $t = p-1$, where p is a prime. When $t = 3$, we get $n = 3$ or 6. Then n satisfies the condition (iv). When $t = p-1$, we have $S(n) = p-1$ and

$$(10) \qquad S(kn) = p,$$

by (8). Since p is a prime, by Lemma 2, we see from (10) that $p \mid kn$. If $p \mid k$, then k/p is a positive integer and $t = p - 1 = kn/p$ by (6). However, by Lemmas 1 and 3, it implies that

$p - 1 > S(p - 1) = S(kn/p) \geq S(n) = t = p - 1$, a contradiction.

If $S(t) > S(t+1)$, then from (17) we get

$$(11) \qquad S(kn) = S(t).$$

Since $S(kn) \geq S(n) = t$, by Lemmas 1 and 3, we see from (11) that $S(t) = t$. Since $n \neq 1$ or 4, by Lemma 4, we get $t = p$, where p is a prime. Hence, by (6), we obtain

$$(12) \qquad p(p+1) = kn.$$

Further, since $S(n) = p$, by Lemma 2, we have $p \mid n$ and n/p is a positive integer. Then, by (12) we get $p \equiv -1 \pmod k$. Furthermore, since $n \neq 4$, we get from (12) that $p > 3$, for $k = 1$. This implies that n satisfies the condition (iii) of (iv). Thus, the theorem is proved.

Proof of Corollaries 1 and 2. Let $\Pi(x)$ and $\Pi(x;a,b)$ be defined as in Lemmas 5 and 6 respectively. By Theorem, we have

$$N(k,x) = \begin{cases} \Pi\left(\sqrt{(x+1/4)} - 1/2\right) - 2, & \text{if } k = 1, \\ \Pi\left(\sqrt{(2x+1/4)} - 1/2\right), & \text{if } k = 2, \\ \Pi\left(\sqrt{(5x+1/4)} - 1/2; 5,-1\right) + 1, & \text{if } k = 5, \\ \Pi\left(\sqrt{(kx+1/4)} - 1/2; k,-1\right), & \text{otherwise.} \end{cases}$$

Therefore, by Lemmas 5 and 6, we get the corollaries immediately.

References

1. C. Ashbacher, "Unsolved Problems", **Smarandache Notions J.**, 9(1998), 152-155.
2. C. Dumitrescu and V. Seleacu, **Some Notions and Questions in Number Theory**, Erhus Univ. Press, Glendale, 1994.
3. J. Hadamard, "Sur la distribution des zeros de la fonction $\zeta(s)$ et des consequences arithmetiques", **Bull. Soc. Math. France**, 24 (1896), 199-220.
4. M. H. Le, "On the Diophantine Equation S(n) = n", **Smarandache Notions J.**, 10(1999), 142-143.
5. A. Page, "On the Number of Primes In An Arithmetic Progression", **Proc. Math. Soc.**, (2), 39(1935), 116-141.

Rongi Chen
Department of Mathematics
Maoming Educational College
Maoming, Guangdong
P. R. China

Maohua Le
Department of Mathematics
Zhanjiang Normal College
Zhanjiang, Guangdong
P. R. China

On The Functional Equation $Z(n) + \varphi(n) = d(n)$

Zhong Li and Maohua Le

Abstract: For any positive integer n, let d(n), $\varphi(n)$ and Z(n) denote the divisor function, the Euler function and the pseudo-Smarandache function of n respectively. In this paper, we prove that the functional equation $Z(n) + \varphi(n) = d(n)$ has no solution n.

Key words: divisor function, Euler function, pseudo-Smarandache function.

Let N be the set of all positive integers. For any positive integer n, let

$$(1) \qquad d(n) = \sum_{d|n} 1,$$

$$(2) \qquad \varphi(n) = \sum_{\substack{1 \le m \le n \\ \gcd(m,n)=1}} 1,$$

$$(3) \qquad Z(n) = \min \left\{ a \mid a \in N, n \mid \sum_{j=1}^{a} j \right\}.$$

Then d(n), $\varphi(n)$ and Z(n) are called the divisor function, the Euler function and the Pseudo-Smarandache function of n respectively. In [1], Ashbacher posed the following unsolved question:

Question: How many solutions n are there to the functional equation

$$(4) \qquad Z(n) + \varphi(n) = d(n), \quad n \in N\,?$$

In this paper, we completely solve the above-mentioned question as follows:

Theorem: The equation $Z(n) + \varphi(n) = d(n)$, $n \in N$ has no solution.

Proof: Let n be a solution of (4). A computer search showed that (4) has no solution with $n \le 10000$ (see [1]). So we have $n > 10000$. Let

$$(5) \qquad n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$$

be the prime factorization of n. By [2, theorems 62 and 273], we see from (1), (2) and (5) that

$$(6) \qquad d(n) = (r_1 + 1)(r_2 + 1) \dots (r_k + 1)$$

$$(7) \qquad \varphi(n) = n \prod_{i=1}^{k} (1 - 1/p_i)$$

On the other hand, it is a well-known fact that

(8) $n \mid \frac{1}{2} Z(n)(Z(n)+1)$

(see [1]). From (8) we get

$$Z(n) \geq \sqrt{(2n + \tfrac{1}{4} - \tfrac{1}{2})}.$$

Therefore, by (4), (5), (6), (7) and (9), we obtain

(10) $1 \geq f(n) + g(n)$

where

(11) $$f(n) = \prod_{i=1}^{k} (1 - 1/p_i)(p^{r_i}/(r_i+1)),$$

(12) $$g(n) = \sqrt{2 \prod_{i=1}^{k} (p_i^{r_i/2}/(r_i+1))} - \tfrac{1}{2} \prod_{i=1}^{k} 1/(r_i+1).$$

Clearly, we see from (12) that $g(n) > 0$ for any positive integer n with $n > 1$. Hence, we get from (10) that

(13) $f(n) < 1$.

If $k = 1$, then $n = p_1^{r_1}$ and $Z(n) \geq p_1^{r_1} - 1$ by (3). Hence, by (1) and (2), n is not a solution of (4). This implies that $k \geq 2$.

If $k \geq 3$, then $p_k \geq 5$ and $f(n) \geq 1$, by (11). This contradicts with (13). So we have $k = 2$. Then (11) can be written as

(14) $$f(n) = (1 - 1/p_1)(1 - 1/p_2)\left((p_1^{r_1} p_2^{r_2})/((r_1+1)(r_2+1))\right).$$

If $p_2 > 3$, then from (14) we get $f(n) \geq 1$, a contradiction. Hence, we deduce that $p_1 = 2$ and $p_2 = 3$. Then, by (13) and (14), we obtain

(15) $$f(n) = (2^{r_1} 3^{r_2})/(3(r_1+1)(r_2+1)) < 1.$$

From (15), we can calculate that $(r_1, r_2) = (1,1)$ or $(2,1)$. This implies that $n \leq 12$, a contradiction. Thus, (4) has no solution n. The theorem is proved.

References

(1) C. Ashbacher, 'The Pseudo-Smarandache Function and the Classical Functions of Number Theory", **Smarandache Notions J.**, 9(1995), 78-81.

(2) G. H. Hardy and E. M. Wright, **An Introduction to the Theory of Numbers**, Oxford University Press, 1937.

SMARANDACHE RELATIONSHIPS AND SUBSEQUENCES*

M. Bencze
2212 Sacele, Grasov, Romania

ABSTRACT

Some Smarandache relationships between the terms of a given sequence are studied in the first paragraph. In the second paragraph, are studied Smarandache subsequences (whose terms have the same property as the initial sequence). In the third paragraph are studied the Smarandache magic squares and cubes of order n and some conjectures in number theory.

Key Words: Smarandache p-q relationships, Smarandache p-q-$\diamond$-subsequence, Smarandache type subsequences, Smarandache type partition, Smarandache type definitions, Smarandache type conjectures in number theory.

1) Smarandache Relationships

Let $\{a_n\}$, $n \geq 1$ be a sequence of numbers and p, q integers ≥ 1. Then we say that the terms

$$a_{k+1}, a_{k+2}, \ldots, a_{k+p}, a_{k+p+1}, a_{k+p+2}, \ldots, a_{k+p+q}$$

verify a *Smarandache p-q relationship* if

$$a_{k+1} \diamond a_{k+2} \diamond \ldots \diamond a_{k+p} = a_{k+p+1} \diamond a_{k+p+2} \diamond \ldots \diamond a_{k+p+q}$$

where "$\diamond$" may be any arithmetic or algebraic or analytic operation (generally a binary law on $\{a_1, a_2, a_3, \ldots\}$).

If this relationship is verified for any $k \geq 1$ (i.e. by all terms of the sequence), then

$\{a_n\}$, $n \geq 1$ is called a *Smarandache p-q-<> sequence*

where "$\diamond$" is replaced by "additive" if $\diamond = +$, "multiplicative" if $\diamond = *$, etc. [according to the operation ($\diamond$) used].

As a particular case, we can easily see that Fibonacci/Lucas sequence

$(a_n + a_{n+1} = a_{n+2})$, for $n \geq 1$

is a Smarandache 2-1 additive sequence.

A Tribonacci sequence $(a_n + a_{n+1} + a_{n+2} = a_{n+3})$, $n \geq 1$ is a Smarandache 3-1 additive sequence. Etc.

Now, if we consider the sequence of Smarandache numbers,

1, 2, 3, 4, 5, 3, 7,4 , 6, 5, 11, 4, 13, 7, 5, 6, 17, . . . , i.e. for each n the smallest number S(n) such that S(n)! is divisible by n [See(1)] (the values of the Smarandache Function), it raises the questions:

(a) How many quadruplets verify a Smarandache 2-2 additive relationship i.e.

$S(n+1) + S(n+2) = S(n+3) + S(n+4)$?

I found: $S(6) + S(7) = S(8) + S(9)$, $3 + 7 = 4 + 6$;

$S(7) + S(8) = S(9) + S(10)$, $7 + 4 = 6 + 5$;

$S(28) + S(29) = S(30) + S(31)$, $7 + 29 = 5 + 31$.

But, what about others? I am not able to tell you if there exist a finite or infinite number (?)

(b) How many quadruplets verify a Smarandache 2-2-subtractive relationship, i.e.

$S(n+1) - S(n+2) = S(n+3) - S(n+4)$?

I found: $S(1) - S(2) = S(3) - S(4)$, $1 - 2 = 3 - 4$;

$S(2) - S(3) = S(4) - S(5)$, $2 - 3 = 4 - 5$;

$S(49) - S(50) = S(51) - S(52)$, $14 - 10 = 17 - 13$.

(c) How many sextuplets verify a Smarandache 3-3 additive relationship, i.e.

$S(n+1) + S(n+2) + S(n+3) = S(n+4) + S(n+5) + S(n+6)$?

I found: $S(5) + S(6) + S(7) = S(8) + S(9) + S(10)$, $5 + 3 + 7 = 4 + 6 + 5$.

I read that Charles Ashbacher has a computer program that calculates the Smarandache Function's values, therefore he may be able to add more solutions to mine.

More generally:

If f_p is a p-ary relation and g_q is a q-ary relation, both of them defined on

$\{ a_1, a_2, a_3, \ldots \}$, then

$a_{i1}, a_{i2}, \ldots, a_{ip}, a_{j1}, a_{j2}, \ldots, a_{jq}$

verify a *Smarandache f_p - g_q - relationship* if

$f_p(a_{i1}, a_{i2}, \ldots, a_{ip}) = g_q(a_{j1}, a_{j2}, \ldots, a_{jq})$.

If this relationship is verified by all terms of the sequence, then $\{a_n\}$, $n \geq 1$ is called a *Smarandache f_p -g_q -sequence.*

Study some Smarandache f_p -g_q - relationships for well-known sequences (perfect numbers, Ulam numbers, abundant numbers, Catalan numbers, Cullen numbers, etc.).

For example: a Smarandache 2-2-additive, or subtractive, or multiplicative relationship, etc.

If f_p is a p-ary relation on $\{ a_1, a_2, a_3, \ldots \}$ and

$f_p(a_{i1}, a_{i2}, \ldots, a_{ip}) = f_p\ a_{j1}, (a_{j2}, \ldots, a_{jp})$

for all a_{ik}, a_{jk}, where $k = 1, 2, \ldots, p$, and for all $p \geq 1$, then $\{a_n\}$, $n \geq 1$, is called a *Smarandache perfect f sequence.*

If not all p-plets $(a_{i1}, a_{i2}, \ldots, a_{ip})$ and $(a_{j1}, a_{j2}, \ldots, a_{jp})$ verify the f_p relation, or not for all $p \geq 1$, the relation f_p is verified, then $\{a_n\}$, $n \geq 1$ is called a *Smarandache partial perfect f-sequence.*

An example: a *Smarandache partial perfect additive sequence*:

1, 1, 0, 2, -1, 1, 1, 3, -2, 0, 0, 2, 1, 1, 3, 5, -4, -2, -1, 1, -1, 1, 1, 3, 0, 2, . . .

This sequence has the property that

$$\sum_{i=1}^{p} a_i = \sum_{j=p+1}^{2p} a_j,$$

for all $p \geq 1$.

It is constructed in the following way:

$$a_1 = a_2 = 1$$

$$a_{2p+1} = a_{p+1} - 1$$

$$a_{2p+2} = a_{p+1} + 1$$

for all $p \geq 1$.

(a) Can you, readers, find a general expression of a_n (as function of n)?

It is periodical, or convergent or bounded?

(b) Please design (invent) yourselves other Smarandache perfect (or partial perfect) sequences.

Think about a multiplicative sequence of this type.

2) Smarandache Subsequences

Let $\{a_n\}$, $n \geq 1$ be a sequence defined by a property (or a relationship involving its terms) P.

Now, we screen this sequence, selecting only its terms those digits hold the property (or relationship involving the digits) P.

The new sequence obtained is called:

(1) **Smarandache P-digital subsequences.**

For example:

(a) *Smarandache square-digital subsequence:*

0, 1, 4, 9, 49, 100, 144, 400, 441, . . .

i.e. from 0, 1, 4, 9, 16, 25, 36, ..., n^2, ... we choose only the terms whose digits are all perfect squares (therefore only 0, 1, 4, and 9).

Disregarding the square numbers of the form $\overline{N\underbrace{0\ \ldots\ 0}_{2k\text{ zeros}}}$, where N is also a perfect square, how many other numbers belong to this sequence?

(b) *Smarandache cube-digital subsequence:*

0, 1, 8, 1000, 8000, . . .

i.e. from 0, 1, 8, 27, 64, 125, 216, . . . , n^3, . . . we choose only the terms whose digits are all perfect cubes (therefore only 0, 1 and 8).

Similar question, disregarding the cube numbers of the form $\overline{M\underbrace{0\ \ldots\ 0}_{3k\text{ zeros}}}$ where M is a perfect cube.

(c) *Smarandache prime digital subsequence*:

2, 3, 5, 7, 23, 37, 53, 73, . . .

i.e. the prime numbers whose digits are all primes.

Conjecture: this sequence is infinite.

In the same general conditions of a given sequence, we screen it selecting only its terms whose groups of digits hold the property (or relationship involving the groups of digits) P.

[A group of digits may contain one or more digits, but not the whole term.]

The new sequence obtained is called:

(2) *Smarandache P-partial digital subsequence.*

Similar examples:

(a) *Smarandache square-partial-digital subsequence:*

49, 100, 144, 169, 361, 400, 441, . . .

i.e. the square members that is to be partitioned into groups of digits which are also perfect squares. (169 can be partitioned as $16 = 4^2$ and $9 = 3^2$, etc.)

Disregarding the square numbers of the form

$\overline{N\underbrace{0\ \ldots\ 0}_{2k\text{ zeros}}}$, where N is also a perfect square,

how many other numbers belong to this sequence?

(b) *Smarandache cube-partial digital subsequence*:

1000, 8000, 10648, 27000, . . .

i.e. the cube numbers that can be partitioned into groups of digits which are also perfect cubes.

(10648 can be partitioned as $1 = 1^3$, $0 = 0^3$, $64 = 4^3$, and $8 = 2^3$).

Same question: disregarding the cube numbers of the form:

$\overline{M0 \; . \; . \; . \; 0}$ where M is also a perfect cube, how many other numbers belong
3k zeros
to this sequence?

(c) *Smarandache prime-partial digital subsequence*:

23, 37, 53, 73, 113, 137, 173, 193, 197, . . .

i.e. prime numbers, that can be partitioned into groups of digits which are also prime,

(113 can be partitioned as 11 and 3, both primes).

Conjecture: this sequence is infinite.

(d) *Smarandache Lucas-partial digital sunsequence*

123, . . .

i.e. the sum of the two first groups of digits is equal to the last group of digits, and the whole number belongs to Lucas numbers:

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, . . .

(beginning at 2 and $L(n+2) = L(n+1) + L(n)$, $n \geq 1$) (123 is partitioned as 1, 2 and 3, then $3 = 2 + 1$).

Is 123 the only Lucas number that verifies a *Smarandache type partition*?

Study some Smarandache P - (partial) - digital subsequences associated to:

- Fibonacci numbers (we were not able to find any Fibonacci number verifying a Smarandache type partition, but we could not investigate large numbers; can you? Do you think none of them would belong to a Smarandache F - partial-digital subsequence?

- Smith numbers, Eulerian numbers, Bernoulli numbers, Mock theta numbers, Smarandache type sequences etc.

Remark: Some sequences may not be smarandachely partitioned (i.e. their associated Smarandache type subsequences are empty).

If a sequence $\{a_n\}$, $n \geq 1$ is defined by $a_n = f(n)$ (a function of n), then a *Smarandache f-digital subsequence* is obtained by screening the sequence and selecting only its terms that can be partitioned in two groups of digits g_1 and g_2 such that $g_2 = f(g_1)$.

(3) Study similar questions for this case, which is more complex.

An interesting law may be

$l(a_1, a_2, \ldots, a_n) = a_1 + a_2 - a_3 + a_4 - a_5 + \ldots$

Smarandache prime conjecture:

Any odd number can be expressed as the sum of two primes minus a third prime (not including the trivial solution p = p + q - q when the odd number is the prime itself).

For example:

1 = 3 + 5 - 7 = 5 + 7 - 11 = 7 + 11 - 17 = 11 + 13 - 24 = . . .
3 = 5 + 11 - 13 = 7 + 19 - 23 = 17 + 23 - 37 = . . .
5 = 3 + 13 - 11 = . . .
7 = 11 + 13 - 17 = . . .
9 = 5 + 7 - 3 = . . .
11 = 7 + 17 - 13 = . . .

(a) Is this conjecture equivalent to Goldbach's conjecture (any odd number ≥ 9 can be expressed as a sum of three primes - finally solved by Vinogradov in 1937 for any odd number greater than $3^{3^{15}}$)?

(b) The number of times each odd number can be expressed as a sum of two primes minus a third prime are called *Smarandache prime conjecture numbers*. None of them are known!

(c) Write a computer program to check this conjecture for as many positive odd numbers as possible.

(2) There are infinitely many numbers that cannot be expressed as the difference between a cube and a square (in absolute value).

They are called *Smarandache bad numbers*(!)

For example: 5, 6, 7, 10, 13, 14, . . . are probably such bad numbers (F. Smarandache has conjectured, see[1]), while

1, 2, 3, 4, 8, 9, 11, 12, 15, . . . are not, because

$1 = | 2^3 - 3^2 |$

$2 = | 3^3 - 5^2 |$

$3 = | 1^3 - 2^2 |$

$4 = | 5^3 - 11^2 |$

$8 = | 1^3 - 3^2 |$

$9 = | 6^3 - 15^2 |$

$11 = | 3^3 - 4^2 |$

$12 = | 13^3 - 47^2 |$

$15 = | 4^3 - 7^2 |$, etc.

(a) Write a computer program to get as many non Smarandache bad numbers (it's easier this way!) as possible,

i.e. find an ordered array of a's such that

$a = | x^3 - y^2 |$, for x and y integers ≥ 1.

REFERENCES

1. Smarandache, F. (1975). "Properties of Numbers", University of Craiova Archives, (see also Arizona State Universitym Special Collections, Tempe, AZ, U. S. A.)

2. Sloane, N. J. A. and Simon, Plouffe, (1995). **The Encyclopedia of Integer Sequences**, Academic Press, San Diego, New York, Boston, London, Sydney, Tokyo, (M0453).

* This paper first appeared in **Bulletin of Pure and Applied Sciences**, Vol. 17E(No.1) 1998; p. 55-62.

A SET OF CONJECTURES ON SMARANDACHE SEQUENCES*

Sylvester Smith
Department of Mathematics, Yuma Community College

ABSTRACT

Searching through the Archives of the Arizona State University, I found interesting sequences of numbers and problems related to them. I display some of them, and the readers are welcome to contribute with solutions or ideas.

Key words: Smarandache P-digital subsequences, Smarandache P-partial subsequences, Smarandache type partition, Smarandache S-sequences, Smarandache uniform sequences, Smarandache operation sequences.

Let $\{ a_n \}$, $n > 1$ be a sequence defined by a property (or a relationship involving its terms P.)

Now, we screen this sequence, selecting only its terms whose digits hold the property (or relationship involving the digits) P.

The new sequence obtained is called:

(1) *Smarandache P-digital subsequences.*

For example:

(a) *Smarandache square-digital subsequence:*

0, 1, 4, 9, 49, 100, 144, 400, 441, . . .

i.e. from 0, 1, 4, 9, 16, 25, 36, ..., n^2, ... we choose only the terms whose digits are all perfect squares (therefore only 0, 1, 4, and 9).

Disregarding the square numbers of the form $\overline{N\underbrace{0\ldots0}_{2k \text{ zeros}}}$, where N is also a perfect square, how many other numbers belong to this sequence?

(b) *Smarandache cube-digital subsequence:*

0, 1, 8, 1000, 8000, . . .

i.e. from 0, 1, 8, 27, 64, 125, 216, . . . , n^3, . . . we choose only the terms whose digits are all perfect cubes (therefore only 0, 1 and 8).

Similar question, disregarding the cube numbers of the form $\overline{M\underbrace{0\ldots0}_{3k \text{ zeros}}}$

where M is a perfect cube.

(c) *Smarandache prime digital subsequence*:

2, 3, 5, 7, 23, 37, 53, 73, . . .

i.e. the prime numbers whose digits are all primes.

Conjecture: this sequence is infinite.

In the same general conditions of a given sequence, we screen it selecting only its terms whose groups of digits hold the property (or relationship involving the groups of digits) P.

[A group of digits may contain one or more digits, but not the whole term.]

The new sequence obtained is called:

(2) *Smarandache P-partial digital subsequence.*

Similar examples:

(a) *Smarandache square-partial-digital subsequence:*

49, 100, 144, 169, 361, 400, 441, . . .

i.e. the square members that is to be partitioned into groups of digits which are also perfect squares. (169 can be partitioned as $16 = 4^2$ and $9 = 3^2$, etc.)

Disregarding the square numbers of the form

$\overline{N0 \ldots 0}$, where N is also a perfect square,
2k zeros
how many other numbers belong to this sequence?

(b) *Smarandache cube-partial digital subsequence*:

1000, 8000, 10648, 27000, . . .

i.e. the cube numbers that can be partitioned into groups of digits which are also perfect cubes.

(10648 can be partitioned as $1 = 1^3$, $0 = 0^3$, $64 = 4^3$, and $8 = 2^3$).

Same question: disregarding the cube numbers of the form:

$\overline{M0 \ldots 0}$ where M is also a perfect cube, how many other numbers belong
3k zeros
to this sequence.

(c) *Smarandache prime-partial digital subsequence*:

23, 37, 53, 73, 113, 137, 173, 193, 197, . . .

i.e. prime numbers, that can be partitioned into groups of digits which are also prime,

(113 can be partitioned as 11 and 3, both primes).

Conjecture: this sequence is infinite.

(d) *Smarandache Lucas-partial digital sunsequence*

123, . . .

i.e. the sum of the two first groups of digits is equal to the last group of digits, and the whole number belongs to Lucas numbers:

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, . . .

(beginning at 2 and $L(n+2) = L(n+1) + L(n)$, $n > 1$) (123 is partitioned as 1, 2 and 3, then $3 = 2 + 1$). Is 123 the only Lucas number that verifies a *Smarandache type partition*?

Study some Smarandache P - (partial) - digital subsequences associated to:

- Fibonacci numbers (we were not able to find any Fibonacci number verifying a Smarandache type partition, but we could not investigate large numbers; can you? Do you think none of them would belong to a Smarandache F - partial-digital subsequence?

- Smith numbers, Eulerian numbers, Bernoulli numbers, Mock theta numbers, Smarandache type sequences etc.

Remark: Some sequences may not be smarandachely partitioned (i.e. their associated Smarandache type subsequences are empty).

If a sequence $\{a_n\}$, $n \geq 1$ is defined by $a_n = f(n)$ (a function of n), then

Smarandache f-digital subsequence is obtained by screening the sequence and selecting only its terms that can be partitioned in two groups of digits g_1 and g_2 such that $g_2 = f(g_1)$.

For example:

(a) If $a_n = 2n$, $n \geq 1$, then

Smarandache even-digital subsequence is:

12, 24, 36, 48, 510, 612, 714, 816, 918, 1020, 1122, 1224, . . .

(i.e. 714 can be partitioned as $g_1 = 7$, $g_2 = 14$, such that $14 = 2*7$, etc.)

(b) *Smarandache lucky-digital subsequence*

37, 49, . . .

(i.e. 37 can be partitioned as 3 and 7, and $L_3 = 7$; the lucky numbers are

1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, 49, 51, 63, . . .

How many other numbers belong to this subsequence? Study the Smarandache f-digital subsequence associated to other well-known sequences.

(3) *Smarandache odd sequence*:

1, 3, 135, 1357, 13579, 1357911, 135791113, 13579111315, 1357911131517, ...

How many of them are prime?

(4) *Smarandache even sequence*:

2, 24, 246, 2468, 246810, 24681012, 2468101214, 246810121416, . . .

Conjecture: None of them is a perfect power!

(5) *Smarandache prime sequence*:

2, 23, 235, 2357, 235711, 23571113, 2357111317, 235711131719, 23571113171923, . . .

How many of them are prime?

(Conjecture: a finite number).

(6) *Smarandache S-sequence*:

General definition:

Let $S_1, S_2, S_3, \ldots, S_n, \ldots$ be an infinite integer sequence (noted by S). Then

$$S_1, \overline{S_1 S_2}, \overline{S_1 S_2 S_3}, \ldots, \overline{S_1 S_2 S_3 \ldots S_n}, \ldots$$

is called the Smarandache S-sequence.

Question:

(a) How many of the Smarandache S-sequence belong to the initial S sequence?

(b) Or, how many of the Smarandache S-sequence verify the relation of other given sequences?

For example:

If S is the sequence of odd numbers 1, 3, 5, 7, 9, . . . then the Smarandache S-sequence is 1, 13, 135, 1357, . . . [(i.e.1)] and all the other terms are odd;

Same if S is the sequence of even numbers [(i.e. 2)]

The question (a) is trivial in this case.

But,when S is the sequence of primes [i.e. 3], the question becomes much harder.

Study the case when S (replaced by F) is the Fibonacci sequence (for one example):

1, 1, 2, 3, 5, 8, 13, 21, ...

Then the Smarandache F - sequence

1, 11, 112, 1123, 11235, 112358, . . .

How many primes does it contain?

(7) *Smarandache uniform sequences*:

General definition:

Let n be an integer not equal to zero and $d_1, d_2, \ldots, d_r$ digits in a base B (of course $r < B$).

Then: multiples of n, written with digits $d_1, d_2, \ldots, d_r$ only (but all r of them), in base B, increasingly ordered, are called the Smarandache uniform sequence.

As a particular case it's important to study the multiples written with one digit only (when $r = 1$).

Some examples (in base 10):

(a) Multiples of 7 written with digit 1 only:

111111, 111111,111111, 111111,111111,111111, 111111,111111,111111,111111, ...

(b) Multiples of 7 written with digit 2 only:

222222, 222222222222, 222222222222222222, 222222222222222222222222, ...

(c) Multiples of 79365 written with digit 5 only:

555555, 555555555555, 555555555555555555, 555555555555555555555555, ...

For some cases, the Smarandache uniform sequence may be empty (impossible):

(d) Multiples of 79365 written with digit 6 only (because any multiple of 79365 will end in 0 or 5.

Remark: If there exists at least a multiple m of n, written with digits $d_1, d_2, \ldots, d_r$ only, in base B, then there exists an infinite number of multiples of n (they have the form:

$m, \overline{mm}, \overline{mmm}, \overline{mmmm}, \ldots$).

With a computer program it's easy to select all multiples (written with certain digits) of a given number - up to some limit.

Exercise: Find the general term expression for multiples of 7 written with digits 1, 3, 5 only in base 10.

(8) *Smarandache operation sequences*:

General definition:

Let E be an ordered set of elements, $E = \{ e_1, e_2, \ldots \}$ and θ a set of binary

operations well-defined for these elements. Then:

a_1 is an element of $\{ e_1, e_2, \ldots \}$.

$a_{n+1} = \min \{ e_1 \theta_1 e_2 \theta_2 \ldots \theta_n e_n \} > a_n$, for $n > 1$.

where all θ_i are operations belonging to θ, is called the *Smarandache operation sequence*.

Some examples:

(a) When E is the natural number set, and θ is formed by the four arithmetic operations: +, -, *, /.

Then: $a_1 = 1$

$$a_{n-1} = \min \{ 1 \, \theta_1 \, 2 \, \theta_2 \dots \theta_n \, (n+1) \} > a_n \text{, for } n > 1,$$

(therefore, all θ_i may be chosen among addition, subtraction, multiplication or division in a convenient way).

Questions: Find this *Smarandache arithmetics operation infinite sequence*. Is it possible to get a general expression formula for this sequence (which starts with 1, 2, 3, 5, 4,?

(b) A finite sequence

$a_1 = 1$

$$a_{n+1} = \min \{ 1 \, \theta_1 \, 2 \, \theta_2 \dots \theta_{98} \, 99 \} > a_n$$

for $n > 1$, where all θ_i are elements of { +, -, *, / }.

Same questions for this *Smarandache arithmetics operation finite sequence*.

(c) Similarly for *Smarandache algebraic operation infinite sequence*

$a_1 = 1$

$$a_{n+1} = \min \{ 1 \, \theta_1 \, 2 \, \theta_2 \dots \theta_n \, (n+1) \} > a_n \text{ for } n > 1,$$

where all θ_i are elements of { +, -, *, /, **, $^y\sqrt{\ }$ }

(X**Y means X^Y and $^y\sqrt{x}$ means the yth root of x).

The same questions become harder but more exciting.

(d) Similarly for *Smarandache algebraic operation finite sequence*:

$a_1 = 1$

$$a_{n+1} = \min \{ 1 \, \theta_1 \, 2 \, \theta_2 \dots \theta_{98} \, 99\} > a \text{, for } n > 1,$$

where all θ_i are elements of { +, -, *, /, **, $^y\sqrt{\ }$ }

(X**Y means X^Y and $^y\sqrt{x}$ means the yth root of x).

Same questions.

More generally: one replaces "binary operations" by "K_i -ary operations" where all K_i are integers ≥ 2). Therefore,

$a_i \in \{ e_1, e_2, \dots \}$,

$a_{n+1} = \min\{ 1\, \theta_1^{(K_1)} \theta_1^{(K_1)} \ldots \theta_1^{(K_1)} K_1$
$\theta_1^{(K_1)}$ is K_1 - ary

$\theta_2^{(K_2)} (K_2 + 1)\theta_2^{(K_2)} \; . \; . \; . \; \theta_2^{(K_2)} (K_1 + K_2 - 1) \; . \; . \; .$
$\theta_2^{(K_2)}$ is K_2 - ary

$(n + 2 - K_r)\, \theta_r^{(K_r)} \; . \; . \; . \; \theta_r^{(K_r)}(n + 1) \} > a_n$, for $n \geq 1$.

Of course $K_1 + (K_2 - 1) + \ldots (K_r - 1) = n+1$.

Remark: The questions are much easier when $\theta = \{ +,- \}$; study the Smarandache operation type sequences in this case.

(9) *Smarandache operation sequences at random*:

Same definitions and questions as for the previous sequences, except that

$a_{n+1} = \{ e_1 \theta_1 e_2 \theta_2 \ldots \theta_n e_{n+1} \} > a_n$, for $n > 1$,

(i.e. it's no "min" any more, therefore a_{n+1} will be chosen at random, but greater than a_n, for any $n > 1$). Study these sequences with a computer program for random variables (under weak conditions).

REFERENCES

1. Smarandache, F. (1975) "Properties of the Numbers", University of Craiova Archives, [see also Arizona State University, Special Collections, Tempe, Arizona, USA].

* Originally appeared in **Bulletin of Pure and Applied Sciences**, Vol. 15 E(No. 1) 1996; p. 101-107

SMARANDACHE PARTITION TYPE AND OTHER SEQUENCES*

Eng. Dr. Fanel IACOBESCU
Electrotechnic Faculty of Craiova, Romania

ABSTRACT

Thanks to C. Dumitrescu and Dr. V. Seleacu of the University of Craiova, Department of Mathematics, I became familiar with some of the Smarandache Sequences. I list some of them, as well as questions related to them. Now I'm working in a few conjectures involving these sequences.

Examples of Smarandache Partition type sequences:

A. 1, 1, 1, 2, 2, 2, 2, 3, 4, 4, ...

(How many times is n written as a sum of non-null squares, disregarding the order of the terms:
for example:

$$\begin{aligned}9 &= 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 \\ &= 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 2^2 \\ &= 1^2 + 2^2 + 2^2 \\ &= 3^2,\end{aligned}$$

therefore ns(9) = 4.)

B. 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, ...

(How many times is n written as a sum of non-null cubes, disregarding the order of the terms:
for example:

$$\begin{aligned}9 &= 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 \\ &= 1^3 + 2^3,\end{aligned}$$

therefore, nc(9) = 2.)

C. General-partition type sequence:

Let f be an arithmetic function and R a relation among numbers.
(How many times can n be written under the form:

$$n = R(f(n_1), f(n_2), ..., f(n_k))$$

for some k and n_1, n_2, ..., n_k such that

$$n_1 + n_2 + \ldots + n_k = n? \}$$

Examples of other sequences:

(1) Smarandache Anti-symmetric sequence:

11, 1212, 123123, 12341234, 1234512345, 123456123456, 12345671234567, 1234567812345678, 123456789, 123456789, 12345678910123456 78910, 1234567891011, 1234567891011, ...

(2) Smarandache Triangular base:

1, 2, 10, 11, 12, 100, 101, 102, 110, 1000, 1001, 1002, 1010, 1011, 10000, 10001, 10002, 10010, 10011, 10012, 100000, 100001, 100002, 100010, 100011, 100012, 100100, 1000000, 1000001, 1000002, 1000010, 1000011, 1000012, 1000100, ...

(Numbers written in the triangular base, defined as follows:

$t(n) = (n(n+1))/2$, for $n \geq 1$.)

(3) Smarandache Double factorial base:

1, 10, 100, 101, 110, 200, 201, 1000, 1001, 1010, 1100, 1101, 1110, 1200, 10000, 10001, 10010, 10100, 10101, 10110, 10200, 10201, 11000, 11001, 11010, 11100, 11101, 11110, 11200, 11201, 12000, ...

(Numbers written in the double factorial base, defined as follows:

$df(n) = n!!$)

(4) Smarandache Non-multiplicative sequence:

General definition: Let $m_1, m_2, \ldots, m_k$ be the first k terms of the sequence, where $k \geq 2$;

then m_i, for $i >= k+1$, is the smallest number not equal to the product of k previous distinct terms.

(5) Smarandache Non-arithmetic progression:

1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41, 64, ...

General definition: if m_1 , m_2 , are the first two terms of the sequence,

then m_k, for $k \geq 3$, is the smallest number such that no 3-term arithmetic

progression is in the sequence.

In our case the first two terms are 1, respectively 2.

Generalization: same initial conditions, but no i-term arithmetic progression in the sequence (for a given $i \geq 3$).

(6) Smarandache Prime product sequence:

2, 7, 31, 211, 2311, 30031, 510511, 9699691, 223092871, 6469693231, 200560490131, 7420738134811, 304250263527211, ...

$P_n = 1 + p_1 p_2 \ldots p_k$, where p_k is the k-th prime.

Question: How many of them are prime?

(7) Smarandache Square product sequence:

2, 5, 37, 577, 14401, 518401, 25401601, 1625702401, 131681894401, 13168189440001, 1593350922240001, ...

$S_k = 1 + s_1 s_2 \ldots s_k$, where s_k is the k-th square number.

Question: How many of them are prime?

(8) Smarandache Cubic product sequence:

2, 9, 217, 13825, 1728001, 373248001, 128024064001, 65548320768001, ...

$C_k = 1 + c_1 c_2 \ldots c_k$, where c_k is the k-th cubic number.

Question: How many of them are prime?

(9) Smarandache Factorial product sequence:

2, 3, 13, 289, 34561, 24883201, 125411328001, 5056584744960001, ...

$F_k = 1 + f_1 f_2 \ldots f_k$, where f_k is the k-th factorial number.

Question: How many of them are prime?

(10) Smarandache U-product sequence {generalization}:

Let u_n, $n \geq 1$, be a positive integer sequence. Then we define a U-sequence as follows:

$Un = 1 + u_1 u_2 \ldots u_n$.

(11) Smarandache Non-geometric progression.

1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 24, 26, 27, 29, 30, 31, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 45, 47, 48, 50, 51, 53, . . .

General definition: if m_1 ,m_2 , are the first two terms of the sequence, then m_k , for $k \geq 3$, is the smallest number such that no 3-term geometric progression is in the sequence. In our case the first two terms are 1, respectively 2.

(12) Smarandache Unary sequence:

11, 111, 11111, 1111111, 11111111111, 1111111111111, 11111111111111111,
1111111111111111111, 11111111111111111111111,
11111111111111111111111111111, 1111111111111111111111111111111, ...

$u(n) = 11...1$, p_n digits of "1", where p_n is the n-th prime.

The old question: are there are infinite number of primes belonging to the sequence?

(13) Smarandache No-prime-digit sequence:

1, 4, 6, 8, 9, 10. 11, 1, 1, 14, 1, 16. 1, 18, 19, 0, 1, 4, 6, 8, 9,
0, 1, 4, 6, 8, 9, 40, 41, 42, 4, 44, 4, 46, 48, 49, 0, ...

(Take out all prime digits of n.)

(14) Smarandache No-square-digit-sequence.

2, 3, 5, 6, 7, 8, 2, 3, 5, 6, 7, 8, 2, 2, 22, 23, 2, 25, 26, 27, 28,
2, 3, 3, 32, 33, 3, 35, 36, 37, 38, 3, 2, 3, 5, 6, 7, 8, 5, 5, 52, 53,
5, 55, 56, 57, 58, 5, 6, 6, 62, ...

(Take out all square digits of n.)

* This paper first appeared in **Bulletin of Pure and Applied Sciences**, Vol. 16 E(No. 2) 1997; P. 237-240.

SMARANDACHE CONCATENATE TYPE SEQUENCES*

Helen Marimutha
Northland Pioneer College (USA)

ABSTRACT

Professor Anthony Begay of Navajo Community College influenced me in writing this paper. I enjoyed the Smarandache concatenation. The sequences shown here have been extracted from the Arizona State University(Tempe) Archives. They are defined as follows:

(1) Smarandache Concatenated natural sequence:

1, 22, 333, 4444, 55555, 666666, 7777777, 88888888, 999999999, 10101010101010101010, 1111111111111111111111, 121212121212121212121212, 13131313131313131313131313, 1414141414141414141414141414, 151515151515151515151515151515, . . .

(2) Smarandache Concatenated prime sequence:

2, 23, 235, 2357, 235711, 23571113, 2357111317, 235711131719, 23571113171923, ...

Conjecture: there are infinitely many primes among these numbers!

(3) Smarandache Concatenated odd sequence:

1, 13, 135, 1357, 13579, 1357911, 135791113, 13579111315, 1357911131517, ...

Conjecture: there are infinitely many primes among these numbers!

(4) Smarandache Concatenated even sequence:

2, 24, 246, 2468, 246810, 24681012, 2468101214, 246810121416, ...

Conjecture: none of them is a perfect power!

(5) Smarandache Concatenated S-sequence { generalization}:

Let $s_1, s_2, s_3, s_4, \ldots, s_n, \ldots$ be an infinite integer sequence (noted by S). Then:

$$s_1, \overline{s_1 s_2}, \overline{s_1 s_2 s_3}, \overline{s_1 s_2 s_3 s_4}, \ldots, \overline{s_1 s_2 s_3 \ldots s_n}, \ldots$$

is called the Concatenated S-sequence.

Questions: (a) How many terms of the Concatenated S-sequence belong to the initial S-sequence?

(b) Or, how many terms of the Concatenated S-sequence verify the relation of other given sequences?

The first three cases are particular.

Look now at some other examples, when S is a sequence of squares, cubes, Fibonacci respectively (and one can go so on).

(6) Smarandache Concatenated Square sequence:

1, 14, 149, 14916, 1491625, 149162536, 14916253649, 1491625364964, ...

How many of them are perfect squares?

(7) Smarandache Concatenated Cubic sequence:

1, 18, 1827, 182764, 182764125, 182764125216, 182764125216343, ...

How many of them are perfect cubes?

(8) Smarandache Concatenated Fibonacci sequence:

1, 11, 112, 1123, 11235, 112358, 11235813, 1123581321, 112358132134, ...

Does any of these numbers is a Fibonacci number?

REFERENCES

1. Smarandache, F. (1997). **Collected Papers Vol. II**, University of Kishinev.
2. Smarandache, F. (1975). "Properties of the Numbers", University of Craiova Archives. [See also Arizona State University Special Collections, Tempe, Arizona, USA].

* This paper originally appeared in **Bulletin of Pure and Applied Sciences**, Vol. 16 E(No. 2) 1997; p. 225-226

SMARANDACHE RECURRENCE TYPE SEQUENCES*

Mihaly Beneze
Brasov, Romania

ABSTRACT

Eight particular, Smarandache Recurrence Sequences and a Smarandache General-Recurrence Sequence are defined below and exemplified (found in State Archives, Rm, Valcea, Romania).

A. 1, 2, 5, 26, 29, 677, 680, 701, 842, 845, 866, 1517, 458330, 458333, 458354, ...

(ss2(n) is the smallest number, strictly greater than the previous one, which is the squares sum of two previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.)

Recurrence definition:

(1) The numbers $a \le b$ belong to SS2;
(2) If b, c belong to SS2, then $b^2 + c^2$ belongs to SS2 too;
(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to SS2.

The sequence (set) SS2 is increasingly ordered.

[Rule (1) may be changed by: the given numbers $a_1, a_2, a_3, \ldots, a_k$, where $k \ge 2$, belongs to SS2.]

B. 1, 1, 2, 4, 5, 6, 16, 17, 18, 20, 21, 22, 25, 26, 27, 29, 30, 31, 36, 37, 38, 40, 41, 42, 43, 45, 46, ...

(SS1(n) is the smallest number, strictly greater than the previous one, (for $n \ge 3$), which is the squares sum of one or more previous distinct terms of the sequence; in our particular case the first term is 1.)

Recurrence definition:

(1) The number a belongs to SS1;
(2) If $b_1, b_2, \ldots, b_k$ belong to SS1, where $k \ge 1$, then $b_1^2 + b_2^2 + \ldots + b_k^2$ belongs to SS1 too;
(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to SS1.

The sequence (set) SS1 is increasingly ordered.

[Rule (1) may be changed by: the given numbers $a_1, a_2, \ldots, a_k$, where $k \ge 1$, belong to SS1.]

C. 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 21, ...

(NSS2(n) is the smallest number, strictly greater than the previous one, which is NOT the squares sum of two previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.)

Recurrence definition:

(1) The numbers $a \le b$ belong to NSS2;
(2) If b, c belong to NSS2, then $b^2 + c^2$ DOES NOT belong to NSS2; any other numbers belong to NSS2;
(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to NSS2.

The sequence (set) NSS2 is increasingly ordered.

[Rule (1) may be changed by; the given numbers $a_1, a_2, ..., a_k$, where $k \geq 2$, belong to NSS2.]

D. 1, 2, 3, 6, 7, 8, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 38, 39, 42, 43, 44, 47, ...

(NSS1(n) is the smallest number, strictly greater than the previous one, which is NOT the squares sum of one or more of the previous distinct terms of the sequence; in our particular case the first term is 1.)

Recurrence definition:

(1) The number a belongs to NSS1;
(2) If $b_1, b_2, ..., b_k$ belong to NSS1, where $k \geq 1$, then $b_1^2 + b_2^2 + ... + b_k^2$ DOES NOT belong to NSS1; any other numbers belong to NSS1;

(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to NSS1.

[Rule (1) may be changed by: the given numbers $a_1, a_2, ..., a_k$, where $k \geq 1$, belong to NSS1.]

E. 1, 2, 9, 730, 737, 389017001, 389017008,389017729, ...

(CS2(n) is the smallest number, strictly greater than the previous one, which is the cubes sum of two previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.)

Recurrence definition:

(1) The numbers $a \leq b$ belong to CS2;
(2) If c,d belong to CS2, then $c^3 + d^3$ belongs to CS2 too;
(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to CS2.

The sequence (set) CS2 is increasingly ordered.

[Rule (1) may be changed by: the given numbers $a_1, a_2, ..., a_k$, where $k \geq 2$, belong to CS2.]

F. 1, 1, 2, 8, 9, 10, 512, 513, 514, 520, 521, 522, 729, 730, 731, 737, 738, 739, 1241, ...

(CS1(n) is the smallest number, strictly greater than the previous one (for $n \geq 3$), which is the cubes sum of one or more previous distinct terms of the sequence; in our particular case the first term is 1;

Recurrence definition:

(1) The number a belongs to CS1;
(2) If $b_1, b_2, ..., b_k$ belong to CS1, where $k \geq 1$, then $b_1^3 + b_2^3 + ... + b_k^3$ belongs to CS2 too;
(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to CS1.

The sequence (set) CS1 is increasingly ordered.

[Rule (1) may be changed by: the given numbers $a_1, a_2, ..., a_k$, where $k \geq 2$, belong to CS1.]

G. 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 36, 37, 38, ...

(NCS2(n) is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of two previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.)

Recurrence definition:

(1) The numbers $a \leq b$ belong to NCS2.
(2) If c,d belong to NCS2, then $c^3 + d^3$ DOES NOT belong to NCS2; any other numbers do belong to NCS2.
(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to NCS2.

The sequence (set) NCS2 is increasingly ordered.

[Rule (1) may be changed by: the given numbers $a_1, a_2, ..., a_k$, where $k \geq 2$, belong to NCS2.]

H. 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31, 32, 33, 34, 37, 38, 39, ...

(NCS1(n) is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of one or more previous distinct terms of the sequence; in our particular case the first term is 1.)

Recurrence definition:

(1) The number a belongs to NCS1.
(2) If $b_1, b_2, ..., b_k$ belong to NCS1, where $k \geq 1$, then $b_1^2 + b_2^2 + ... + b_k^2$ DOES NOT belong to NCS1.
(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to NCS1.

The sequence (set) NCS1 is increasingly ordered.

[Rule (1) may be changed by: the given numbers $a_1, a_2, ..., a_k$, where $k \geq 2$, belong to NCS1.]

I. General recurrence type sequence:

General recurrence definition:

Let $k \geq j$ be natural numbers, and $a_1, a_2, ..., a_k$ be given elements, and R a j-relationship (relation among j elements).

Then:

(1) The elements $a_1, a_2, ..., a_k$ belong to SGR.
(2) If $m_1, m_2, ..., m_j$ belong to SGR, then $R(m_1, m_2, ..., m_j)$ belongs to SGR too.
(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to SGR.

The sequence (set) SGR is increasingly ordered.

Method of construction of the general recurrence sequence:

-level 1: the given elements $a_1, a_2, ..., a_k$ belong to SGR;

-level 2: apply the relationship R for all combinations of j elements among $a_1, a_2, ..., a_k$; the results belong to SGR too;

order all elements of levels 1 and 2 together,

-level i+1:

if $b_1, b_2, \ldots, b_m$ are all elements of levels 1, 2, ..., i-1 and $c_1, c_2, \ldots, c_n$ are all elements of level i, then apply the relationship R for all combinations of j elements among $b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_n$ such that at least an element is from the level i;

the results belong to SGR too;

order all elements of levels i and i+1 together;

and so on . . .

* Originally appeared in **Bulletin of Pure and Applied Sciences**, Vol. 16 E(No. 2) 1997; P. 231-236.

About Smarandache-Multiplicative Functions

Sabin Tabirca
Bucks University College, Computing Department, England.

The main objective of this note is to introduce the notion of the S-multiplicative function and to give some simple properties concerning it. The name of S-multiplicative is short for Smarandache-multiplicative and reflects the main equation of the Smarandache function.

Definition 1. A function $f: N^* \to N^*$ is called S-multiplicative if :

(1) $(a,b) = 1 \Rightarrow f(a * b) = \max\{ f(a), f(b) \}$

The following functions are obviously S-multiplicative:

1. The constant function $f: N^* \to N^*$, $f(n) = 1$.
2. The Erdos function $f: N^* \to N^*$, $f(n) = \max\{ p \mid p \text{ is prime and } n \dot{:} p \}$. [1].
3. The Smarandache function $S: N^* \to N$, $S(n) = \max\{ p \mid p! \dot{:} n \}$. [3].

Certainly, many properties of multiplicative functions[2] can be translated for S-multiplicative functions. The main important property of this function is presented in the following.

Definition 2. If $f: N^* \to N$ is a function, then $\bar{f}: N^* \to N$ is defined by

$$\bar{f}(n) = \min\{ f(d) \mid n \dot{:} d \}.$$

Theorem 1. If f is S-multiplicative function, then $\bar{f}$ is S-multiplicative.

Proof. This proof is made using the following simple remark:

(2). $(d|a*b \wedge (a,b)=1) \Rightarrow ((\exists d_1 \mid a)(\exists d_2 \mid b)(d_1,d_2) = 1 \wedge d = (d_1 * d_2)$

If d_1 and d_2 satisfy (2), then $f(d_1 * d_2) = \max\{f(d_1), f(d_2)\}$.

Let a,b be two natural numbers, such that $(a,b) = 1$. Therefore, we have

(3) $\bar{f}(a * b) = \min\limits_{d|a*b} f(d) = \min\limits_{d_1|a, d_2|a} f(d_1 * d_2) = \min\limits_{d_1|a} \min\limits_{d_2|a} \max\{ f(d_1), f(d_2) \}$.

Applying the distributing property of the max and min functions, equation (3) is transformed as follows:

$\bar{f}(a*b) = \max\{ \min\limits_{d_1|a} f(d_1), \min\limits_{d_2|a} f(d_2) \} = \max\{ \bar{f}(a), \bar{f}(b) \}$. Therefore,

the function $\bar{f}$ is S-multiplicative.

We believe that many other properties can be deduced for S-multiplicative functions. Therefore, it will be in our attention to further investigate these functions.

References

1. Erdos, P.: (1974) **Problems and Result in Combinatorial Number Theory**, Bordaux.

2. Hardy, G. H. and Wright, E. M.,:(1979) **An Introduction to Number Theory**, Clarendon Press, Oxford.
3. F. Smarandache:(1980) 'A Function in Number Theory', Analele Univ. Timisoara, XVIII.

ON SMARANDACHE MIXED NON-EUCLIDEAN GEOMETRIES

Professor Emeritus Anghel N. Rugina
Northeastern University, Boston, MA 02130, USA

Abstract.
In this short paper I compare the Smarandache's Non-Euclidean Geometries [2] with my Orientation Table For Any Science [1].

Introduction:
Here it is <u>An Orientation Table For Any Science</u> **(Natural or Social)**

<u>Building blocks</u>:

S = stable (equilibrium) elements, forces, values, behavior

U = unstable (disequilibrium) elements, forces, values, behavior

<u>Models</u>:	**S**	**U**	**<u>Description</u>**
M1 =	**100% S**	**0**	**A system of general stable equilibrium at its limit of perfection** **The methodological habitat for truths in the abstract or the pure classical model in science, in the sense of Newton (physics) or Walras (economics)**
M2 =	**95% S**	**+ 5% U**	**A system of stable equilibrium but with minor deviations . This is the methodological habitat for truths in the concrete. It is the case for special relativity. (Einstein and Newton)**
M3 =	**65% S**	**+ 35% U**	**A mixed system of simple anomalies or relativity of the first order. The equilibrium elements still prevail. Habitat for truths in the concrete.**
M4 =	**50% S**	**+ 50% U**	**A mixed system of unstable equilibrium. In economics it represents the Keynesian model of "equilibrium with unemployment but adding the prefix of "unstable". It is the usual model in modern science guided by unstable equilibrium or "stable disequilibrium".**
M5 =	**35% S**	**+ 65% U**	**A mixed model of compound anomalies or relativity of the second order where disequilibrium elements prevail. A weak major disequilibrium.**
M6 =	**5% S**	**+ 95% U**	**A borderline mixed system where disequilibrium elements dominate to a**

	very large degree. A strong major disequilibrium.
M7 = 100% U	A system of total disequilibrium dominated completely by pure contradictions, real chaos.

My non-understandings are:

1. "Mixed Non-Euclidean Geometries" cannot mean the same thing with "Anti-Geometry". Tis would involve that all Non-Euclidean Geometries deny each other.

2. The"Euclidean Geometry" is just one model, specifically Model M1 on my Orietation Table. Indeed, a similar Orientation Table can be constructed for Geometry. See: enclosure:p.5

3. Independent of Model M1 (Euclidean), there is an unlimited number of possible mixed, Noneuclidean, concentrated just for study purposes in 6 other models. Only Model M7 which represents the Geometry of total disequilibrium or chaos negates model M1 and therefore may be called the Anti-Euclidean or Anti-Classical system of Geometry. Actually this is the only case when we can talk about M7 Anti-Geometry with specification.

4. The Non-Euclidean M2,M3,M4, M5 and M6 which represent a minor disequilibrium, a neutral disequilibrium (M4) or unstable equilibrium and major disequilibria (M5,M6) systems of Geometry do not "run counter to the classical ones" (M1 with truth in the abstract and M2 with truth in the concrete) but they are just different in various degrees. There is no contradiction here or,if there is one then it is partial or imperfect but not complete.

5. To "transform the apparently unscientific ideas into scientific ones" is a treacherous operation. To me something "unscientific" means being "untrue" and I do not see how you can transform logically something which is not true into something which is true! Unless, one is willing to use a "Hocus-Pocus" logic (just a joke) or incomplete logic" which closer to "Fuzzy Logic" (a more recent term). I do not know how Bertrand Russell would react to the "Fuzzy-Logic" name!

6. The term "Anti-Geometry" is not quite correct, at least not complete Anti-Classical or -Euclidean Geometry is O.k. but with the understanding that it refers to Model M7.

References:

[1] Anghel N. Rugina, *Prolegomena to any Future Study in Economics, Finance and Other Social Sciences: The Road to a Third Revolution in Economics, Financial, Social, Ethical, Logical and Political Thinking,* International Journal of Social Economics, MCB University Press, West Yorkshire, England, Vol. 25, No. 5, 1-388, 1998.

[2] Florentin Smarandache, *Mixed Non-Euclidean Geometries (Anti-Geometry),* Proceedings of the First Symposium (editor Serban C. Andronescu), Romanian Academy of Scientists, The American Branch, American Institute for Writing Research, New York, 109-127, 1998.

SMARANDACHE NUMBER RELATED TRIANGLES

K. R. S. Sastry
Jeevan Sandhya, Doddakalsandra Post, Raghuvana Halli
Bangalore 560 062, INDIA

ABSTRACT

Given a triangle in Euclidean geometry it is well known that there exist an infinity of triangles each of which is similar to the given one. In Section I we make certain observations on Smarandache numbers. This enables us to impose a constraint on the lenghts of the corresponding sides of similar triangles. In Section II we do this to see that infinite class of similar triangles reduces to a finite one. In Section III we disregard the similarity requirement. Finally, in Section IV we pose a set of open problems.

I SMARANDACHE NUMBERS: SOME OBSERVATIONS

Suppose a natural number n is given. The Smarandache number of n is the least number denoted by S(n) whith the following property: n divides S(n)! but not (S(n)-1)!. Below is a short table containing n and S(n) for $1 \le n \le 12$.

n	1	2	3	4	5	6	7	8	9	10	11	12
S(n)	1	2	3	4	5	3	7	4	6	5	11	4

A look at the above table shows that S(3) = S(6) = 3, S(4) = S(8) = S(12) = 4, ...

Let a natural number k be given. Then the equation S(x) = k cannot have an infinity of solutions x. This is because the largest solution is x=k!. This observation enables us to impose a restriction on the lengths of the corresponding sides of similar triangles. In the next section we shall see how to do this. Throughout this paper the triangles are assumed to have natural number side lengths. Also, the triangles are non degenerate.

II SMARANDACHE SIMILAR TRIANGLES

Let us denote by T(a,b,c) the triangle ABC with side lengths a, b, c. Then the two similar triangles T(a,b,c) and T'(a',b',c') are said to be <u>Smarandache Similar</u> if S(a) = S(a'), S(b) = S(b'), S(c) = S(c'). Trivially, a given triangle is Smarandache similar to itself. Non trivially the two Pytagorean triangles (right triangles with natural number side lengths) T(3,4,5) and T'(6,8,10)

are Smarandache similar because
$S(3) = S(6) = 3$, $S(4) = S(8) = 4$, $S(5) = S(10) = 5$.
However, the Similar triangles (3,4,5) and (9,12,15) are not Smarandache Similar because $S(3) = 3 \neq S(9)$ which is 6. In fact the class of Smarandache Similar triangles generated by T(3,4,5) contains just two: T(3,4,5) and T'(6,8,10) in view of the fact that the solution set of the equation $S(x) = 3$ consists of just two members $x = 3, 6$.

For another illustration let us determine the class of Smarandache Similar Triangles generated by the 60° triangle $T(a,b,c) = (5,7,8)$. The algorithm to do this is as follows: First we calculate $S(5) = 5$, $S(7) = 7$, $S(8) = 4$. Next we solve the equations $S(a') = 5$, $S(b') = 7$, $S(c') = 4$. Let us solve the last equation first.

$S(c') = 4 \rightarrow c' = 4, 8, 12, 24$.

Here the largest value $c' = 24 = 3c$. Hence we need not to solve the other two equations beyond the solutions $a' = 3a$, $b' = 3b$. This observation therefore gives us

$S(a') = 5 \rightarrow a' = 5, 10, 15$ and

$S(b') = 7 \rightarrow b' = 7, 14, 21$.

It is now clear that the class of Smarandache similar triangles contains just two members: (5,7,8) and (15,21,24).

III SMARANDACHE RELATED TRIANGLES

In this section we do not insist on the similarirty requirement that we had in Section II. Hence the definition: Given a triangle T(a,b,c) we say that a triangle T'(a',b',c') is Smarandache related to T if $S(a') = S(a)$, $S(b') = S(b)$, $S(c') = S(c)$. Note that the triangles T and T' may or may not be similar. As an illustration let us determine all the triangles that are Smarandache related to T(3,4,5). To do this we follow the same algorithm that we mentioned in Section II but we have to find all the solutions of the equations $S(a') = 3$, $S(b') = 4$, $S(c') = 5$. Therefore

$S(a') = 3 \rightarrow a' = 3, 6;$

$S(b') = 4 \rightarrow b' = 4, 8, 12, 24;$

$S(c') = 5 \rightarrow c' = 5, 10, 15, 20, 30, 40, 60, 120.$

This gives us the complete solution $(a',b',c') =$ (3,4,5); (3,8,10); (3,12,10); (6,4,5); (6,8,5); (6,8,10); (6,12,10); (6,12,15); (6,24,20).

IV CONCLUSION

In the present discussion I have used small natural numbers k so that the solution of the equations $S(x) = k$ can be easily determined. I do not know if this interesting converse problem of determining all natural numbers x for given k of the Smarandache equation $S(x)=k$ has been disscused by someone already. In case if this has not been already considered, I invite the reader to devise efficient methods to solve the preceding equation. We conclude this section by posing the following open problems to the reader.

(A) Are there two distinct dissimilar Phytagorean triangles that are Smarandache related? i.e. both $T(a,b,c)$ and $T'(a',b',c')$ are Pythagorean such that $S(a') = S(a)$, $S(b') = S(b)$, $S(c') = S(c)$ but T and T' are *not* similar.

(B) Are there two distinct and dissimilar 60° triangles (120° triangles) that are Smarandache related?

(c) Given a triangle $T(a,b,c)$. Is it possible to give either an exact formula or an uper bound for the total number of triangles (without actually determining all of them) that are Smarandache related to T?

(D) Consider other ways of relating two triangles in the Smarandache number sense. For example, are there two triplets of natural numbers (α,β,γ) and (α',β',γ') such that $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma' = 180$ and $S(\alpha) = S(\alpha')$, $S(\beta) = S(\beta')$, $S(\gamma) = S(\gamma')$.
If such distinct triplets exist the two triangles would be Smarandache related via their angles. Of course in this relationship the side lengths of the triangles may not be natural numbers.

Solutions To Some Sastry Problems On Smarandache Number Related Triangles

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
Hiawatha, IA 52233 USA
e-mail 71603.522@compuserve.com

In his recent paper[1], Sastry defines two triangles T(a,b,c) and T(a',b',c') to be Smarandache related if S(a) = S(a'), S(b) = S(b') and S(c) = S(c'). The function S is known as the Smarandache function and is defined in the following way.

For n any integer greater than zero, the value of the Smarandache function S(n) is the smallest integer m such that n divides m!.

He closes the paper by asking the following questions:

A) Are there two distinct dissimilar Pythagorean triangles that are Smarandache related? A triangle T(x,y,z) is Pythagorean if x*x + y*y = z*z.

B)Are there two distinct and dissimilar 60(120) degrees triangles that are Smarandache related? A 60(120) degrees triangle is one containing an angle of 60(120) degrees.

C) Given a triangle T(a,b,c), is it possible to give either an exact formula or an upper bound for the total number of triangles (without actually determining them), which are Smarandache related to T?

D) Consider other ways of relating two triangles in the Smarandache number sense. For example, are there two triplets of natural numbers (a,b,c) and (a',b',c') such that a + b + c = a' + b' + c' = 180 and S(a) = S(a'), S(b) = S(b') and S(c) = S(c')? If this were true, then the angles, in degrees, of the triangles would be Smarandache related.

In this paper, we will consider and answer questions (A), (B) and (D). Furthermore, we will also explore these questions using the Pseudo Smarandache function Z(n).

Given any integer n > 0, the value of the Pseudo Smarandache function is the smallest integer m such that n evenly divides

$$\sum_{k=1}^{m} k .$$

A) The following theorem is easy to prove.

Theorem: There are an infinite family of pairs of dissimilar Pythagorean triangles that are Smarandache related.

Proof:
Start with the two Pythagorean triangles

T(3,4,5) and T(5,12,13)

Clearly, these two triangles are not similar. Now, let p be an odd prime greater than 13 and form the triples

T(3p,4p,5p) T(5p,12p,13p)

Obviously, these triples are also Pythagorean. It is well-known that if $n = kp$, where $k < p$ and p is a prime, then $S(kp) = p$. Therefore,

$$S(3p) = S(4p) = S(5p) = S(12p) = S(13p) = p$$

and the triples form triangles that are not similar since the originals were not. Therefore, we have the desired infinite family of solutions.

Definition: Given two triangles T(a,b,c) and T(a',b',c'), we say that they are Pseudo Smarandache related if $Z(a) = Z(a')$, $Z(b) = Z(b')$ and $Z(c) = Z(c')$.

A computer program was written to search for dissimilar pairs of Pythagorean triples T(x,y,z) and T(u,v,w) that are also Pseudo Smarandache related. Several were found and a few are given below.

$x = 49$, $y = 168$, $z = 175$
$Z(x) = 48$, $Z(y) = 48$, $Z(z) = 49$
$u = 147$, $v = 196$, $w = 245$
$Z(u) = 48$, $Z(v)=48$, $Z(w) = 49$

$x = 96$, $y = 128$, $z = 160$
$Z(x) = 63$, $Z(y) = 255$, $Z(z) = 64$
$u = 128$, $v = 504$, $w = 520$
$Z(u) = 255$, $Z(v) = 63$, $Z(w) = 64$

$x = 185$, $y = 444$, $z = 481$
$Z(x) = 74$, $Z(y) = 111$, $Z(z) = 221$
$u = 296$, $v = 555$, $w = 629$
$Z(u) = 111$, $Z(v) = 74$, $Z(w) = 221$

x = 238, y = 816, z = 850
Z(x) = 84, Z(y) = 255, Z(z) = 424
u = 510, v = 680, w = 850
Z(u) = 84, Z(v) = 255, Z(w) = 424

While these numbers do not readily display the pattern of an infinite family of solutions, there is no real reason to think that there is only a finite number of solutions.

Conjecture: There are an infinite number of pairs of Pythagorean triples T(x,y,z) and T(u,v,w) that are Pseudo Smarandache related.

C) A computer program was written to search for two dissimilar 60 degrees triangles T(a,b,c) and T(a1,b1,c1) that are Smarandache related and several solutions were found.

a = 10, b = 14, c = 16, S(a) = 5, S(b) = 7, S(c) = 6
a1 = 30, b1 = 70, c1 = 80, S(a1) = 5, S(b1) = 7, S(c1) = 6

a = 10, b = 14, c = 16, S(a) = 5, S(b) = 7, S(c) = 6
a1 = 45, b1 = 105, c1 = 120, S(a1) = 6, S(b1) = 7, S(c1) = 5

Note that the triangles T(30,70,80) and T(45, 105,120) are similar.

a = 16, b = 19, c = 21, S(a) = 6, S(b) = 19, S(c) = 7
a1 = 80, b1 = 304, c1 = 336, S(a1) = 6, S(b1) = 19, S(c1) = 7

a = 20, b = 28, c = 32, S(a) = 5, S(b) = 7, S(c) = 8
a1 = 60, b1 = 140, c1 = 160, S(a1) = 5, S(b1) = 7, S(c1) = 8

a = 20, b = 28, c = 32, S(a) = 5, S(b) = 7, S(c) = 8
a1 = 120, b1 = 280, c1 = 320, S(a1) = 5, S(b1) = 7, S(c1) = 8

Note again that the triangles T(60,140,160) and T(120, 280, 320) are similar. Given the number of solutions found in this limited search, the following conjecture seems safe.

Conjecture: There are an infinite number of dissimilar 60 degrees triangles that are Smarandache related.

Another computer program was written to search for dissimilar 60 degree triangles T(a,b,c) and T(a1,b1,c1) that are Pseudo Smarandache related. Only four pairs were found in a limited search and they are given below.

a = 24, b = 56, c = 64, Z(a) = 15, Z(b) = 48, Z(c) = 127
a1 = 40, b1 = 56, c1 = 64, Z(a1) = 15, Z(b1) = 48, Z(c1) = 64

a = 49, b = 91, c = 105, Z(a) = 48, Z(b) = 13, Z(c) = 14
a1 = 56, b1 = 91, c1 = 105, Z(a1) = 48, Z(b1) = 13, Z(c1) = 14

a = 42, b = 98, c = 112, Z(a) = 20, Z(b) = 48, Z(c) = 63
a1 = 70, b1 = 98, c1 = 112, Z(a1) = 20, Z(b1) = 48, Z(c1) = 63

a = 42, b = 98, c = 112, Z(a) = 20, Z(b) = 48, Z(c) = 63
a1 = 210, b1 = 294, c1 = 336, Z(a1) = 20, Z(b1) = 48, Z(c1) = 63

Question: Are there an infinite number of dissimilar 60 degrees triangles that are Pseudo Smarandache related?

Solutions to the corresponding problem for dissimilar 120 degrees triangles T(a,b,c) and T(a1,b1,c1) that are Smarandache related were also searched for using another computer program. Several were found, although they appear to be sparser than the corresponding 60 degrees triangles. The solutions that were found are as follows.

a = 32, b = 98, c = 78, S(a) = 8, S(b) = 14, S(c) = 13
a1 = 196, b1 = 364, c1 = 224, S(a1) = 14, S(b1) = 13, S(c1) = 8

a = 32, b = 98, c = 78, S(a) = 8, S(b) = 14, S(c) = 13
a1 = 392, b1 = 728, c1 = 448, S(a1) = 14, S(b1) = 13, S(c1) = 8

a = 51, b = 119, c = 17, S(a) = 17, S(b) = 17, S(c) = 17
a1 = 119, b1 = 221, c1 = 136, S(a1) = 17, S(b1) = 17, S(c1) = 17

a = 51, b = 119, c = 17, S(a) = 17, S(b) = 17, S(c) = 17
a1 = 238, b1 = 442, c1 = 272, S(a1) = 17, S(b1) = 17, S(c1) = 17

a = 51, b = 119, c = 17, S(a) = 17, S(b) = 17, S(c) = 17
a1 = 357, b1 = 663, c1 = 408, S(a1) = 17, S(b1) = 17, S(c1) = 17

a = 51, b = 119, c = 17, S(a) = 17, S(b) = 17, S(c) = 17
a1 = 272, b1 = 833, c1 = 663, S(a1) = 17, S(b1) = 17, S(c1) = 17

a = 51, b = 119, c = 17, S(a) = 17, S(b) = 17, S(c) = 17
a1 = 476, b1 = 884, c1 = 544, S(a1) = 17, S(b1) = 17, S(c1) = 17

Note the cases where the second triangles of pairs are similar.

Question: Is there an infinite family of dissimilar 120 degrees triangles that are Smarandache related?

Finding dissimilar 120 degrees triangles that are Pseudo Smarandache related proved to be more difficult. In a search for all $b \leq 337$, $a \leq 1000$, $c \leq 1000$, $a1 \leq 1000$,

$b1 \le 1000$ and $c1 \le 1000$, only one solution,

a = 168, b = 312, c = 192, Z(a) = 48, Z(b) = 143, Z(c) = 128
a1 = 192, b1 = 588, c1 = 468, Z(a) = 128, Z(b1) = 48, Z(c1) = 143

was found.

Question: Is there an infinite number of dissimilar 120 degrees triangles that are Pseudo Smarandache related?

D) A computer program was written to check for triplets of natural numbers (a,b,c) and (a',b',c') such that a + b + c = a' + b' + c' = 180 and S(a) = S(a'), S(b) = S(b') and S(c) = S(c') and many such pairs of triplets were found. While it is obvious that the number is finite, the following list is not exhaustive.

a = 1, b = 11, c = 168, S(a) = 0, S(b) = 11, S(c) = 7
a' = 1, b' = 14, c' = 165, S(a') = 0, S(b') = 7, S(c') = 11

a = 2, b = 7, c = 171, S(a) = 2, S(b) = 7, S(c) = 19
a' = 2, b' = 38, c' = 140, S(a') = 2, S(b') = 19, S(c') = 7

a = 3, b = 7, c = 170, S(a) = 3, S(b) = 7, S(c) = 17
a' = 6, b' = 21, c' = 153, S(a') = 3, S(b') = 7, S(c') = 17

The last being an example of a pair of triples where there is no number in common.

An exhaustive computer search revealed that all possible angle measures 1 through 178 can be an angle in such a pair of triangles except 83, 97, 107, 113, 121, 127, 137, 139, 149, 151, 163, 166, 167, 169, 172, 173, 174, 175, 176, 177, and 178.

The corresponding problem using the Pseudo Smarandache function is as follows.

Are there two triplets of natural numbers (a,b,c) and (a',b',c') such that a + b + c = a' + b' + c' = 180 and Z(a) = Z(a'), Z(b) = Z(b') and Z(c) = Z(c')?

Another computer program was written that used Z(n) rather than S(n) in the search for such triples. Many solutions exist and some are given below.

a = 2, b = 24, c = 154, Z(2) = 3, Z(24) = 15, Z(154) = 55
a' = 6, b' = 20, c' = 154, Z(6) = 3, Z(20) = 15, Z(154) = 55

a= 4, b = 8, c = 168, Z(4) = 7, Z(8) = 15, Z(168) = 48
a' = 4, b' = 56, c' = 120, Z(4) = 7, Z(56) = 48, Z(120) = 15

$a = 4$, $b = 16$, $c = 160$, $Z(4) = 7$, $Z(16) = 31$, $Z(160) = 64$
$a' = 14$, $b' = 62$, $c' = 104$, $Z(14) = 7$, $Z(62) = 31$, $Z(104) = 64$

The last solution shows us that there are solutions where there are no numbers common to the triples.

There are many solutions to this expression. An exhaustive computer search was performed for all possible values $1 \leq a \leq 178$ and the following numbers did not appear in any triple.

1, 15, 23, 35, 41, 45, 51, 59, 65, 67, 71, 73, 77, 79, 82, 83, 86, 87, 89, 90, 91, 97, 101, 102, 105, 107, 109, 113, 115, 116, 118, 121, 123, 125, 126, 127, 131, 134, 135, 137, 139, 141, 142, 143, 148, 149, 151, 152, 153, 157, 158, 159, 161, 163, 164, 166, 167, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178.

Reference

1. K. R. S. Sastry, 'Smarandache Number Related Triangles'. Appeared on the world wide web at http://www.gallup.unm.edu/~smarandache.

On the Difference $S(Z(n)) - Z(S(n)$

Maohua Le

Abstract: In this paper, we prove that there exist infinitely many positive integers n satisfying $S(Z(n)) > Z(S(n))$ or $S(Z(n)) < Z(S(n))$.

Key words: Smarandache function, Pseudo-Smarandache function, composite function, difference.

For any positive integer n, let $S(n)$, $Z(n)$ denote the Smarandache function and the Pseudo-Smarandache function of n respectively. In this paper, we prove the following results:

Theorem 1: There exist infinitely many n satisfying $S(Z(n)) > Z(S(n))$.

Theorem 2: There exist infinitely many n satisfying $S(Z(n)) < Z(S(n))$.

The above mentioned results solve Problem 21 of [1].

Proof of Theorem 1.
Let p be an odd prime. If $n = (1/2)p(p+1)$, then we have

(1) $S(Z(n)) = S(Z((1/2)p(p+1))) = S(p) = p$

and

(2) $Z(S(n)) = Z(S((1/2)p(p+1))) = Z(p) = p-1$.

We see from (1) and (2) that $S(Z(n)) > Z(S(n))$ for any odd prime p. It is a well-known fact that there exist infinitely many odd primes p. Thus, the theorem is proved.

Proof of Theorem 2.
If $n = p$, where p is an odd prime, then we have

(3) $S(Z(n)) = S(Z(p)) = S(p-1) < p-1$

and

(4) $Z(S(n)) = Z(S(p)) = Z(p) = p-1$.

By (3) and (4), we get $S(Z(n)) < Z(S(n))$ for any p. Thus, the theorem is proved.

Reference

[1] C. Ashbacher, Problems, **Smarandache Notions Journal**, 9(1998), 144-151.

Maohua Le
Department of Mathematics
Zhanjiang Normal College
Zhanjiang, Guangdong
P.R. China

SMARANDACHE CEIL FUNCTIONS*

Anthony Begay
Navajo Community College, Arizona (USA)

ABSTRACT

In this paper some definitions, examples and conjectures are exposed related to the Smarandache type functions, found in the Archives of the Arizona State University, Tempe, USA Special Collections.

(1) Smarandache Ceil Function of Second Order:

2, 4, 3, 6, 4, 6, 10, 12, 5, 9, 14, 8, 6, 20, 22, 15, 12, 7, 10, 26, 18, 28, 30, 21, 8, 34, 12, 15, 38, 20, 9, 42, 44, 30, 46, 24, 14, 33, 10, 52, 18, 28, 58, 39, 60, 11, 62, 25, 42, 16, 66, 45, 68, 70, 12, 21, 74, 30, 76, 51, 78, 40, 18, 82, 84, 13, 57, 86, ...

($S_2(n) = m$, where m is the smallest positive integer for which n divides m^2.)

Reference:

(a) **Surfing on the Ocean of Numbers -- a few Smarandache Notions and Similar Topics**, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 27-30.

(2) Smarandache Ceil Function of Third Order:

2, 2, 3, 6, 4, 6, 10, 6, 5, 3, 14, 4, 6, 10, 22, 15, 12, 7, 10, 26, 6, 14, 30, 21, 4, 34, 6, 15, 38, 20, 9, 42, 22, 30, 46, 12, 14, 33, 10, 26, 6, 28, 58, 39, 30, 11, 62, 5, 42, 8, 66, 15, 34, 70, 12, 21, 74, 30, 38, 51, 78, 20, 18, 82, 42, 13, 57, 86, ...

($S_3(n) = m$, where m is the smallest positive integer for which n divides m^3.)

Reference:

(a) **Surfing on the Ocean of Numbers -- a few Smarandache Notions and Similar Topics**, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 27-30.

(3) Smarandache Ceil Function of Fourth Order:

2, 2, 3, 6, 2, 6, 10, 6, 5, 3, 14, 4, 6, 10, 22, 15, 6, 7, 10, 26, 6, 14, 30, 21, 4, 34, 6, 15, 38, 10, 3, 42, 22, 30, 46, 12, 14, 33, 10, 26, 6, 14, 58, 39, 30, 11, 62, 5, 42, 4, 66, 15, 34, 70, 6, 21, 74, 30, 38, 51, 78, 20, 6, 82, 42, 13, 57, 86, ...

($S_4(n) = m$, where m is the smallest positive integer for which n divides m^4.)

Reference:

(a) **Surfing on the Ocean of Numbers -- a few Smarandache Notions and Similar Topics**, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 27-30.

(4) Smarandache Ceil Function of Fifth Order:

2, 2, 3, 6, 2, 6, 10, 6, 5, 3, 14, 2, 6, 10, 22, 15, 6, 7, 10, 26, 6, 14, 30, 21, 4, 34, 6, 15, 38, 10, 3, 42, 22, 30, 46, 6, 14, 33, 10, 26, 6, 14, 58, 39, 30, 11, 62, 5, 42, 4, 66, 15, 34, 70, 6, 21, 74, 30, 38, 51, 78, 10, 6, 82, 42, 13, 57, 86, ...

(S (n) =m, where m is the smallest positive integer for which n divides m^5.)
5

Reference:

(a) **Surfing on the Ocean of Numbers -- a few Smarandache Notions and Similar Topics**, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 27-30.

(5) Smarandache Ceil Function of Sixth Order:

2, 2, 3, 6, 2, 6, 10, 6, 5, 3, 14, 2, 6, 10, 22, 15, 6, 7, 10, 26, 6, 14, 30, 21, 2, 34, 6, 15, 38, 10, 3, 42, 22, 30, 46, 6, 14, 33, 10, 26, 6, 14, 58, 39, 30, 11, 62, 5, 42, 4, 66, 15, 34, 70, 6, 21, 74, 30, 38, 51, 78, 10, 6, 82, 42, 13, 57, 86, ...

(S (n) = m, where m is the smallest positive integer for which n divides m^6.)
6

Reference:

(a) **Surfing on the Ocean of Numbers -- a few Smarandache Notions and Similar Topics**, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 27-30.

(6) Smarandache - Fibonacci triplets:

11, 121, 4902, 26245, 32112, 64010, 368140, 415664, 2091206, 2519648, 4573053, 7783364, 79269727, 136193976, 321022289, 445810543, 559199345, 670994143, 836250239, 893950202, 937203749, 1041478032, 1148788154, ...

(An integer n such that S(n) = S(n-1) + S(n-2) where S(k) is the Smarandache function: the smallest number k such that S(k)! is divisible by k.)

Remarks:

It is not known if this sequence has infinitely or finitely many terms.

H. Ibstedt and C. Ashbacher independently conjectured that there are infinitely many.

H. I. found the biggest known number: 19 448 047 080 036.

References:

(a) **Surfing on the Ocean of Numbers -- a few Smarandache Notions and Similar Topics**, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 19-23.

(b) C. Ashbacher and M. Mudge, <Personal Computer World>, London, October 1995; p. 302.

(7) Smarandache-Radu duplets

224, 2057, 265225, 843637, 6530355, 24652435, 35558770, 40201975, 45388758, 46297822, 67697937, 138852445, 157906534, 171531580, 299441785, 551787925, 1223918824, 1276553470, 1655870629, 1853717287, 1994004499, 2256222280, ...

(An integer n such that between S(n) and S(n+1) there is no prime [S(n) and S(n + 1) included].

where S(k) is the Smarandache function: the smallest number k such that S(k)! is divisible by k.)

Remarks:

It is not known if this sequence has infinitely or finitely many terms.

H. Ibstedt conjectured that there are infinitely many.

H. I. found the biggest known number:

270 329 975 921 205 253 634 707 051 822 848 570 391 313!

References:

(a) **Surfing on the Ocean of Numbers -- a few Smarandache Notions and Similar Topics**, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 19-23.

(b) I. M. Radu, <Mathematical Spectrum>, Sheffield University, UK, Vol. 27, (No. 2), 1994/5; p. 43.

* Originally appeared in **Bulletin of Pure and Applied Sciences**, Vol. 16E(No. 2), 1997; p. 227-229.

Some Problems Concerning The Smarandache Deconstructive Sequence*

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
Hiawatha, IA 52233
e-mail 71603.522@compuserve.com

The Smarandache Deconstructive Sequence (SDS(n)) of integers is constructed by sequentially repeating the digits 1-9 in the following way:

1, 23, 456, 7891, 23456, 789123, 4567891, 23456789, 123456789, 1234567891, ...

and first appeared in the collection by Smarandache[1]. In a later collection by Kashihara[2], the question was asked:

How many primes are there in this sequence?

In this article, we will briefly explore that question and raise a few others concerning this sequence.

The values of the first thirty elements of this sequence appear in Table 1. From the list, it seems clear that the trailing digits repeat the pattern,

1, 3, 6, 1, 6, 3, 1, 9, 9, 1, 3, 6, 1, 6, 3, 1, 9, 9, 1, . . .

and it is simple to prove that this is indeed the case. Given the rules used in the construction of this sequence, the remaining columns also have similar patterns.

It is also simple to prove that every third element in the sequence is evenly divisible by 3. More specifically, $3 \mid SDS(n)$ if and only if $3 \mid n$.

The list contains the eight primes

23, 4567891, 23456789, 1234567891, 23456789123456789, 23456789123456789123, 4567891234567891234567891, 1234567891234567891234567891.

If we do not consider the first element in the list, the percentage of primes is $\frac{8}{29} = 0.276$. Given this, admittedly slim, numeric evidence and the regular nature of the digits, the author is confident enough to offer the following conjecture.

Conjecture 1: The Smarandache Deconstructive Sequence contains an infinite number of primes.

Two out of every nine numbers end in 6. In examining the factorizations of these numbers, we see that 456 is divisible by 2^3, 23456 by 2^5, and all others by 2^7. This prompts the question:

Table 1.

1	1
2	23
3	456
4	7891
5	23456
6	789123
7	4567891
8	23456789
9	123456789
10	1234567891
11	23456789123
12	456789123456
13	7891234567891
14	23456789123456
15	789123456789123
16	4567891234567891
17	23456789123456789
18	123456789123456789
19	1234567891234567891
20	23456789123456789123
21	456789123456789123456
22	7891234567891234567891
23	23456789123456789123456
24	789123456789123456789123
25	4567891234567891234567891
26	23456789123456789123456789
27	123456789123456789123456789
28	1234567891234567891234567891
29	23456789123456789123456789123
30	456789123456789123456789123456

Question 1: Does every even element of the Smarandache Deconstructive Sequence contain at least three instances of the prime 2 as a factor?

Even more specifically,

Question 2: If we form a sequence from the elements of SDS(n) that end in a 6, do the powers of 2 that divide them form a monotonically increasing sequence?

The following is prompted by examining the divisors of the elements of the sequence.

Question 3: Let k be the largest integer such that $3^k \mid n$ and j the largest integer such that $3^j \mid SDS(n)$. Is it true that k is always equal to j?

And we close with the question

Question 4: Are there any other patterns of divisibility in this sequence?

* This paper originally appeared in **Journal of Recreational Mathematics**, Vol. 29, No. 2.

References

1. F. Smarandache, *Only Problems, Not Solutions*, Xiquan Publishing House, Phoenix, Arizona, 1993.
2. K. Kashihara, *Comments and Topics on Smarandache Notions and Problems*, Erhus University Press, Vail, Arizona, 1996.

S-PRIMALITY DEGREE OF A NUMBER AND S-PRIME NUMBERS

by Emil Burton
Department of Mathematics
Babeş-Bolyai University
3400 Cluj-Napoca, Romania

Abstract.
In this paper we define the S-Primality Degree of a Number, the S-Prime Numbers, and make some considerations on them.

The depths involved by the Smarandache function are far from being exhausted or completely explored.
If one takes $S(1) = 1$ then

$$\sum_{1\le n\le x} \lfloor S(n)/n \rfloor = \begin{cases} \pi(x)+1, & \text{if } 1\le x<4; \\ \pi(x)+2, & \text{if } x\ge 4; \end{cases}$$

where $S(n)$ is the Smarandache function, $\pi(x)$ the number of primes less than or equal to x, and $\lfloor a \rfloor$ the greatest integer less than or equal to a (integer part).

The ratio $S(n)/n$ measures the **S-Primality Degree** (S stands for Smarandache) of the number n.
Whereas n is called **S-Prime** if $S(n)/n = 1$.
Therefore, the set of S-Prime numbers is $P \cup \{1, 4\}$, with $P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$ the set of traditional prime numbers.

Traversing the natural number set $N^* = \{1, 2, 3, 4, 5, 6, \ldots\}$ we meet "the most composite" numbers (= the most "broken up"), i. e. those of the form $n = k!$ for which $S(k!)/k! = k/k! = 1/(k-1)!$
The philosophy of this clasification of the natural numbers is that number 4, for example, appears as a prime (S-Prime) and in the same time composite (broken up).
It is not surprising that in the approachment of Fermat Last THeorem's proof, $x^n+y^n=z^n$ doesn't have nonzero integer solutions for $n \ge 3$, it had had to treat besides the cases $n \in \{3, 5, 7, 11, 13, 17, \ldots\}$ the special case n=4 as well because, for example, $x^8+y^8=z^8$ is reduceable to $(x^2)^4+(y^2)^4=(z^2)^4$.
Also, it is not surprising that Einstein (intuitevily) chosed the R^4 space to treat the relativity theory.
It is not surprising that the multiplication of triplets (a,b,c)(m,n,p) does not really work when we want to sink R^2 into R^3, while the multiplication of quadruplets (a,b,c,d)(m,n,p,q) led to the quaternions theory.

Some Elementary Algebraic Considerations Inspired by Smarandache Type Functions (II)

E. Rădescu

University of Craiova, Department of Mathematics,
1100 Craiova, Romania

Abstract

The paper presents new properties for some functions constructed similarly to the function $\eta : \mathrm{N}^* \to \mathrm{N}^*$, the Smarandache function, defined by $\forall n \in \mathrm{N}^*$, $\eta(n) = \min\{k | k! \text{ is divisible by } n\}$ – "Smarandache's type function".

The Smarandache η function and its principal properties are already known in the literature of speciality. Other functions were built analogously, among which the following ones.

The function η_1. Starting from a sequence of positive integers $\sigma : \mathrm{N}^* \to \mathrm{N}^*$ satisfying the condition

$$\forall n \in \mathrm{N}^*, \exists m_n \in \mathrm{N}^*, \forall m \geq m_n \Rightarrow n / \sigma(m) \tag{1}$$

an associated function was built $\eta_1 : \mathrm{N}^* \to \mathrm{N}^*$, defined by

$$\eta_1(n) = \min\{m_n | m_n \text{ is given by (1) }\}, \forall n \in \mathrm{N}^*. \tag{2}$$

Such sequences - possibly satisfying an extra condition - considered by G. Christol to generalise the p-adic numbers were called also multiplicative convergent to zero (m.c.z.). An example is $\sigma : \mathrm{N}^* \to \mathrm{N}^*$ with $\sigma(n) = n!$. For $n = 6$, there is $m_6' = 4$ such that $\forall m \geq 4 \Rightarrow 6/m!$ ($6/4!$ for $m = 4$; $6/5!$ for $m = 5$) but there is and $m_6'' = 7$ such that $\forall m \geq 7 \Rightarrow 6/m!$; because the smallest of them is $m_6 = 3$ such that $\forall m \geq 3 \Rightarrow 6/3!$, it results $\eta_1(6) = 3$.

We note that for $\sigma(n) = n!$ the associated function η_1 is just the η function - from where the ideea of building the η_i functions (by generalization of the sequence).

The function η_2. A sequence of positive integers $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ is called "of divisibility sequence (d.s.)" if:

$$m/n \Rightarrow \sigma(m)/\sigma(n), \tag{3}$$

and "of strong divisibility sequence (s.d.s.)" if

$$\sigma((m,n)) = (\sigma(m), \sigma(n)), \ \forall m, n \in \mathbf{N}^*, \tag{4}$$

(m, n) being the greatest common factor.

(Strong divisibility sequences are studied for instance by N. Jensen in [5]. It is known that the Fibonacci sequence is a s.d.s.).

Starting from a sequence $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ satisfying the condition

$$\forall n \in \mathbf{N}^*, \exists m_n \in \mathbf{N}^*, \forall m \in \mathbf{N}^*, \ m_n/m \Rightarrow n/\sigma(m) \tag{5}$$

an associated function was built that is $\eta_2 : \mathbf{N}^* \to \mathbf{N}^*$ defined by

$$\eta_2(n) = \min\{m_n | m_n \text{ is given by } (5)\}, \forall n \in \mathbf{N}^*. \tag{6}$$

If the sequence σ is d.s. or s.d.s., the function η_2 has new properties with interesting algebraic interpretations.

We observe that in (1) appeared both the natural order $(m \geq m_n)$ and the divisibility as relation of order on $\mathbf{N}^*$ $(n/\sigma(m))$ and in (5), only the divisibility as relation of order on $\mathbf{N}^*$. From the alternation of the two relations of order on $\mathbf{N}^*$ can be defined analogously two more functions η_3 and η_4. (see [1])

Starting from a sequence $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ satisfying the condition

$$\forall n \in \mathbf{N}^*, \exists m_n \in \mathbf{N}^*, \forall m \in \mathbf{N}^*, m_n/m \Rightarrow n \leq \sigma(m) \tag{7}$$

an associated function was built that is $\eta_3 : \mathbf{N}^* \to \mathbf{N}^*$, defined by

$$\eta_3(n) = \min\{m_n | m_n \text{ is given by } (7)\}, \forall n \in \mathbf{N}^*. \tag{8}$$

Also, starting from a sequence $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ satisfying the condition

$$\forall n \in \mathbf{N}^*, \exists m_n \in \mathbf{N}^*, \forall m \in \mathbf{N}^*, m_n \leq m \Rightarrow n \leq \sigma(m) \tag{9}$$

an associated function was built that is $\eta_4 : \mathbf{N}^* \to \mathbf{N}^*$, defined by

$$\eta_4(n) = \min\{m_n | m_n \text{ is given by } (9)\}, \forall n \in \mathbf{N}^*. \tag{10}$$

The principal properties of the functions above are divided in three groups:

I The arithmetical properties of the proper function.

II The properties of sumatory function associated to each of the numerical functions above. (see [3])

III The algebraical properties of the proper function. Thanks to the arithmetical properties, every function can be viewed as morphism (endomorphism) between certain universal algebras (we can be obtain several situations considering various operations of $\mathbf{N}^*$). (see [2], [4])

This paper presents a construction from group III which guides to a prolongation s_4 of the function η_4 for more complexe universal algebras.

If the initial sequence is s.d.s., the associated function η_4 has a series of important properties from which we retain:

$$\eta_4(\max\{a,b\}) = \max\{\eta_4(a), \eta_4(b)\}\ ; \tag{11}$$

$$\eta_4(\min\{a,b\}) = \min\{\eta_4(a), \eta_4(b)\}\, \forall a,b \in \mathbf{N}^*\,. \tag{12}$$

We may stand out, from other possible structures on $\mathbf{N}^*$, the universal algebra $(\mathbf{N}^*, \Omega)$ where the set of operations is $\Omega = \{\vee, \wedge, \psi_0\}$ with $\vee, \wedge : (\mathbf{N}^*)^2 \to \mathbf{N}^*$ defined by $a \vee b = \sup\{a,b\}$, $a \wedge b = \inf\{a,b\}$, $\forall a,b \in \mathbf{N}^*$ ($\mathbf{N}^*$ is a lattice with the natural order) and $\psi_0 : (\mathbf{N}^*)^0 \to \mathbf{N}^*$ - a null operation that fixes 1, the unique particular element with the role of neutral element for "$\vee$": $1 = e_\vee$.

Therefore, the universal algebra $(\mathbf{N}^*, \Omega)$ is of type $\tau = \begin{pmatrix} \vee & \wedge & \psi_0 \\ 2 & 2 & 0 \end{pmatrix} = (2,2,0)$.

With the properties (11) and (12) the function η_4 is endomorphism for the universal algebra above. It can be stated

Teorema 1 *If $\eta_4 : \mathbf{N}^* \to \mathbf{N}^*$ is the function defined by (10), endomorphism for the universal algebra $(\mathbf{N}^*, \Omega)$ and I is a set, then there is a unique $s_4 : (\mathbf{N}^*)^I \to (\mathbf{N}^*)^I$, endomorphism for the universal algebra $\left((\mathbf{N}^*)^I, \Omega\right)$ so that $p_i \circ s_4 = \eta_4 \circ p_i$, $\forall i \in I$, where $p_j : (\mathbf{N}^*)^I \to \mathbf{N}^*$ with $\forall a = \{a_i\}_{i \in I} \in (\mathbf{N}^*)^I$, $p_j(a) = a_j$, $\forall j \in I$, are the canonical projections, morphisms between $\left((\mathbf{N}^*)^I, \Omega\right)$ and $(\mathbf{N}^*, \Omega)$.*

The proof can be done directly: it is shown that the correspondence η_4 is a function, endomorphism and complies with the required conditions. The operations of Ω for the universal algebra $\left((\mathbf{N}^*)^I, \Omega\right)$ are made "on components".

The algebraic properties of s_4 - the prolongation to more ampler universal algebra of the function η_4 - for its restriction to $\mathbf{N}^*$, could bring new properties for the function η_4 that we considered above.

The paper contents, in completion, a formula of calcul for the sumatory function F_{η_2} of function η_2.

If the initial sequence is s.d.s., this formula is:

$$F_{\eta_2}(n) = \eta_2(1) + \sum_{\substack{h,t=1 \\ h\neq t}}^{k} [\eta_2(p_h), \eta_2(p_t)] + \sum_{\substack{h,t,q=1 \\ h\neq t\neq q}}^{k} [\eta_2(p_h), \eta_2(p_t), \eta_2(p_q)] +$$

$$+ \cdots + \eta_2(n)\,,\ \forall n = p_1 \cdot p_2 \cdots p_k,\ p_i,\ i = \overline{1,k} \text{ - prime numbers}$$

and $\eta_2(p^a) = F_{\eta_2}(p^a) - F_{\eta_2}(p^{a-1})$.

References

[1] Dumitrescu, C., Rocşoreanu, C., *Some connection between the Smarandache function and the Fibonacci sequence*, Smarandache Notions Journal, Vol. 10, No.1-2-3, Spring, 1995, p.50-59

[2] Rădescu, E., Rădescu, N., Dumitrescu, C., *Some elementary algebraic considerations inspired by the Smarandache function*, Smarandache Founction Journal, Vol.6,No.1.1995,p.50-54

[3] Rădescu, E., Rădescu, N., *Some considerations concerning the sumatory function associated to Smarandache function*, Smarandache Notions Journal, Vol.7,No.1-2-3,1996,p.63-69

[4] Rădescu, E., Rădescu, N., *Some elementary algebraic considerations inspired by Smarandache's function (II)*, Smarandache Notions Journal, Vol.7,No.1-2-3,1996,p.70-77

[5] Jensen N., "Some Results on Divisibility Sequences", Appl. of Fibonacci Numbers, Vol. 3, G.E. Bergum and al. Editors, Kluwer Academic Press, 1990, pp. 181-189.

ON THE FUNCTIONAL EQUATION $(S(n))^r + (S(n))^{r-1} + \cdots + S(n) = n$

Rongji Chen

Abstract For any positive integer n, let $S(n)$ be the Smarandache function of n. Let r be a fixed positive integer with $r \geqslant 3$. In this paper we give a necessary and sufficient condition for the functional equation $(S(n))^r + (S(n))^{r-1} + \cdots + S(n) = n$ to have positive integer solutions n.

Key words Smarandache function, functional equation, solvability.

1 Introduction

Let $\mathbb{N}$ be the set of all positive integers. For any $n \in \mathbb{N}$, let the arithmetic function

(1) $\quad S(n) = \min\{a \mid a \in \mathbb{N},\ n \mid a!\}$

Then $S(n)$ is called the Smarandache function of n For a fixed $r \in \mathbb{N}$ with $r \geqslant 3$, we discuss the solvability of the functional equation

(2) $\quad (S(n))^r + (S(n))^{r-1} + \cdots + S(n) = n, n \in \mathbb{N}$

There are many unsolved questions concerned this equation(see [1]). A computer search showed that if $r = 3$, then (2) has no solution n with $n \leqslant 10000$. In this paper we prove a general result as follows.

Theorem For any fixed $r \in \mathbb{N}$ with $r \geqslant 3$, a positive integer n is a solution of (2) if and only if $n = p(p^{r-1} + p^{r-2} + \cdots + 1)$, where p is an odd prime satisfying $p^{r-1} + p^{r-2} + \cdots + 1 \mid (p-1)!$.

By our theorem, we find that if $r = 3$, then (2) has exactly two solutions $n = 305319$ and $n = 499359$ with $n < 1000000$.

2 Preliminaries

Lemma 1 For any u, $v \in \mathbb{N}$ with $\gcd(u, v) = 1$, we have $S(uv) = \max(S(u), S(v))$.

Proof Let $a = S(u)$, $b = S(v)$ and $c = S(uv)$. By (1), a, b, c are least positive integers satisfying

(3) $\quad u \mid a!, \; v \mid b!, \; uv \mid c!,$

respectively. We see from (3) that

(4) $\quad c \geqslant \max(a, b)$

If $a \geqslant b$, then $u \mid a!$ and $v \mid a!$ by (3). Since gcd $(u, v) = 1$, we get $uv \mid a!$. It implies that $a \geqslant c$. Therefore, by (4), we obtain $c = a = \max(a, b)$. By the same method, we can prove that if $a \leqslant b$, then $c = b = \max(a, b)$. The lemma is proved.

Lemma 2 If $S(u) = u$, then $u = 1, 4$ or p, where p is a prime.

Proof See [3].

Lemma 3 If $u > 1$, where $u \in \mathbb{N}$, then u has a prime factor p such that $p \mid S(u)$.

Proof Let $u = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ be the factorization of u. It is a well known fact that $S(u) = \max(S(p_1^{r_1}), S(p_2^{r_2}), \cdots, S(p_k^{r_k}))$ and $p_i \mid S(p_i^{r_i})$ for $i = 1, 2, \cdots, k$ (see [2]). The lemma follows immediately.

3 Proof of Theorem

Let $n=p(p^{r-1}+p^{r-2}+\cdots+1)$, where p is an odd prime satisfying $p^{r-1}+p^{r-2}+\cdots+1 \mid (p-1)!$. Then, by (1), we get $S(n)=p$. Therefore, n is a solution of (2).

On the other hand, let n be a solution of (2). Then we have $n>1$. Further, let $t=S(n)$. We get from (2) that

(5) $\quad t(t^{r-1}+t^{r-2}+\cdots+1)=n.$

Since $\gcd(t, t^{r-1}+t^{r-2}+\cdots+1)=1$, by Lemma 1, we see from (5) that

(6) $t=S(n)=S(t(t^{r-1}+t^{r-2}+\cdots+1))$
$\quad =\max(S(t), S(t^{r-1}+t^{r-2}+\cdots+1))$

If $S(t)\leqslant S(t^{r-1}+t^{r-2}+\cdots+1)$, then from (6) we get

(7) $\quad t=S(t^{r-1}+t^{r-2}+\cdots+1)$

Since $t^{r-1}+t^{r-2}+\cdots+1>1$, by Lemma 3, $t^{r-1}+t^{r-2}+\cdots+1$ has a prime factor p such that $p \mid S(t^{r-1}+t^{r-2}+\cdots+1)$. Hence, by (7), we get $p \mid t$. However, since $\gcd(t, t^{r-1}+t^{r-2}+\cdots+1)=1$, it is impossible. So we have

(8) $\quad S(t)>S(t^{r-1}+t^{r-2}+\cdots+1)$

and

(9) $\quad t=S(t),$

by (6)

On applying Lemma 2, we see from (9) that either $t=4$ or $t=p$, where p is a prime. If $t=4$, then $n=4$, 8, 12 or 24. However, since $r\geqslant 3$, we get from (5) that $t^r+t^{r-1}+\cdots+t\geqslant 4^3+4^2+4>24\geqslant n$, a contradiction. If $t=p$, then from (8) and (9) we obtain

(10) $\quad S(p^{r-1}+p^{r-2}+\cdots+1)<S(t)=S(p)=p$

It implies that $p^{r-1}+p^{r-2}+\cdots+1 \mid (p-1)!$ and $p>2$. Therefore, we see from (5) that if n is a solution of (2), then $n=p(p^{r-1}+p^{r-2}+\cdots+1)$, where p is an odd prime satisfying $p^{r-1}+p^{r-2}+\cdots+1 \mid (p-1)!$. Thus, the theorem is proved.

References

[1] C. Ashbacher, Unsolved problems, Smarandache Notions J., 9 (1998), 152-155.

[2] C. Dumitrescu and V. Seleacu, Some Notions and Questions in Number Theory, Erhus Univ. Press, Glendale, 1994.

[3] M.-H. Le, On the diophantine equation S(n) = n, Smarandache Notions J., 10(1999), 142-143.

Department of Mathematics
Maoming Educational College
Maoming, Guangdong
P. R. China

SOLUTION OF TWO QUESTIONS CONCERNING THE DIVISOR FUNCTION AND THE PSEUDO - SMARANDACHE FUNCTION

Zhong Li

Abstract In this paper we completely solve two questions concerning the divisor function and the pseudo - Smarandache function.

Key words divisor function, pseudo - Smarandache function, functional equation

1 Introduction

Let $\mathbb{N}$ be the set of all positive integers . For any $n \in \mathbb{N}$, let

(1) $$d(n)=\sum_{d \mid n} 1,$$

(2) $$Z(n)=\min\{a \mid a \in \mathbb{N}, n \mid \sum_{j=1}^{a} j\}$$

Then $d(n)$ and $Z(n)$ are called the divisor function and the pseudo - Smarandache function of n, respectively, In[1], Ashbacher posed the following unsolved questions.

Question 1 How many solutions n are there to the functional equation.

(3) $$Z(n)=d(n), n \in \mathbb{N}?$$

Question 2 How many solutions n are there to the functional equation.

(4) $$Z(n)+d(n)=n, n\in\mathbb{N}?$$

In this paper we completely solve the above questions as follows.

Theorem 1 The equation (3) has only the solutions $n=1,3$ and 10.

Theorem 2 The equation (4) has only the solution $n=56$.

2 Proof of Theorem 1

A computer search showed that (3) has only the solutions $n=1,3$ and 10 with $n\leqslant 10000$ (see [1])

We now let n be a solution of (3) with $n\neq 1,3$ or 10. Then we have $n>10000$. Let

(5) $$n=p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$$

be the factorization of n. By [2, Theorem 273], we get from (1) and (5) that

(6) $$d(n)=(r_1+1)(r_2+1)\cdots(r_k+1).$$

On the other hand, since $\sum_{j=1}^{a} j=a(a+1)/2$ for any $a\in\mathbb{N}$, we see from (2) that $n\mid Z(n)(Z(n)+1)/2$. It implies that $Z(n)(Z(n)+1)/2\geqslant n$. So we have

(7) $$Z(n)\geqslant\sqrt{2n+\frac{1}{4}}-\frac{1}{2}$$

Hence, by (3), (5), (6) and (7), we get

(8) $$1>\sqrt{2}\prod_{i=1}^{k}\frac{p_i^{r_i/2}}{r_i+1}-\frac{1}{2}\prod_{i=1}^{k}\frac{1}{r_i+1}$$

If $p_1>3$, then from (8) we get $p_1\geqslant 5$ and

$$1\geqslant\sqrt{2}\left(\frac{\sqrt{5}}{2}\right)^k-\frac{1}{2^{k+1}}>1,$$

a contradiction. Therefore, if (8) holds, then either $p_1=2$ or $p_1=3$. By the same method, then n must satisfy one of the following conditions.

(i) $p_1=2$ and $r_1 \leqslant 4$.

(ii) $p_1=3$ and $r_1=1$.

However, by (8), we can calculate that $n<10000$, a contradiction. Thus, the theorem is proved.

3 Proof of Theorem 2

A computer search showed that (4) has only the solution $n=56$ with $n \leqslant 10000$ (see [1]). We now let n be a solution of (4) with $n \neq 56$. Then we have $n>10000$. We see from (4) that

(9) $$Z(n) \equiv -d(n) \pmod n$$

It implies that.

(10) $$Z(n)+1 \equiv 1-d(n) \pmod n$$

By the proof of Theorem 1, we have $n \mid Z(n)(Z(n)+1)/2$, by (2). It can be written as

(11) $$Z(n)(Z(n)+1) \equiv 0 \pmod n.$$

Substituting (9) and (10) into (11), we get

(12) $$d(n)(d(n)-1) \equiv 0 \pmod n.$$

Notice that $d(n)>1$ if $n>1$. We see from (12) that

(13) $$(d(n))^2>n$$

Let (5) be the factorization of n . By (5), (6) and (13), we obtain

(14) $$1>\prod_{i=1}^{k} \frac{p_i^{r_i}}{(r_i+1)^2}$$

On the other hand, it is a well known fact that $Z(p^r) \geq p^r - 1 > (r+1)^2$ for any prime power p^r with $p^r > 32$. We find from (14) that $k \geq 2$.

If $p_1 > 3$, then $p_i^{r_i}/(r_i+1)^2 \geq 5/4 > 1$ for $i = 1, 2, \cdots k$, It implies that if (14) holds, then either $p_1 = 2$ or $p_1 = 3$. By the same method, then n must satisfy one of the following conditions:

(i) $p_1 = 2, p_2 = 3$ and $(r_1, r_2) = (1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (1,2), (2,2), (3,2), (4,2)$ or $(5,2)$.

(ii) $p_1 = 2, p_2 > 3$ and $r_1 \leq 5$.

(iii) $p_1 = 3$ and $r_1 = 1$.

However, by (14), we can calculate that $n < 10000$, a contradiction. Thus, the theorem is proved.

References

[1]C. Ashbacher, The pseudo – Smarandache function and the classical functions of number theory, Smarandache Notions J., 9(1998), 78 – 81.

[2]G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, Oxford Univ. Press, 1937.

Department of Mathematics
Maoming Educational College
Maoming, Guangdong
P. R. China

ON A CONJECTURE OF SMARANDACHE ON PRIME NUMBERS

J. Sándor
Department of Mathematics
Babeş-Bolyai University
Cluj-Napoca, Romania

Let p_n denote the n-th prime number. One of Smarandache's conjectures in [3] is the following inequality:

$$p_{n+1}/p_n \le 5/3, \text{ with equality for } n = 2. \tag{1}$$

Clearly, for n = 1, 2, 3, 4 this is true, and for n = 2 there is equality. Let n > 4. Then we prove that (1) holds true with strict inequality. Indeed, by a result of Dressler, Pigno and Young (see [1] or [2]) we have

$$p_{n+1}^2 \le 2p_n^2. \tag{2}$$

Thus $p_{n+1}/p_n \le \sqrt{2} \le 5/3$, since $3\sqrt{2} < 5$ (i.e. 18 < 25). This finishes the proof of (1).

References:

[1] R. E. Dressler, L. Pigno and R. Young, *Sums of Squares of Primes*, Nordisk Mat. Tidskrift 24(1976), 39.

[2] D. S. Mitrinović and J. Sándor (in coop. with B. Crstici), *Handbook of Number Theory*, Kluwer Acad. Publ., 1995.

[3] M. L. Perez, editor, *Five Smarandache Conjectures on Primes*, http://www.gallup.unm.edu/~smarandache/conjprim.txt.

On the Smarandache Irrationality Conjecture

Florian Luca

The Smarandache Irratioality Conjecture (see [1]) claims:

Conjecture.

Let $a(n)$ be the nth term of a Smarandache sequence. Then, the number

$$0.a(1)a(2)...a(n)...$$

is irrational.

Here is an immediate proof in the following cases:

1. $a(n) = n$;
2. $a(n) = d(n)$ =number of divisors of n;
3. $a(n) = \omega(n)$ =number of distinct prime divisors of n;
4. $a(n) = \Omega(n)$ =number of total prime divisors of n (that is, counted with repetitions);
5. $a(n) = \phi(n)$ =the Euler function of n;
6. $a(n) = \sigma(n)$ =the sum of the divisors of n;
7. $a(n) = p_n$ =the nth prime;
8. $a(n) = \pi(n)$ =the number of primes smaller than n;
9. $a(n) = S(n)$ =the Smarandache function of n;
10. $a(n) = n!$;
11. $a(n) = a^n$, where a is any fixed positive integer coprime to 10 and larger than 1;
12. $a(n)$ =any fixed non-constant polynomial in one of the above;

Here is the argument:
Assume that

$$0.a(1)a(2)...a(n)...$$

is rational. Write it under the form

$$0.a(1)a(2)...a(n)... = 0.ABBBB...,$$

where A is some block of digits and B is some other repeting block of digits. Asume that B has length t. If there exist infinitely many $a(n)$'s such that the decimal representation of $a(n)$ contains at least $2t$ consecutive zeros, then, since B has lenght t, it follows that the block of these $2t$ consecutive zeros will contain a full period B. Hence, $B = 0$ and the number has, in fact, only finitely many nonzero decimals, which is impossible because $a(n)$ is never zero.

All it is left to do is to notice that if $a(n)$ is any of the 12 sequences above, then $a(n)$ has the property that there exist arbitrarily many consecutive zero's in the decimal representation of $a(n)$. This is clear for the sequences 1, 2, 3, 4, 8 and 9 because these functions are onto, hence they have all the positive integers in their range. It is also obvious for the sequence 10 because $n!$ becomes divisible with arbitrarily large powers of 10 when n is large. For the sequence 7, fix any t and choose infinitely many primes from the progression $(10^{2t+2}k + 1)_{k \geq 0}$ whose first term is 1 and whose difference is 10^{2t+2}. This is possible by Dirichlet's theorem. Such a prime will end in ...00000001 with $2t+1$ consecutive zero's. For the sequence 5, notice that the Euler function of the primes constructed above is of the form $10^{2t+2}k$, hence it ends in $2t+2$ zeros, while for the sequence 6, notice that the divisor sum of the above primes is of the form $10^{2t+2}k+2$, hence it ends in ...000002 with $2t+1$ consecutive zeros. For the sequence 11, since a is coprime to 10, it follows that for any t there exist infinitely many n's such that $a^n \equiv 1 \pmod{10^{2t+2}}$. To see why this happens, simply choose n to be any multiple of the Euler function of 10^{2t+2}. What the above congruence says, is that a^n is of the form .....0000001 with at least $2t+1$ consecutive zero's (here is why we don't want a to be 1).

Now 12 should also be obvious. It is also clear that the argument can be extended to any base.

It certainly seems much harder to conclude if any one of those series is transcendal or not.

Reference

[1] *Smarandache Irrationality Conjectures*, at http://www.gallup.unm.edu/ smarandache/

Mathematical Institute
Czech Academy of Sciences
Žitná 25, 115 67 Praha 1
CZECH REPUBLIC
e-mail: luca@matsrv.math.cas.cz

A NOTE ON S(n), WHERE n IS AN EVEN PERFECT NUMBER

J. Sándor
Department of Mathematics
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

In a recent paper [1] the following result is proved: If $n = 2^{k-1}(2^k-1)$, 2^k-1 = prime, is an even perfect number, then $S(n) = 2^k-1$, where $S(n)$ is the well-known Smarandache Function.

Since $S(ab) = \max \{S(a), S(b)\}$ for $(a, b) = 1$, and $S(a) \le a$ with equality for $a = 1, 4$, and a = prime (see [3]), we have the folowwing one-line proof of this result:
$S(\ 2^{k-1}(2^k-1)\ ) = \max \{\ S(2^{k-1}), S(2^k-1)\ \} = 2^k-1$,
since $S(2^{k-1}) \le 2^{k-1} < 2^k-1$ for $k \ge 2$.
On the other hand, if 2^k-1 is prime, then we have $S(2^k-1) \equiv 1 \pmod k$; an interesting table is considered in [2]. Indeed, k must be a prime too, $k = p$; while Fermat's little theorem gives $2^p-1 \equiv 1 \pmod p$. From $2^{2p}-1 = (2^p-1)(2^p+1)$ and $(2^p-1, 2^p+1) = 1$ we can deduce $S(2^{2p}-1) = \max \{\ S(2^p-1), S(2^p+1)\ \} = 2^p-1$ since 2^p+1 is being composite, $S(2^p+1) < 2/3(2^p+1) < 2^p-1$ for $p \ge 3$. Thus, if 2^k-1 is a Mersenne prime, then $S(2^k-1) \equiv S(2^{2k}-1) \equiv 1 \pmod k$. If 2^p-1 and $2^{2p}+1$ are both primes, then
$S(2^{4p}-1) = \max \{\ S(2^{2p}-1), S(2^{2p}+1)\ \} = 2^{2p}+1 \not\equiv 1 \pmod{4p}$.

References:

1. Sebastian Martin Ruiz, "Smarandache's Function Applied to Perfect Numbers", <Smarandache Notions J.>, 10 (1999), No. 1-2-3, 114-115.
2. Sebastian Martin Ruiz, "A Congruence with Smarandache's Function", <Smarandache Notions J.>, 10 (1999), No. 1-2-3, 130-132.
3. Charles Ashbacher, "An Introduction to the Smarandache Function", Erhus Univ. Press, Vail, AZ, 1995.

MOMENTS OF THE SMARANDACHE FUNCTION

Steven R. Finch
MathSoft Inc.
101 Main Street
Cambridge, MA, USA 02142
sfinch@mathsoft.com

Given a positive integer n, let $P(n)$ denote the largest prime factor of n and $S(n)$ denote the smallest integer m such that n divides $m!$

This paper extends earlier work [1] on the average value of the Smarandache function $S(n)$ and is based on a recent asymptotic result [2]:

$$\left|\{n \le N : P(n) < S(n)\}\right| = o\left(\frac{N}{\ln(N)^j}\right) \quad \text{for any positive integer j}$$

due to Ford. We first prove:

Theorem 1. $E(S(N)^k) = \frac{1}{N}\cdot\sum_{n=1}^{N} S(n)^k = \frac{\zeta(k+1)}{k+1}\cdot\frac{N^k}{\ln(k)} + O\left(\frac{N^k}{\ln(N)^2}\right)$

where $\zeta(x)$ is the Riemann zeta function. In particular,

$$\lim_{N\to\infty}\frac{\ln(N)}{N}\cdot E(S(N)) = \frac{\pi^2}{12} = 0.82246703\ldots$$

$$\lim_{N\to\infty}\frac{\ln(N)}{N^2}\cdot Var(S(N)) = \frac{\zeta(3)}{3} = 0.40068563\ldots$$

Sketch of Proof. On one hand,

$$L(k) = \lim_{n\to\infty}\frac{\ln(n)}{n^k}\cdot E(P(n)^k) \le \lim_{n\to\infty}\frac{\ln(n)}{n^k}\cdot E(S(n)^k) = \lim_{N\to\infty}\frac{\ln(N)}{N^{k+1}}\cdot\sum_{n=1}^{N} S(n)^k$$

The above summation, on the other hand, breaks into two parts:

$$\lim_{N\to\infty}\frac{\ln(N)}{N^{k+1}}\cdot\left(\sum_{P(n)=S(n)} P(n)^k + \sum_{P(n)<S(n)} S(n)^k\right)$$

The second part vanishes:

$$\lim_{N\to\infty}\frac{\ln(N)}{N}\cdot\left(\sum_{P(n)<S(n)}\left(\frac{S(n)}{N}\right)^k\right)\le\lim_{N\to\infty}\frac{\ln(N)}{N}\cdot\left(\sum_{P(n)<S(n)}1\right)=\lim_{N\to\infty}\frac{\ln(N)}{N}\cdot o\left(\frac{N}{\ln(N)}\right)=0$$

while the first part is bounded from above:

$$\lim_{N\to\infty}\frac{\ln(N)}{N^{k+1}}\cdot\left(\sum_{P(n)=S(n)}P(n)^k\right)\le\lim_{N\to\infty}\frac{\ln(N)}{N^{k+1}}\cdot\sum_{n=1}^{N}P(n)^k=\lim_{n\to\infty}\frac{\ln(n)}{n^k}\cdot E\left(P(n)^k\right)=L(k)$$

A formula for *L(k)* was found by Knuth and Trabb Pardo [3] and the remaining second-order details follow similarly.

Observe that the ratio $\sqrt{Var(S(N))}\,/\,E(S(N))\to\infty$ as $N\to\infty$, which indicates that the traditional sample moments are unsuitable for estimating the probability distribution of *S(N)*. An alternative estimate involves the relative number of digits in the output of *S* per digit in the input. A proof of the following is similar to [1]; the integral formulas were discovered by Shepp and Lloyd [4].

Theorem 2.

$$\lim_{N\to\infty}E\left(\left\{\frac{\ln(S(N))}{\ln(N)}\right\}^k\right)=\int_0^\infty\frac{x^{k-1}}{k!}\cdot\exp\left(-x-\int_x^\infty\frac{e^{-y}}{y}dy\right)dx=\begin{cases}0.62432998 & \text{if} \quad k=1\\ 0.42669576 & \text{if} \quad k=2\\ 0.31363067 & \text{if} \quad k=3\\ 0.24387660 & \text{if} \quad k=4\\ 0.19792289 & \text{if} \quad k=5\end{cases}$$

The mean output of *S* hence has asymptotically 62.43% of the number of digits of the input, with a standard deviation of 19.21%. A web-based essay on the Golomb-Dickman constant 0.62432998... appears in [5] and has further extensions and references.

References

1. S. R. Finch, The average value of the Smarandache function, *Smarandache Notions Journal* 9 (1998) 95-96.
2. K. Ford, The normal behavior of Smarandache function, *Smarandache Notions Journal* 9 (1998) 81-86.
3. D. E. Knuth and L. Trabb Pardo, Analysis of a simple factorization algorithm, *Theoret. Comp. Sci.* 3 (1976) 321-348.
4. L. A. Shepp and S. P. Lloyd, Ordered cycle lengths in a random permutation, *Trans. Amer. Math. Soc.* 121 (1966) 350-557.
5. S. R. Finch, *Favorite Mathematical Constants*, website URL http://www.mathsoft.com/asolve/constant/constant.html, MathSoft Inc.

THE INTANGIBLE ABSOLUTE TRUTH

by Gheorghe Dinulescu-Câmpina,
"N. Grigorescu" College, Câmpina, Romania

In my own work "The Modelling of the Rationality" under the basis of the MESER licence, I have enlightened a new spiritual doctrine sustained by scientific and logical hypotheses.

The reception of the soundnes of the mentioned notion proceeds from the Einstein's principle concerning "The internal perfection and its external acknowlegment" but, like other responsible "creators", I felt that it was necessary to consider the expression of the feelings of the uncertainty, mine first.

Although I found many external recognitions in our great forerunners' ideas and theories, we have not had a proven substantiation yet (which is not by all means necessary with philosophical hypotheses) of the hypotheses that I have forwarded.

I am willing to belive it was not accidentally that I got knowledge of the ideas set forth by the mathematician and philosopher Florentin Smarandache, the creator of the Neutrosophy, as a branch of Philosophy, that studies the origin , the character, the aim and the interactions of the neutralities from the spectrum of ideality.

I have established that the Neutrosophy Theory, that belongs to the mentioned thinker, sets up as the scientifically demonstrated fundament for the great majority of the hypotheses I have set forth in "The Modelling of Rationality".

Esentially, Professor Smarandache's Neutrosophy stimulates that for any idea <A> there is also an idea <anti A> and another <neut A>.

The fundamental thesis of the Neutrosophy is: if <A> is t% true and f% false, as bivalent extremes, as a matter of course i% is indeterminant, as a result, t+i+f=100 (or t%+i%+f%=1) which gives a meaning, easily altered, to the usual notions as, for example, the one of complementarity.

Consequently, the complement of t is not f, but i+f, and the complement of f is not t, but t+i.

Florentin Smarandache's theory of Neutrosophy suggests also the fact that any hypothesis has a nature of extreme (it allows an anty-hypothesis and a neutro-hypothesis) which is not bad because t+i+f=100 must be considered dialectically, where both t and f tend to be decreasing without annihilating each other in the advantage of i.

Far from the idea that any hypothesis should not have a nature of extreme, just such a nature is desirable to generate polemics which, in case of confrontation, draws nearer t and f aiming at the neutral equilibrium of the t+f+i=100 relationship, that provides the opportunity of accomplishment.

The theory of Neutrosophy makes obvious the relative nature of the truth and the false, only the neutral nature tending to the absolute owing to its force of accomplishment.

Thanks to the specifications that are stimulated in Smarandache's Neutrosophy, the hypothesis of the MESER concept as: the complementarity between the sacred and the profane, between the divine creation and the intra-specific evolution, the non-contradiction between science and religion, materialism(substantialism) and idealism, between gnosticism and agnosticism, prove to be rational and therefore real and the paradoxes become justified.

Related to the sense of knowledge the MESER concept identifies two modalities: the scientific knowledge that specialises knowledge "more and more from that <<less and less>>" and the philosophic, encyclopaedic knowledge "less and less from that<<more and more>>".

If the first modality is limited especially by the posibilities of communication, the second one is also limited by the insufficient power of comprehensibility of the human mind.

The equilibrium between the two directions which, in the last analysis signifies the way to the truth, is ddetermined by the divine laws of the dissociations, purification (the selection and the dissolving of what is settled, established for good) and those of monadic recomposition, laws that ascertain for the general knowledge a social character, expressed by the syntagme "more and more from <<more and more>>" rendered by the well-known paradox "the more you learn, the less you know."

After all, the fundamental law of Neutrosophy is a succesful attempt for resolving the paradox of the knowledge and confirms that the absolute truth is intangible not in a derogatory way but in an optimistic one, approved and significant by the will of God.

Being operative even in the case of the characteristic interpretations, as the present one, Neutrosophy confirms its viability even by the fact that it suggests methods, modalities of evaluations, and new interpretative views.

References:

1.Dinulescu-Câmpina, Gheorghe, "Modelarea Ratiunii sau Doctrina Sufletului", manuscript.

2.Smarandache, Florentin, "Neutrosophy": Neutrosophic Probability, Set, and Logic", American Research Press, Rehoboth, 105 pages, first edition, 1998; online in five parts at:

a) Neutrosophy, a New Branch of Philosophy:
http://www.gallup.unm.edu/~smarandache/NeutroSo.txt

b) TransDisciplinarity, a Neutrosophic Method:
http://www.gallup.unm.edu/~smarandache/TransDis.txt

c) Neutrosophic Probability, a Generalization of Classical Probability:
http://www.gallup.unm.edu/~smarandache/NeutProb.txt

d) Neutrophic Set, a Generalization of the Fuzzy Set:
http://www.gallup.unm.edu/~smarandache/NeutSet.txt

e) Neutrosophic Logic, a Generalization of the Fuzzy Logic:
http://www.gallup.unm.edu/~smarandache/NeutLoc.txt

ADVANCE OF SMARANDACHE APPROACH TO SOLVING SYSTEMS OF DIOPHANTINE EQUATIONS

Y. V. CHEBRAKOV

Department of Mathematics, Technical University,
Nevsky 3-11, 191186, St-Petersburg, Russia
E-mail: chebra@phdeg.hop.stu.neva.ru

By developing F. Smarandache (algebraic) approach to solving systems of Diophantine equations we elaborate a set of new computative algorithms and analytical formulae, which may be used for finding numerical solutions of some combinatorial and number-theoretic problems.

Key words: systems of Diophantine equations, algebraic approach, combinatorics, number theory, Magic and Latin squares.

1 Introduction

Let it be required to solve some system of Diophantine equations. In this case [1] algebraic methods can be applied for

a) constructing the total algebraic solution of the system;

b) finding the transformations translating an algebraic solution of the system from one form into another one;

c) elucidating the general legitimacies existing between the elements of the algebraic solution;

d) replacing the total algebraic solution containing L arbitrary selected parameters by a set of algebraic solutions containing less than L parameters.

This paper is devoted to further advance of algebraic approach to solving systems of Diophantine equations. In particular, in this investigation we

1) describe the simple way of obtaining a total solution of systems of Diophantine linear equations in the integer numbers, and show (see Sect. 2) that this way may be considered as some modification of F. Smarandache algorithm 3 from his work [2];

2) demonstrate the effectiveness of the algebraic approach to the elaboration of computative algorithms and analytical formulae, which may be used respectively for obtaining the required numerical solutions of the discussed systems and for counting of the total quantity of solutions from a given class of numbers (Sect. 3);

3) derive analytical formulae available for constructing classical Magic squares of both odd and even orders (Sect. 4).

2 The way of obtaining the total solution of systems of Diophantine linear equations in the integer numbers

Let it be required to solve some system of linear Diophantine equations in the integer numbers. It seems to be evident, that there is no complication in solving this problem at present. For instance, one may find in the work[2] as many as five different algorithms to obtain a total solution of this problem, which correctness are proved by mathematical methods and illustrated by concrete examples. In particular, to illustrate the correctness of an algorithm 3, the system from three following equations

$$\begin{cases} 3x_1 + 4x_2 \quad + \quad 22x_4 - 8x_5 = 25 \\ 6x_1 \qquad + \qquad 46x_4 - 12x_5 = 2 \\ \qquad 4x_2 + 3x_3 - x_4 + 9x_5 = 26 \end{cases} \tag{1}$$

are solved in the work[2]. The final solution, obtained for (1) by the mentioned algorithm, has form

$$x_1 = -40k_1 - 92k_2 + 27; \quad x_2 = 3k_1 + 3k_2 + 4; \quad x_3 = -11k_1 + 8; \tag{2}$$
$$x_4 = 6k_1 + 12k_2 - 4; \quad x_5 = 3k_1 - 2,$$

where k_1 and k_2 are any integer numbers.

Let us clear up a question whether (2) is the total solution of system (1) in the integer numbers. To make it we will solve (2) by the algebraic methods with testing their correctness on every step of our computations.

1. As well-known[1-3], the total algebraic solution of the system (1) may be found by standard algebraic methods (for instance, by Gauss method). In our case it has the form

$$\begin{cases} x_1 = -(23x_4 - 6x_5 - 1)/3 \\ x_2 = (x_4 + 2x_5 + 24)/4 \\ x_3 = (-11x_5 + 2)/3 \end{cases} \tag{3}$$

that coincides with the solution found on the first step of the algorithm 3 of the work[2].

2. As well-known from the theory of comparison[3,4], the total solution in the integer numbers for the last equation of the system (3) has the form $x_3 = \{-11(3m_1 + m_0) + 2\}/3$, where m_1 is any integer number; the value of m_0 is equal 0, ±1 or ±2 and is chosen from the condition that the number $(-11m_0 + 2)/3$ must be integer. Thus, we find on second step of our computations that $x_5 = 3m_1 + 1$ and $x_3 = -11m_1 - 3$.

We note that the solution $x_5 = 3k_1 - 2$ of (2) may be obtained from our solution by change of the variable m_1 to $k_1 - 1$. Thus, both values of x_5 are identical solutions.

3. Let us get to solving the second equation of the system (3) in the integer numbers. Replacing the value of x_5 by $3m_1+1$ in this equation we obtain that $x_2 = 6 + (6m_1 + x_4 + 2)/4$. Hence it appears (see point 2) that $x_4 = (-2+4l_1)m_1 - 2 + 4l_2$, where l_1 and l_2 are any integer numbers.

4. Replacing the value of x_4 by $(-2+4l_1)m_1 - 2 + 4l_2$ in the first equation of the system (3) we obtain that $x_1 = 2x_5 - \{23(-2+4l_1)m_1 + 92l_2 - 47\}/3$. Hence it appears (see points 2 and 3) that $l_1 = 3m_2 + 2$ and $l_2 = 3m_3 + 1$ and, consequently, the total solution of the system (1) in the integer numbers has the form

$$x_1 = -4m_1(23m_2+10) - 92m_3 - 13; \quad x_2 = 3m_1(m_2+1) + 3m_3 + 7; \tag{4}$$
$$x_3 = -11m_1 - 3; \quad x_4 = 6m_1(2m_2+1) + 2(6m_3+1); \quad x_5 = 3m_1 + 1,$$

where m_1, m_2 and m_3 are any integer numbers.

Comparing (4) with (2) we find that

a) the solution (4) contains greater by one parameter than solution (2);

b) if $m_2 = 0$, $m_1 = k_1 - 1$ and $m_3 = k_2$ in the solution (4) then the solution (4) coincides with the solution (2).

Thus, the solution (4) contains all numerical solutions of the system (1), which may be obtained from (2), but a part of numerical solutions, which may be obtained by (4), can not obtain from (2) or, in other words, (2) is not the total solution of the system (1) in the integer numbers.

We add that, in general, a partial loss of numerical solutions of systems of linear Diophantine equations may have more serious consequences than in the discussed case. For instance, as it has been proved in the work[5] by using the algebraic approach to solving systems of Diophantine equations,

if Magic squares of 4th order contain in its cells 8 even and 8 odd numbers then they can not have structure patterns another than 12 ones, adduced in works[1, 3, 5–7] *for Magic squares, contained integer numbers from 1 to 16.*

In reality, this statement is incorrect because yet several new structure patterns may exist for Magic squares from 8 even and 8 odd numbers[1, 3].

3 Analysing a system from 8 linear Diophantine equations

To demonstrate the effectiveness of the algebraic approach to solving some combinatorial and number-theoretic problems, presented in the form of systems of Diophantine equations, in this section we will analyse the following system from 8 linear Diophantine equations

$$\begin{array}{lll} 1.\ a_1 + a_2 + a_3 = S, & 4.\ a_1 + a_4 + a_7 = S, & 7.\ a_1 + a_5 + a_9 = S, \\ 2.\ a_4 + a_5 + a_6 = S, & 5.\ a_2 + a_5 + a_8 = S, & 8.\ a_3 + a_5 + a_7 = S. \\ 3.\ a_7 + a_8 + a_9 = S, & 6.\ a_3 + a_6 + a_9 = S, & \end{array} \tag{5}$$

We note if symbols $a_1, a_2, \ldots, a_9$ are arranged as in the table 1, shown in figure, and their values are replaced by ones, which are taken from some total algebraic solution of the system (5), then table 1 will be transformed into the total algebraic formula of Magic squares of 3rd order. In other words, the discussed problem on solving the system (5) connects direct with the well-know ancient mathematical problem on constructing numerical examples of Magic squares of 3rd order.

3.1 Requirements to a set of numbers, which is the solution of the system (5)

Proposition 1. *A set of nine numbers is a solution of the system (5) only in the case if one succeeds to represent these nine numbers in the form of such three arithmetic* *progressions from 3 numbers whose differences are identical and the first terms of all three progressions are also forming an arithmetic progression.*

Proof. Using standard algebraic methods (for instance, Gauss method) we find that the total algebraic solution of the system (5) has the form

$$a_1 = 2a_5 - a_9;\quad a_2 = 2a_9 + a_6 - 2a_5;\quad a_3 = 3a_5 - a_6 - a_9;\quad a_4 = 2a_5 - a_6;\quad a_7 = a_9 + a_6 - a_5;\quad a_8 = 4a_5 - 2a_9 - a_6, \tag{6}$$

where values of parameters a_5, a_6 and a_9 are chosen arbitrarily. Arranging solutions (4) in order, shown in the table 2 (see figure), we obtain the table 3. It is noteworthy that arithmetic progressions with the difference $2a_5 - a_6 - a_9$ place in the rows of the table 3, whereas ones, having the difference $a_5 - a_9$, place in its columns. If one introduces three new parameters a, b and c by the equalities $a_5 = a + b + c$, $a_6 = a + 2c$ and $a_9 = a + b$ into the table 3, then this table will acquire more elegant form, which it has in table 4, and so the fact of existing of the arithmetic progressions in it will receive more visual impression. Thus, the proof of Proposition 1 follows directly from the construction of tables 3 and/or 4 and it is appeared as a result of using the algebraic methods, mentioned in the points (a) and (c) of Sect. 1.

a_1	a_2	a_3
a_4	a_5	a_6
a_7	a_8	a_9

(1)

$2a_9 + a_6 - 2a_5$	a_9	$2a_5 - a_6$
$a_9 + a_6 - a_5$	a_5	$3a_5 - a_6 - a_9$
a_6	$2a_5 - a_9$	$4a_5 - 2a_9 - a_6$

(3)

a_2	a_9	a_4
a_7	a_5	a_3
a_6	a_1	a_8

(2)

a	$a + b$	$a + 2b$
$a + c$	$a + b + c$	$a + 2b + c$
$a + 2c$	$a + b + 2c$	$a + 2b + 2c$

(4)

Figure. Elucidating the general legitimacies existing between the elements of the solution (6).

3.2 Elaboration of a universal algorithm for finding all numerical solutions of the system (5) from a given class of numbers

Let it be required to find all numerical solutions of the system (5), which belong to the given class of numbers and has $a_5 = f$. For elaboration of a universal algorithm, solving this problem, we first write out all possible decompositions of the number $2f$ in the two summands of the following form

$$2f = x_1(j) + x_2(j), \tag{7}$$

where j is the number of a decomposition; $x_1(j)$ and $x_2(j)$ are two such numbers that $x_1(j) < x_2(j)$ and both ones belong to the given class of numbers. In a complete set of various decompositions of the kind, we fix only one, having, for instance, the number k. Determine for it the number $d(k)$

$$d(k) = 2f - x_1(j). \tag{8}$$

Proposition 2. *The desirable numerical solution of the system (5), which contains $a_5 = f$ and numbers $x_1(k)$ and $x_2(k)$ of (7), can be found only in the case, if one succeeds to find, among the remaining numbers of the form $x_1(k)$, an arithmetic progression from three numbers with the difference $d(k)$.*

Proof. The truth of Proposition 2 follows from the construction of the tables 3 and/or 4, shown in figure.

It is evident also, to obtain a complete set of solutions of the system (5) from the given class of numbers, one should repeat the foregoing actions for all the differences $d(k)$.

3.3 Deriving an analytical formula for counting the quantity of various solutions of the system (5) from natural numbers

Proposition 3. *If $A(m)$ is the total number of various solutions of the system (5) from natural numbers and $a_5 = m$ then its value may be computed by the formula*

$$\mathrm{A}(m) = 9[m/6]^2 + \{3\,(m \bmod 6) - 8\}[m/6] + 2 - 2\,[\{(m \bmod 6) + 5\}/6] + [(m \bmod 6)/5]. \tag{9}$$

Proof. We first write out all possible decompositions of the number $2m$ in two distinct terms.

$$\begin{array}{lll}
2m = 1 + (2m-1) & (j = 1, & d(1) = m-1), \\
2m = 2 + (2m-2) & (j = 2, & d(2) = m-2), \\
\dots\dots\dots\dots & \dots\dots\dots & \dots\dots\dots\dots \\
2m = m-2 + (m+2) & (j = m-2, & d(m-2) = 2), \\
2m = m-1 + (m+1) & (j = m-1, & d(m-1) = 1).
\end{array} \tag{10}$$

The problem on counting total number of various solutions of the system (5) with $a_5 = m$ is now reduced, in accordance with the universal algorithm of Sect. 3.2, to counting a total number of various arithmetic progressions consisting of three numbers, which may be composed from the numbers $1, 2, \dots, m-2, m-1$ and such that the differences in these progressions are respectively equal to $d(m-1)$, $d(m-2)$, ..., $d(1)$.

To simplify this new problem we shall deduce a recurrence relation which will link the total numbers of various solutions having $a_5 = m$ and $a_5 = m-1$. For this aim we decompose all the solutions with $a_5 = m$ in two groups. The solutions, having number 1, will be attributed to the first group. A total number of such solutions will be denoted by $\mathrm{A}_1(m)$. All the remaining solutions we shall attribute to the second group. We decrease now each number by 1 in all solutions of the second group. After this operation, a lot of the second group solutions will represent by themselves a complete set of various solutions from natural numbers with $a_5 = m-1$. Thus, the following relation

$$\mathrm{A}(m) = \mathrm{A}_1(m) + \mathrm{A}(m-1) \tag{11}$$

is valid or, in other words, if we know a value of $A(m-1)$ then for finding the value $A(m)$ it will be sufficient to count the number of the solutions containing $a_5 = m$ and the number 1. This new combinatorial problem can be reformulated as the following one

to find a total number of various arithmetic progressions from three numbers which can be composed from the sequence of numbers 1, 2, …, m – 2, m – 1 and such that the first number of these progressions is number 1 and the differences of the progressions are respectively equal to d(m – 1), d(m – 2), …, d(1).

It seems to be evident, that a total number of the desired progressions coincides with the maximal difference value of the progression D_{max} for which one can still find an arithmetic progression of the required form from the set of numbers 1, 2, …, $m-2$, $m-1$. The value of D_{max} can be found from the correlation $1 + 2D_{max} = m - 1$, whence $D_{max} = [(m-2)/2]$, where square brackets denote the integer part. But in reality this value of D_{max} is not always coinciding with the value of $A_1(m)$: to eliminate this non-coincidence we must decrease the total number of arithmetic progressions by one if numbers $1 + d(k)$ or $1 + 2d(k)$ coincide with the number $x_1(k)$ of (7).

Let us determine at which values of $d(k)$ this coincidence occurs:

$$1 + d(k_1) = k_1 = m - d(k_1); \quad 1 + 2d(k_2) = k_2 = m - d(k_2), \tag{12}$$

whence $d(k_1) = (m-1)/2$, and $d(k_2) = (m-1)/3$. If $d(k_1) = (m-1)/2$ the number $1 + 2d(k_1) > m - 1$. Consequently, this case is never fulfilled. The coincidence occurs in the second case if $m - 1$ is multiple of 3.

If we decompose all m-numbers in six classes so that the numbers of the form $6k$ will be attributed to the first class and those of the form $6k + 1$ — to the second one and so on, where $k = 1, 2, \ldots$, then for all six classes of the m-numbers one can write out in the explicit form the values of D_{max} and $A_1(m)$:

$$\begin{array}{lll} m = 6k, & D_{max} = 3k-1, & A_1(6k) = 3k-1; \\ m = 6k+1, & D_{max} = 3k-1, & A_1(6k+1) = 3k-2; \\ m = 6k+2, & D_{max} = 3k, & A_1(6k+2) = 3k; \\ m = 6k+3, & D_{max} = 3k, & A_1(6k+3) = 3k; \\ m = 6k+4, & D_{max} = 3k+1, & A_1(6k+4) = 3k; \\ m = 6k+5, & D_{max} = 3k+1, & A_1(6k+5) = 3k+1. \end{array} \tag{13}$$

Further we shall need the value of the difference $\Delta A(k, i)$ of the following form

$$\Delta A(k, i) = A(6(k+1)+i) - A(6k+i), \tag{14}$$

where $i = 0, 1, \ldots, 5$. Using (14), (13) and (11) we may find an explicit expressions for $\Delta A(k, i)$. Let, for instance, $i = 0$. Then

$$\begin{aligned} \Delta A(k, 0) = A(6(k+1)) - A(6k) &= A_1(6(k+1)) + A_1(6k+5) + \\ &+ A_1(6k+4) + A_1(6k+3)) + A_1(6k+2) + A_1(6k+1) = \\ &= \{3(k+1)-1\} + (3k+1) + 3k + 3k + 3k + (3k-2) = 18k+1. \end{aligned} \tag{15}$$

Remaining values of $\Delta A(k, i)$ for $i = 0, 1, \ldots, 5$ can be found analogously:

$$\Delta A(k, i) = 18k + 1 + 3i. \qquad (16)$$

It is evident, since the $\Delta A(k, i)$ is a linear function from k, values $A(6k + i)$ may be obtained from the second degree polynomial

$$A(6k + i) = b_2(i)\, k^2 + b_1(i)\, k + b_0(i), \qquad (17)$$

where $b_2(i) = 9$, $b_1(i) = 3i - 8$, $b_0(i) = 2 - 2[(i + 5)/6] + [i/5]$; square brackets mean the integer part. Taking into account that $i = (m \bmod 6)$, $k = [m/6]$ and $6k + i = m$ we may obtain (9) from (17).

It should be noted that using regression analysis methods one may appreciably simplify [8, 9] the expression (9):

$$A(m) = g\{(3m^2 - 16m + 18.5)/12\}, \qquad (18)$$

where the notation $g\{a\}$ means the nearest integer to a.

4 Algebraic approach to deriving analytical formulae available for constructing classical Magic squares of the n-th order

We remind that in the general case [1, 3] *Magic* square*s* represent by themselves numerical or analytical square tables, whose elements satisfy a set of definite basic and additional relations. The basic relations therewith assign some constant property for the elements located in the rows, columns and two main diagonals of a square table, and additional relations, assign additional characteristics for some other sets of its elements. In particular, when the constant property is a significance of sum of various elements in rows, columns or main diagonals of the square, then this square is an *Additive* one. If an Additive square is composed of successive natural numbers from 1 to n^2, then it is a *Classical* one.

It is evident [3, 10, 11] that, from the point of view of mathematics, the analytical solution of the problem on constructing Classical squares of the n-th order consists of determining a form of f and g functions, which permit to compute the position for any natural number N from 1 to n^2 in cells of these squares: $x = f(N, n)$ and $y = g(N, n)$.

In this section we

1) adduce two types of analytical functions, by which one may construct Classical squares of odd orders;

2) reveal a connection between these analytical functions and Latin squares;

3) give an algebraic generalisation of the notion "Latin square";

4) derive analytical formulae available for constructing Classical squares of both odd and even orders.

4.1 Classical approach to deriving analytical formulae available for constructing Magic squares of odd order from natural numbers

For any linear algorithmic methods of constructing Classical squares of odd orders, the functions f and g have the following forms [3, 10, 11]:

$$f(N, n) \equiv a_1(N-1) + b_1[(N-1)/n] + c_1 \pmod{n}, \tag{19}$$
$$g(N, n) \equiv a_2(N-1) + b_2[(N-1)/n] + c_2 \pmod{n},$$

where square brackets mean the integer part; a sign "$\equiv$" is the modulo n equality; N is any natural number from 1 to n^2; a_1, b_1, c_1 and a_2, b_2, c_2 are such integral coefficients, that the numbers a_1, a_2, b_1, b_2; $a_1b_2 - a_2b_1$; $a_2 - a_1, b_2 - b_1, a_2 + a_1$ and $b_2 + b_1$ are mutually disjoint with n.

There is no difficulty in counting that in formula (19) the coefficients $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ are equal[3, 10, 11]

— $\{1, 1, -[n/2]; 1, -1, [n/2]\}$ for Terrace algorithmic method of constructing Classical squares;

— $\{1, -1, n/2]; 1, -2, n-1\}$ for Siamese method;

— $\{1, -1, [n/2]; 2, 2, 0\}$ for Knight method;

— $\{(3-a)q + (a+1)/2, (3-a)q + (a-1)/2, 0; (3-a)q + (a-1)/2, (3-a)q + (a+1)/2, 0\}$ (where $q = [(n+1)/6]$, $a = n - 6q$) for the classical square of the n-th order, which, if n is an odd number, non-divisible by three, can be formed also from a pair of orthogonal Latin squares, constructed by the pair of comparisons $x + 2y \pmod{n}$ and $2x + y \pmod{n}$; and so on.

It should be noted, that the above conditions for coefficients of functions f and g become contradictory for even n. For example, by the conditions, the coefficients a_1, a_2, b_1 and b_2 of the functions f and g of (19) should be mutually disjoint with n, and consequently, if n is even, they must be odd. The same requirement must be the true for the number $d = a_1b_2 - a_2b_1$. But if a_1, a_2, b_1 and b_2 are odd, the number d, which is the difference of the two odd numbers, will be an even number.

Thus, an essential fault of linear formulae of (19) is the impossibility of using them for constructing Classical squares of even orders.

4.2 Revealing a connection between Latin squares and analytical formulae of (19)

Proposition 4. *If a Classical square of the n-th order is constructed by formulae (19), then it may be constructed also by the formula*

$$N(x, y) = n\,p(x, y) + r(x, y) + 1, \tag{20}$$

where $p(x, y) \equiv \alpha_1 x + \beta_1 y + \sigma_1$ *and* $r(x, y) \equiv \alpha_2 x + \beta_2 y + \sigma_2$.

Proof. The equivalence of formulae (19) and (20) appears from their linearity and the fact, that (20) are inverse formulae to (19). In particular, if values of coefficients $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ of formulae (19) are known then values of $\{\alpha_1, \beta_1, \sigma_1, \alpha_2, \beta_2, \sigma_2\}$ of formulae (2) may be computed from following linear equations[11]

$$m\alpha_1 \equiv -a_2;\ m\beta_1 \equiv a_1;\ m\sigma_1 \equiv a_2c_1 - a_1c_2; \tag{21}$$
$$m\alpha_2 \equiv b_2;\ m\beta_2 \equiv -b_1;\ m\sigma_2 \equiv b_1c_2 - b_2c_1;$$
$$m = a_1b_2 - a_2b_1$$

and, reciprocally, at the reverse task, the values of $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ may be computed from equations:

$\mu a_1 \equiv -\beta_1;\ \ \mu b_1 \equiv \beta_2;\ \ \mu c_1 \equiv \beta_1\sigma_2 - \beta_2\sigma_1;$ (22)

$\mu a_2 \equiv \alpha_1;\ \ \mu b_2 \equiv -\alpha_2;\ \ \mu c_2 \equiv \alpha_2\sigma_1 - \alpha_1\sigma_2;$

$\mu = \alpha_1\beta_2 - \alpha_2\beta_1.$

For instance, values of $\{\alpha_1, \beta_1, \sigma_1, \alpha_2, \beta_2, \sigma_2\}$ of formulae (20) are equal

— $\{(n+1)/2, (n-1)/2, (n-1)/2; (n+1)/2, (n+1)/2, 0\}$ for Terrace method;

— $\{1, n-1, (n-1)/2;\ 2, n-1, (n-1)/2\}$ for Siamese method;

— $\{[n/2], a, n-a; (n+1)/2, a, a\}$ for Knight method, where $a = dc + (n-d)(1-c)$, $d = [(n+1)/4]$, $c = [(n \bmod 4)/2]$.

We remind[3, 12] that, a quadratic table $n \times n$ in size is *Latin* square of n-th order if only n elements of this table are different and each of these n elements occurs only one time in each row and column of the table. The two Latin squares P and R of the same order n are called *orthogonal* if all the pairs, formed by their elements p_{ij} and r_{ij} (i is the number of a row; j is the number of a column) are different.

Proposition 5. *If elements of a Latin square of the n-th order are numbers 0, 1, ..., n − 1, then, for constructing such Latin square, one may use a linear comparison*

$$L(x, y) \equiv \alpha x + \beta y + \sigma, \tag{23}$$

where α and β are integer numbers, which are to be mutually disjoint with n; σ is any integer number.

Proof. Let the numbers $L(x, y)$ of (23) are located in each cells of a quadratic table $n \times n$ in size. We consider n numbers, which are located in row y_0 of this table. Since the discussed numbers are obtained from the linear comparison (23) at $x = 0, 1, \ldots, n-1$, to show that all they are different, we should demonstrate that they belong to different modulo n classes. Let $x_1 > x_2$ and $\alpha x_1 + \beta y_0 + \sigma \equiv \alpha x_2 + \beta y_0 + \sigma$. Since $\beta y_0 + \sigma$ is a constant, in accordance with the properties of comparisons[3, 4], we obtain the new equality $\alpha x_1 \equiv \alpha x_2$. Hence, since α is mutually disjoint with n, $x_1 \equiv x_2$. But this equality contradicts our assumption. Thus, each of numbers $0, 1, \ldots, n-1$ occurs only one time in each row and column of the discussed table and so this table is Latin square of n-th order.

Proposition 6. *Every Classical square of the odd order, decomposed on two orthogonal Latin squares, may be constructed by the formulae (19) and otherwise.*

Proof. The truth of Proposition 6 follows directly from Propositions 4 and 5 and conditions for coefficients of functions f and g of (19).

4.3 Deriving analytical formulae available for constructing Classical squares of both odd and even orders

The way 1. Let us give an algebraic generalisation of the notion "Latin square":

a quadratic table n×n in size is the generalised Latin square of n-th order if only n elements of this table are different and each of these n elements occurs only n times in this table.

Proposition 7. *Every Classical square of a order n may be decomposed on two orthogonal generalised Latin squares P and R of the order n.*

Proof. To prove Proposition 7, it is sufficient to note that

a) any integer number N from 1 to n^2 may be presented in the form

$$N = np + r + 1, \tag{24}$$

where p and r can take values only $0, 1, \ldots, n-1$;

b) each of values $0, 1, \ldots, n-1$ of parameters p and r occurs n times precisely in the decomposition (24) of numbers N.

Thus, to construct two orthogonal generalised Latin squares P and R from a Classical square of a order n, one should replace in the Classical square all numbers N by respectively $(N-1) \bmod n$ and $[(N-1)/n]$.

Proposition 8. *Every Classical square of order n may be constructed by the formula (20), in which functions p(x, y) and r(x, y) may belong, in general case, to both linear and non-linear classes of ones.*

Proof. The truth of Proposition 8 follows directly from Propositions 7 and materials of Sect. 4.1.

We note, in particular, one may construct Classical squares of even-even orders n ($n = 4k$; $k = 1, 2, \ldots$) by the analytical formula (20), in which functions $p(x, y)$ and $r(x, y)$ have the following forms [3, 11, 13]

$$p(x, y) = cx + (1-c)(n-x-1) \text{ and } r(x, y) = (1-c)y + c(n-y-1), \tag{25}$$

where $c = \{[(x+1)/2] + [(y+1)/2]\} \bmod 2$; or

$$p(x, y) = cd - x - 1 + (1-c)(n-d) \text{ and } r(x, y) = by + (1-b)(n-y-1), \tag{26}$$

where $c = (x + y + a) \bmod 2$; $d = (1-a)y + a(n-y-1)$; $b = \{[(x+3)/2] + [y/2] + a\} \bmod 2$; $a = [2y/n]$; and so on.

The way 2. It is evident, we may consider Classical squares not only as the sum of two orthogonal generalised Latin squares (see the way 1) but, for instance, as quadratic tables whose rows contain certain numerical sequences. Let us look into the problem on finding universal analytical formulae for constructing Classical squares from this new point of view.

Proposition 9. *If a Classical square of the n-th order is constructed by formulae (19), then it may be constructed also by the formula*

$$N(x, y) \equiv a + b - \lambda c, \tag{27}$$

where a, b and c are any integer numbers; λ is 0 or 1, the sign "≡" is the modulo n^2 equality.

Proof. Let a Classical square of an odd order is constructed by formulae (19). It follows from Proposition 4 that this square may be constructed also by formulae (20). We deduct x-th element of first row from every x-th element of all y-th rows of the Classical square. It is evident that the number

$$\langle\ n\{(\alpha_1 x+\beta_1 y+\sigma_1) \bmod n\} - n\{(\alpha_1 x+\sigma_1) \bmod n\} + (\alpha_2 x+\beta_2 y+\sigma_2) \bmod n - (\alpha_2 x+\sigma_2) \bmod n\ \rangle \bmod n^2. \tag{28}$$

will be located in the cell (x, y) of the reformed Classical square {see (20)}. Using the equality $(dn) \bmod n^2 = n\,(d \bmod n)$ we present (28) as the sum of two summands

$$n\{(\beta_1 y)\} \bmod n + \{(\alpha_2 x+\beta_2 y+\sigma_2) \bmod n - (\alpha_2 x+\sigma_2) \bmod n\}. \tag{29}$$

The second summand of (29) may have only two values: $(\beta_2 y) \bmod n$ or $n-(\beta_2 y) \bmod n$. Thus, we obtain that numbers of any y-th row of the reformed Classical square may have only two values. By using the mentioned method of constructing formula (27), we find that parameters of this formula a, b, c and λ are connected with parameters of the formula (20) by correlations

$$a = n\{(\alpha_1 x+\sigma_1) \bmod n\} + (\alpha_2 x+\sigma_2) \bmod n, \tag{30}$$

$$b = n\{(\beta_1 y) \bmod n\} + (\beta_2 y) \bmod n, \quad c = n,$$

$$\lambda = [\ 1 - \operatorname{sign}\{(\alpha_2 x+\beta_2 y+(64)) \bmod n - (\alpha_2 x+(64)) \bmod n\ \}\ ]/2,$$

where $\operatorname{sign}(x) = |x|/x$ if $x \neq 0$ and $\operatorname{sign}(0) = 0$.

It should be noted, if we get off the sign "$\equiv$" in the formula (27) and translate correlations (30) into language of numerical sequences {see the point (c) of Sect. 1}, we obtain that, for algorithmic methods which mentioned in Sect. 4.1 and 4.2, the parameters of formula (23) are determined by correlations

$$a = -(-1)^k n(n-1)/4 + k(n+1)/2 + \{n(n-3)+2\}/4, \tag{31}$$

$$b = n-1-y, \quad c = n, \quad \lambda = [\ \operatorname{sign}\{(h-y)+2\}/2\ ],$$

$$h = [\ z/2\ ] - 1 + (n+1)\,(\ z/2 - [\ z/2\ ]\ ),$$

$$\sigma_y(z) = y+z+2 - n\,[\ \operatorname{sign}\{\ (y+z-n+1)+2\ \}/2\ ], \quad k_1 = \sigma_y(x)$$

where the numerical sequence $\{a_k\}$, if its values are computed at $k = 0, 1, \ldots, n-1$, coincides with the numerical sequence, located in the first row of the Classical square; $\sigma_y(t)$ is a permutation operator of numbers $0, 1, \ldots, n-1$ and for

— Terrace method $k = k_1$, $z = k_1 - 1$;

— Siamese method $z \equiv \{n - 2(k_1-1)\} \bmod n$, $k = z+1$;

— Knight method $k_2 = \sigma_y(k_1-1)$, $z = -(-1)^{k_2} n/4 + k_2/2 + (n+4)/4$, $k = z+1$.

It is evident, using the formula (27) with parameters (31), one can have no difficulty in discovering "genetic connections" between different Classical squares and constructing methods and in generating a set of new methods. For instance, if n is an odd number, non-divisible by three, the new algorithmic methods for constructing Classical squares of odd orders appear when $k_1 = \sigma_y^{\,4}(x)$ or $k_1 = \sigma_y^{\,7}(x)$ in (31), or the form of σ_y and/or the numerical sequence $\{a_k\}$ is changed.

It remains for us to add that parameters of the formula (27) are determined by correlations

$$a = nk, \quad b = w, \quad c = n-2w-1, \quad \lambda = [((k+2) \bmod 4)/2], \tag{32}$$

$$\sigma_w(z) = 1 + \{\ z - h\,(\ 2\,(\ (z+h) \bmod 2\ ) - 1\ )\ \} \bmod n,$$

$h = y + c\,[\,y/[\,y/2\,]\,]$, $\quad k_1 = \sigma_w(z)$

for formulae (25) and (26), where $\sigma_w(z) = 1 + \{z - h\,(2((z+h) \bmod 2) - 1)\} \bmod n$; $h = y + c\,[\,y/[\,y/2]\,]$; $\quad k_1 = \sigma_w(z)$ and for

— the formula (26) $z = x$; $\quad w = y$; $\quad k = k_1$;

— the formula (25) $\lambda_y = [((\,y+1) \bmod 4\,)/2]$; $\quad k = k_1$; $\lambda_x = [((x+1+2\lambda_y) \bmod 4)/2]$; $\quad z \equiv (x_s + n - h\,(1 - 2(x_s \bmod 2))) \bmod n$; $\quad x_s = \lambda_x x + (1 - \lambda_x)\,(\,n - x - 1)$; $w = \lambda_y\, y + (1 - \lambda_x)\,(\,n - y - 1)$.

References

1. Chebrakov Y V *Smarandache Notions J.* (1997) **8** (1-2-3) 184
2. Smarandache F *Collected papers* (Bucharest, Editura Tempus România SRL, 1996) 145
3. Chebrakov Y V *Magic squares. Number theory, algebra, combinatorial analysis* (St. Petersburg, St. Petersburg State Technical Univ. Press, 1998) (In Russian)
4. Vinogradov I M *Foundation of the theory of numbers* (Moscow, Nauka, 1981) (In Russian)
5. Ollerenshaw K and Bondi H *Phil. Trans. R. Soc.* (1982) **A 306**, 443
6. Andrews W S *Magic squares and cubes* (Chicago, Open court Pub. Co., 1917)
7. Dudeney H E *Amusement in mathematics* (London, T Nelson & Sons, 1917)
8. Chebrakov Y V *The parameters estimation theory in the measuring experiments* (St. Petersburg, St. Petersburg State Technical Univ. Press, 1997) (In Russian)
9. Chebrakov Y V and Shmagin V V *Some non-traditional approaches to solving a few data analysis problems* In: Integral methods in science and engineering (Edinburgh, Longman, 1997) **2** 69
10. Postnikov M M *Magic squares* (Moscow, Nauka, 1964) (In Russian)
11. Chebrakov Y V *System analysis methods in experimental inquiry* (St. Petersburg, St. Petersburg State Technical Univ. Press, 1997) (In Russian)
12. Denes J and Keedwell A D *Latin Squares: New Developments in the Theory and Applications* (North-Holland, Amsterdam, 1991)
13. Chebrakov Y V and Shmagin V V *Different types of formulae for the analytical description of Magic squares constructing methods* In: Proc. of Conf. on Analytic and Elementary Number Theory (Vienna, 1996) 200

ANALYTICAL APPROACH TO DESCRIPTION OF SOME COMBINATORIAL AND NUMBER-THEORETIC COMPUTATIVE ALGORITHMS

Y. V. CHEBRAKOV and V. V. SHMAGIN
Department of Mathematics, Technical University,
Nevsky 3-11, 191186, St-Petersburg, Russia
E-mail: chebra@phdeg.hop.stu.neva.ru

We discuss the theme on translating different descriptions of computative algorithms into high-level programming languages, enumerate some advantages of analytical descriptions and demonstrate that logical functions may be used effectively to create analytical formulae available for describing a set of combinatorial and number-theoretic computative algorithms. In particular, we adduce analytical formulae to generate l-th prime numbers p_l, permutations of order m, k-th numbers of Smarandache sequences of 1st and 2nd kinds and classical Magic squares of an order n.

Key words: computative algorithms, analytical approach, logical functions, combinatorics, number theory.

1 Introduction

As well-known[1,2] verbal and diagram (graph-diagram) techniques available for describing computative algorithms are the most wide-spread at present. For instance, *Euclidean algorithm*, allowing to find the greatest common divisor (GCD) of the positive integers a and b ($a > b$) has the following verbal description[3]

1. Assign $m = a$, $n = b$;
2. Find $r = m \bmod n$;
3. If $r > 0$, then pass to 4. Otherwise, pass to 5;
4. Assign $m = n$, $n = r$ and pass to 2;
5. Answer: GCD(a, b) = n.

Since all computative algorithms are realised, as the rule, on computer at present, the main fault of the verbal description of computative algorithms is the necessity of translating this description into one of special computer-oriented languages.

The diagram form of the description of computative algorithms allows to simplify slightly the process of such translation. In particular, the diagram form of Euclidean algorithm is shown in figure 1, where squares with digits 1, 2, 4, 5 and the rhomb with the condition $r > 0$ mean respectively to points 1, 2, 4, 5 and 3 of the verbal description.

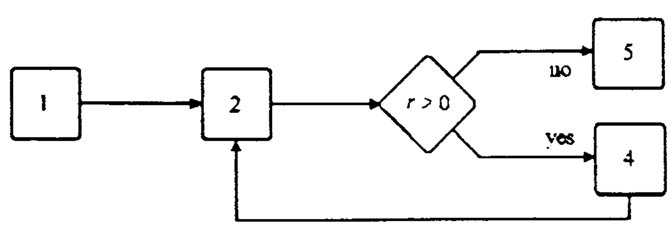


Figure 1. Diagram form of the description of Euclidean algorithm.

The logical technique[4] available for describing computative algorithms is less known than verbal and diagram descriptions, but just it gives easier way to obtain a program code. In particular, we may present the logical description of Euclidean algorithm in the form

$$A_1 \downarrow^2 A_2 \alpha_3 \uparrow_1 A_4 \beta \uparrow_2 \downarrow^1 A_5, \tag{1}$$

where A_i are elements of the vector operator $A = \{\langle m = a,\ n = b\rangle, \langle r = m \bmod n\rangle, \langle\varnothing\rangle, \langle m = n,\ n = r\rangle, \langle n\rangle\}$; α_i are elements of the vector conditional jump $\alpha = \{\langle\varnothing\rangle, \langle\varnothing\rangle, \langle r > 0\rangle, \langle\varnothing\rangle, \langle\varnothing\rangle\}$; $\langle\varnothing\rangle$ is the blank in A and α; $\uparrow_i$ and $\downarrow^i$ are arrows indicating respectively points of departures and destinations; β is the unconditional jump instruction.

We note that, generally speaking, an one-to-one correspondence exists between three foregoing techniques for describing computative algorithms. In other words these techniques are identical in substance.

One of modern techniques available for describing computative algorithms is using the built-in predicates calculus, realised, for instance, in Prolog[5]. In particular, we may represent Prolog description of Euclidean algorithm by three statements

GCD(0,V,V). (2)
GCD(NS,VS,V):–NS1 is VS mod NS, GCD(NS1,NS,V).
?–GCD(b, a, V).

Where the second statement is the direct record of the recursive computative procedure, allowing to find GCD(a, b); the first one determines the condition to finish this procedure; the third one is constructed to introduce the concrete values of numbers of Euclidean algorithm; NS, NS1, V and VS are internal variables of the procedure and V = GCD(a, b) after calculations. The main obstacle of this technique spreading is necessity of preliminary good knowledge of predicates calculus theory.

This paper is devoted to an advance of analytical approach to describing some combinatorial and number-theoretic computative algorithms. Since at present any analytical description of the computative algorithms allows to automate the process of obtaining the program code, we suppose that the discussed theme appears to be interesting.

2 Constructing analytical formulae by using logical functions

2.1 Formulae to generate n-th prime number p_n

In our view, the most impressive application of logical functions in elementary number theory is the formula[3, 6] to generate n-th prime number p_n:

$$p_n = \sum_{m=0}^{(n+1)^2+1} \operatorname{sg}(n+1-\sum_{k=2}^{m}(\{(k\text{-}1)!\}^2 - k[\{(k-1)!\}^2/k])), \tag{3}$$

where $p_0 = 2$, $p_1 = 3$, ...; square brackets mean the integer part; sg is a logical function: $\operatorname{sg}(x) = 1$ if $x > 0$ and $\operatorname{sg}(x) = 0$ if $x \le 0$. Let us find another analytical formula for p_n without factorials.

As well-known[3, 7], any prime number has exactly two divisors: the unit and itself. Thus, any integer number a is a prime one if it has not divisors among integer numbers from 2 to $[\sqrt{a}]$ or, in the language of analytical formulae, if

$$\chi_a = \prod_{j=2}^{[\sqrt{a}]} \{\mathrm{sg}(a - j[i / j])\}, \tag{4}$$

then a is a prime only when $\chi_a = 1$. It appears directly from (4) and (3) that the desirable formula for p_n has form

$$p_n = \sum_{m=0}^{(n+1)^2+1} \mathrm{sg}(n - 1 - \sum_{a=3}^{m} \chi_a), \tag{5}$$

where $p_2 = 2, p_3 = 3, p_4 = 5, \dots$.

2.2 The analytical description of the permutations generator

As well-known[2, 3]

a) *the permutation of order m* is called any arrangement of m various objects in a series;

b) the verbal description of the simple algorithm available for constructing all the permutations from m objects, if all the permutations from $m - 1$ objects have been already constructed, has form

Enumerate $m - 1$ various objects by the numbers $1, 2, \dots, m - 1$. For each permutation of $a_1, a_2, \dots, a_{m-1}$, containing the numbers $1, 2, \dots, m - 1$, form m other permutations by putting the number m in all the possible places. As a result obtain the permutations:

$$\begin{array}{l} m,\ a_1,\ a_2,\ \dots,\ a_{m-1}; \\ a_1,\ m,\ a_2,\ \dots,\ a_{m-1}; \\ \dots\dots\dots\dots\dots; \\ a_1,\ a_2,\ \dots,\ m,\ a_{m-1}; \\ a_1,\ a_2,\ \dots,\ a_{m-1},\ m. \end{array} \tag{6}$$

It is evident that one can obtain all the permutations of order m by this algorithm and none of the permutations of (6) may be obtained more than once. If this verbal description we translate into one of special computer-oriented languages, for instance, into Pascal, then we obtain the program code, shown in table 1. This program works correctly at initial conditions m4 = 1; n1 = m and the array nb3 contains such numbers in the first m cells, which should be rearranged, and has following advantages over the verbal description

a) the knowledge of all permutations from $m - 1$ objects is not required for generating the permutations of order m;

b) permutations are realised with any set of numbers, contained in the first m cells of the array nb3.

Table 1. The representation of the permutations generator as Pascal program.

```
Procedure Perm(Var m4,n1,n:Integer;
Var nb3,nb4,nb5:Ten);
Label A28,A29,A30;
Var nt,k,m2:Integer;
Begin
   If m4=1 Then
   Begin
      m4:=0;n:=n1;
      For k:=2 to n do
      Begin
         nb4[k]:=0;
         nb5[k]:=1;
      End;
      Exit;
   End;
   k:=0; n:=n1;
A28:
   m2:=nb4[n]+nb5[n];nb4[n]:=m2;
   If m2=n Then
      Begin
         nb5[n]:=-1;Goto A29;
      End;
      If Abs(m2)>0 Then Goto A30;
      nb5[n]:=1;Inc(k);
A29:
   If n>2 Then
   Begin
      Dec(n);Goto A28;
   End;
   Inc(m2);m4:=1;
A30:
   m2:=m2+k;
   nt:=nb3[m2];
   nb3[m2]:=nb3[m2+1];
   nb3[m2+1]:=nt
End;
```

It should be noted that the main fault of both the verbal description and the program code is the fact that the knowledge of the previous permutation of order m is required for constructing the next permutation from m objects. To eliminate this fault one may use a set of analytical formulae [6]

$$r_j = p_j - z_1 + 1, \quad p_j = j - 1 + f(1 - c_j) + c_j\,(m - j - f), \tag{7}$$

$$f = t_{j-1} - (m - j + 1)\,[\,t_{j-1}/(m - j + 1)],$$

$$t_j = [\,k \,/ \prod_{q=1}^{j}(m - q + 1)\,], \qquad c_j = |\,(-1)^{t_j} - 1\,|/2,$$

$$z_1 = \mathrm{sg}(1 + p_{j-1} - p_j - z_2) + z_2, \;\; z_2 = \mathrm{sg}(1 + p_{j-2} - p_j - z_3) + z_3, \;\ldots,$$

$$z_{j-1} = \mathrm{sg}(1 + p_1 - p_j),$$

where k is a number of permutation, generated of (7); r_j is a number, which j-th element of the initial sequence nb3 has in k-th permutation; the all another parameters in (7) are auxiliary.

2.3 The formula for counting the value of GCD(a, b)

We may present one of possible formulae available for counting the value of GCD(a, b) {see Sect. 1} as

$$\mathrm{GCD}(a, b) = b\{1 - \mathrm{sign}(r)\} + k\,\mathrm{sign}(r), \qquad r = a - b[a/b], \tag{8}$$

$$k = \underset{i=2}{\overset{[\sqrt{b}]}{\mathrm{MAX}}}\ \{i(1 - d)\}, \quad d = \mathrm{sign}\{a - i[a/i]\} \times \mathrm{sign}\{b - i[b/i]\},$$

where the function $\mathrm{MAX}(a_1, a_2, \ldots, a_i)$ gives the greatest from numbers $a_1, a_2, \ldots, a_i$; $\mathrm{sign}(x) = |x|/x$ if $x \neq 0$ and $\mathrm{sign}(0) = 0$.

2.4 Formulae for the calculation of n-th numbers in Smarandache sequences of 1st and 2nd kinds

As we found earlier[7], the terms of six Smarandache sequences of 1st kind[8] may be computed by means of one general recurrent expression

$$a_{\varphi(n)} = \sigma(a_n 10^{\psi(a_n)} + a_n + 1), \tag{9}$$

where $\varphi(n)$ and $\psi(a_n)$ are some functions; σ is an operator. For instance,

a) if $\varphi(n) = n+1$, $\sigma = 1$ and $\psi(a_n) = [\lg(n+1)] + 1$ then the following sequence of the numbers, denoted as *S_1-sequence*, is generated by (9)

$$1,\ 12,\ 123,\ 1234,\ 12345,\ 123456, \ldots \tag{10}$$

Let each number χ_k, determined as

$$\chi_k = -1 + \sum_{j=0}^{[\lg(k+0,5)]} (k+1-10^j), \tag{11}$$

correspond to each number a_k of sequences (10), where the notation "$[\lg(y)]$" means integer part from decimal logarithm of y. Using (11) we may construct the following analytical formula for the calculation of n-th number in the S_1-sequence:

$$a_n = 10^{\chi_n} \sum_{i=1}^{n} (i/10^{\chi_i}); \tag{12}$$

b) if $\varphi(n) = n+1$; $\sigma = \gamma$ is the operator of mirror-symmetric extending the number $a_{[(n+1)/2]}$ of S_1-sequence from the right with 1-truncating the reflected number from the left, if n is the odd number, and without truncating the reflected number, if n is the even number; $\psi(a_n) = [\lg([(n+1)/2] + 1)] + 1$, then the following sequence of the numbers, denoted as *S_2-sequence*, is generated by (9)

$$1,\ 11,\ 121,\ 1221,\ 12321,\ 123321,\ 1234321, \ldots \tag{13}$$

The analytical formula for the calculation of n-th number in the S_2-sequence has the form

$$a_n = \sum_{i=1}^{[n/2]} i\, 10^{\chi_i - [\lg i]} + \sum_{i=1}^{[(n+1)/2]} i\, 10^{d}, \tag{14}$$

where $d = 1 + \chi_{[(n+1)/2]} + \chi_{[n/2]} - \chi_i$; and so on.

We find recently that the terms of Smarandache sequences of 2nd kind[8] may be computed also by the universal analytical formula {compare with formulae (12) and (5)}

$$a_n = \sum_{m=1}^{U_n} \mathrm{sg}(n+2-b-\sum_{i=b}^{m}\chi_i), \qquad (15)$$

where χ_i are the characteristic numbers for Smarandache sequences of the 2nd kind; U_n is an up-estimation for the value a_n; b is a constant. For instance, if

$$U_n = n^2,\ b=2,\ \chi_i = \mathrm{sg}\,\{\sum_{k=1}^{g!}\ \prod_{q=1}^{[\sqrt{c}]}\mathrm{sg}\,(c-q[c/q])\,\},\ g=[\lg i]+1, \qquad (16)$$

$$c = 10^g\sum_{p=1}^{g}\{\,[i/10^{g-p}]-10\,[i/10^{g-p+1}]\,\}/10^{r_p},$$

the following sequence of Smarandache numbers of 2nd kind is generated by (15)

$$1, 2, 3, 5, 7, 11, 13, 14, 16, 17, 19, 20, 23, 29, 30, 31, 32, 34, \ \ldots \qquad (17)$$

We note that

a) in formula (16): k is a number of the permutation, which is generated from digits of the number i; r_j is a number, which j-th digit of the number i has in k-th permutation {see (7)};

b) only such integer numbers belong to the sequence (17), which are prime numbers or can be derived to prime numbers by a permutation of digits in the initial natural numbers {the number 1 is related to prime numbers in this sequence}.

2.5 Formulae for analytical description of Magic squares constructing methods

As we discovered earlier[3, 11], logical functions may be used effectively to create analytical formulae available for describing computative algorithms on constructing Magic squares of any order n. For instance, let us consider a well-known "Method of two squares", whose verbal description has the form[3, 12]:

1. Make a drawing of two square tables of any order $n = 4k + 2$ ($k = 1, 2, \ldots$). Divide every table in four equal squares which we shall call A, B, C and D squares respectively {see figure 2(1)};

2. Place a Magic square of order $m = 2k + 1$ in the A, B, C and D squares of the first table. It is obvious {see figure 2(2)}, the first table will have the same sum of numbers in its rows, columns and main diagonals;

3. Fill the cells of the second table: all cells of A square are to have zeros; cells of D square — numbers $u = m^2$; cells of B square — numbers $2u$ and cells of C square — numbers $3u$. The obtained table {see figure 2(3)} will have the same sum of numbers only in its columns;

4. Perform such rearrangement of the numbers in the table 2(3) that the new table will have the same sum of numbers in its rows, columns and main diagonals. It can be achieved, for instance, by operations

a) underline in the square A of the second table

– k zeros, located in the extreme left positions of all rows, excepting the middle row {see figure 2(3)};

– the zero, located in the central cell of the middle row, and another $k-1$ zeros, located left of the central cell.

Exchange all marked zeros against the respective numbers of the square C {see figure 2(4)} and otherwise;

b) mark $k-1$ numbers $2u$ in the extreme right positions of every row of square B {see figure 2(3)}, and then exchange ones against corresponding numbers of the square D (see figure 2(4)) and otherwise.

5. Add (cell-wise) two auxiliary tables {the Magic square of order 10, obtained as a result of adding auxiliary squares 2(2) and 2(4) is shown in figure 2(5)}.

A	B
C	D

(1)

15	14	6	22	3	15	14	6	22	3
21	2	18	10	9	21	2	18	10	9
13	5	24	1	17	13	5	24	1	17
4	16	12	8	20	4	16	12	8	20
7	23	0	19	11	7	23	0	19	11
15	14	6	22	3	15	14	6	22	3
21	2	18	10	9	21	2	18	10	9
13	5	24	1	17	13	5	24	1	17
4	16	12	8	20	4	16	12	8	20
7	23	0	19	11	7	23	0	19	11

(2)

0	0	0	0	0	50	50	50	50	50
0	0	0	0	0	50	50	50	50	50
0	0	0	0	0	50	50	50	50	50
0	0	0	0	0	50	50	50	50	50
0	0	0	0	0	50	50	50	50	50
75	75	75	75	75	25	25	25	25	25
75	75	75	75	75	25	25	25	25	25
75	75	75	75	75	25	25	25	25	25
75	75	75	75	75	25	25	25	25	25
75	75	75	75	75	25	25	25	25	25

(3)

75	75	0	0	0	50	50	50	50	25
75	75	0	0	0	50	50	50	50	25
0	75	75	0	0	50	50	50	50	25
75	75	0	0	0	50	50	50	50	25
75	75	0	0	0	50	50	50	50	25
0	0	75	75	75	25	25	25	25	50
0	0	75	75	75	25	25	25	25	50
75	0	0	75	75	25	25	25	25	50
0	0	75	75	75	25	25	25	25	50
0	0	75	75	75	25	25	25	25	50

(4)

90	89	6	22	3	65	64	56	72	28
96	77	18	10	9	71	52	68	60	34
13	80	99	1	17	63	55	74	51	42
79	91	12	8	20	54	66	62	58	45
82	98	0	19	11	57	73	50	69	36
15	14	81	97	78	40	39	31	47	53
21	2	93	85	84	46	27	43	35	59
88	5	24	76	92	38	30	49	26	67
4	16	87	83	95	29	41	37	33	70
7	23	75	94	86	32	48	25	44	61

(5)

Figure 2. Method of two squares.

Table 2. The representation of computative algorithm "Method of two squares" as Pascal program.

```
Procedure BuildMS;
{ Build Magic square of order n=4k+2 }
Var m,i,j,n2:Byte;
  z,x,y,nn:Integer;
Begin
   z:=0;n2:=n;
   n:=n2 Shr 1;
   m:=n Shr 1;
   nn:=Sqr(n);
{ Build an auxiliary square of order n=4k+2}
   { Place a Magic square in A square }
   For z:=1 to nn do
   Begin
      x:=(fX(z)-1+n)Mod n+1;
      y:=(gY(z)-1+n)Mod n+1;
      mk[(x-1)*n2+y]:=z;
   End;
   n:=n2;n2:=n Shr 1;
   For i:=1 to n2 do For j:=1 to n2 do
   Begin
{ Place the  Magic square in B, C and D squares }
      mk[Ind(i,j+n2)]:=mk[Ind(i,j)]+nn Shl 1;
      mk[Ind(i+n2,j)]:=mk[Ind(i,j)]+nn Shl 1+nn;
      mk[Ind(i+n2,j+n2)]:=mk[Ind(i,j)]+nn;
   End;
{Modify the auxiliary square }
    { Modify A and C squares }
   For i:=1 to n2 do
   If i<>n2 Shr 1+1 Then
   Begin
      For j:=1 to m do
      Begin
         Inc(mk[Ind(i,j)],nn*3);
         Dec(mk[Ind(i+n2,j)],nn*3);
      End;
   End;
   i:=n2 Shr 1+1;
   For j:=i Downto i-m+1 do
   Begin
      Inc(mk[Ind(i,j)],nn*3);
      Dec(mk[Ind(i+n2,j)],nn*3);
   End;
    { Modify B and D squares }
   For j:=n Downto n-m+2 do
   For i:=1 to n2 do
   Begin
      Dec(mk[Ind(i,j)],nn);
      Inc(mk[Ind(i+n2,j)],nn);
   End;
End;
```

If the foregoing verbal description we translate into Pascal, we obtain the program code, shown in table 2. In this program code

a) the Magic square of order $m = 2k + 1$, located in the A, B, C and D squares of the auxiliary square, may be built for instance, by the functions [3, 11, 13]

$$f(z,\ m)\ \equiv\ (z-1)\ +\ [\ (z-1)/m\ ]\ -\ [m/2], \tag{18}$$
$$g(z,\ m)\ \equiv\ (z-1)-\ [\ (z-1)/m\ ]\ +[m/2]\ ,$$

where square brackets mean the integer part; a sign "≡" is the modulo m equality; z is any natural number from 1 to m^2; functions f and g afford to compute the position of any natural number z in cells of the Magic square: $x = f(z, n)$ and $y = g(z, n)$. In particular, functions (18) may be presented as following two distinct Pascal-procedures

```
Function fX(z:Integer):Integer;
Begin
   fX:=1+(z-1)+(z-1)div n - n shr 1;
End;

Function gY(z:Integer):Integer;
Begin
   gY:=1+(z-1)-(z-1)Div n + n shr 1;
End;
```

b) two procedures "Ind" and "Sign" are auxiliary and have the form

```
Function Ind(x,y:Integer):Integer;
Begin
   Ind:=(x-1)*n+y;
End;

Function Sign(n:Word):ShortInt;
Begin
   If Odd(n) Then Sign:= -1 Else Sign:=1;
End;
```

The analytical description of "Method of two squares" has the form [3]

$$x = i + c_1 m - 1; \quad y = (1 - c_3 - c_6 - c_5)\,(j + c_2 m) \; + \; (c_3 + c_6 + c_5)\,(1 + \tag{19}$$

$$+ \{(j + (c_2 + 1)m - 1) \bmod n\}) \; - \; 1;$$

$$u = m^2; \quad z = 1 + \{(N - 1) \bmod u\}; \quad i = f(z,\ m) \; + \; 1;$$

$$j = g(z, m) + 1; \quad c_1 = [\{(\,[\,(N-1)/\,\mathrm{u}\,] + 1) \bmod 4\}/2\,];$$

$$c_2 \equiv [\,(N-1)/u\,] \bmod 2, \quad c_3 = [\,(\mathrm{sign}(c_1 m + i - 3k - 4) + 2)/2\,];$$

$$c_4 = \mathrm{asg}(j - k - 1); \quad c_6 = c_4\,[\,(\mathrm{sign}(k - i - c_1 m) + 2)/2\,];$$

$$c_5 = (1 - c_4)(1 - c_1)[\,\{\mathrm{sign}(i-1) + 1\}/2\,]\times[\,(\mathrm{sign}(k - i + 1) + 2)/2\,],$$

where $n = 4k + 2$ is an order of the desirable Magic square, contained natural numbers N from 1 to n^2; $m = 2k + 1$; functions $f(z, m)$ and $g(z, m)$ are determined by the expression (18); $\mathrm{asg}(x) = 1$ if $x \neq 0$ and $\mathrm{asg}(0) = 0$ $\{\mathrm{asg}(x) = |\,\mathrm{sign}(x)\,| = \mathrm{sign}|\,x\,| = \mathrm{sg}|\,x\,|\}$.

References

1. Lyapunov A A *J. Izv. Vysh. Uch. Zav. Radiofizika* (1958) **1** (In Russian)
2. Knuth D E *Art of Computer Programming* (New York, Addison-Wesley, 1973 (1), 1981(**2**)) **1-2**
3. Chebrakov Y V *Magic squares. Number theory, algebra, combinatorial analysis* (St. Petersburg, St. Petersburg State Technical Univ. Press, 1998) (In Russian)
4. Yanov Y I *Problemy Kibernetiki* (1958) **1** (In Russian)
5. Doores J, Reiblein A R and Vadera S *Prolog – Programming for tomorrow* (Wilmslow, Sigma Press, 1987)
6. Schihanovich Y *Kvant* (1976) **5** (In Russian)
7. Vinogradov I M *Foundation of the theory of numbers* (Moscow, Nauka, 1981) (In Russian)
8. Chebrakov Y V and Shmagin V V *Some new methods for constructing analytical formulae in combinatorics* In: Proc. of Research Conf. on Algebra and Discr. Math.: Combinatorics – Algebraic, Geometric and Probabilistic Aspects (San Feliu de Guixols, 1997) 35
9. Chebrakov Y V and Shmagin V V *Smarandache Notions J.* (1997) **8** (1-2-3) 62
10. Dumitrescu C and Seleacu V *Some notions and questions in number theory* (Vail, Erhus University Press, 1995)
11. Chebrakov Y V and Shmagin V V *Different types of formulae for the analytical description of Magic squares constructing methods* In: Proc. of Conf. on Analytic and Elementary Number Theory (Vienna, 1996) 200
12. Ball W W R and Coxeter H S M *Mathematical Recreation and Essay* (Toronto, Univ. of Toronto Press, 1974)
13. Postnikov M M *Magic squares* (Moscow, Nauka, 1964) (In Russian)

NUMERICAL FUNCTIONS AND TRIPLETS

I. Bălăcenoiu, D. Bordea, V. Seleacu

We consider the functions: f_s, f_d, f_p, $F : \mathbf{N}^* \to \mathbf{N}$, where $f_s(k) = n$, $f_d(k) = n$, $f_p(k) = n$, $F(k) = n$, n being, respectively, the least natural number such that $k/n! - 1$, $k/n! + 1$, $k/n! \pm 1$, $k/n!$ or $k/n! \pm 1$. This functions have the next properties:

1. Obviouvsly, from definition of this function, it results:

$$F(k) = \min\{S(k), f_p(k)\} = \min\{S(k), f_s(k), f_d(k)\}$$

   where S is the Smarandache function (see [3]).

2. $F(k) \leq S(k)$, $F(k) \leq f_s(k)$, $F(k) \leq f_d(k)$, $F(k) \leq f_p(k)$

3. $F(k) = S(k)$ if k is even, $k \geq 4$.
   Proof. For any $n \in \mathbf{N}$, $n \geq 2$, $n!$ is even, $n! \pm 1$ are odd. If k is even, then k cannot divide $n! \pm 1$. So $F(k) = S(k) = n \geq 2$ if k is even, $k \geq 4$.

4. If $p > 3$ is prime number, then $F(p) \leq p - 2$.
   Proof. According to Wilson's theorem $(p-1)! + 1 = M_p$. Because $(p-2)! - 1 + (p-1)! + 1 = (p-2)!p$ results for $p > 3$, $(p-2)! - 1 = M_p$ and so $F(p) \leq p - 2$.

5. $F(m!) = F(m! \pm 1) = S(m!) = m$.

6. The equation $F(k) = F(k+1)$ has infinitly many solutions, because, according to the property 5), there is the solutions $k = m!$, $m \in \mathbf{N}^*$.

7. If $F(k) = S(k)$ and n is the least natural number such that $k/n!$, then k not divide $s! \pm 1$ for $s < n$.
   Let $k = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. According to $S(k) = \max_{1 \le i \le r}\{S_{p_i}(\alpha_i)\}$, it results that $S(k) \ge p_h$, where $p_h = \min\{p_1, p_2, \ldots, p_r\}$.
   If k not divide $s! \pm 1$ for $s \le p_h$, then k not divide $t! \pm 1$ for $t > p_h$.
   Consequently, if k not divide $(n-1)!$, $k/n!$ and k not divide $s! \pm 1$ for $s \le \min\{n, p_h\}$, then $F(k) = S(k) = n$.
   Obviously, the numbers $k = 3t$, t being odd, $t \ne 1$, have $p_h = 3$ and they satisfy the condition $3t$ not divide $s! \pm 1$ for $s = 1, 2, 3$.
   Therefore, for $k = 3t$, t odd, $t \ne 1$, $F(3t) = S(3t) = n$, n being the least natural number such that $3t/n!$.

8. The partition "bai" of the odd numbers.

$$\text{Let } A = \{k \in \mathbf{N} | k \text{ odd and } F(k) = S(k)\}$$

$$B = \{k \in \mathbf{N} | k \text{ odd and } F(k) < S(k)\}$$

   (A, B) is the partition "bai" of the odd numbers.
   Into A there are numbers $k = 3t$, t odd, $t \ne 1$. Obviously, A has infinitly many elements.
   Into B there are numbers $k = t! \pm 1$ with $t \ge 3$, $t \in \mathbf{N}$. Obviously, B has infinitly many elements.

Definition 1 *Let $n \in \mathbf{N}^*$. We called triplet $\hat{n}$, the set:*
$n-1$, n, $n+1$.

Definition 2 *Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are separated if*
$k+1 < n-1$, *i.e.* $n-k > 2$.

Definition 3 *The triplets $\hat{k}$, $\hat{n}$ are l_s-relatively prime if*
$(k-1, n-1) = 1$, $(k+1, n+1) \ne 1$.

For example: $\hat{6}$ and $\widehat{72}$ are l_s-relatively prime.

Definition 4 *The triplets $\hat{k}$, $\hat{n}$ are l_d-relatively prime if*
$(k-1, n-1) \ne 1$, $(k+1, n+1) = 1$.

Definition 5 *The triplets $\hat{k}$, $\hat{n}$ are l-relatively prime if*
$(k-1, n-1) = 1$, $(k+1, n+1) = 1$.

Definition 6 *The triplets $\hat{k}$, $\hat{n}$ are d-relatively prime if* $(k-1, n+1) = 1,\ (k+1, n-1) = 1$.

For example: $\hat{2}$ and $\hat{6}$ are d-relatively prime.

Definition 7 *Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are d_s-relatively prime if* $(k-1, n+1) = 1,\ (k+1, n-1) \neq 1$.

For example: $\hat{6}$ and $\widehat{120}$ are d_s-relatively prime.

Definition 8 *Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are d_d-relatively prime if* $(k-1, n+1) \neq 1,\ (k+1, n-1) = 1$.

Example: $\hat{6}$ and $\widehat{24}$ are d_d-relatively prime.

Definition 9 *The triplets $\hat{k}$, $\hat{n}$ are p-relatively prime if* $(k-1, n-1) = 1,\ (k-1, n+1) = 1,\ (k+1, n-1) = 1,\ (k+1, n+1) = 1$.

Obviously, if $\hat{k}$, $\hat{n}$ are p-relatively prime, then they are l and d-relatively prime.
For example: $\widehat{24}$ and $\widehat{120}$ are p-relatively prime.

Definition 10 *Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are F-relatively prime if*

$$(k-1, n-1) = 1,\ (k+1, n-1) = 1,$$
$$(k-1, n) = 1,\ (k+1, n) = 1$$
$$(k-1, n+1) = 1,\ (k+1, n+1) = 1.$$

Definition 11 *The triplets $\hat{k}$, $\hat{n}$ are t-relatively prime if* $(k-1, n-1) \cdot (k-1, n) \cdot (k-1, n+1) \cdot (k, n-1) \cdot (k, n) \cdot (k, n+1) \cdot$ $(k+1, n-1) \cdot (k+1, n) \cdot (k+1, n+1) = 6$.

For example: $\hat{2}$ and $\hat{4}$ and t-relatively prime.

Definition 12 *Let $H \subset \mathbf{N}^*$. The triplet $\hat{n}$, $n \in H$ is, respectively, $l_s, l_d, l, d, d_s, d_d, p, F, t$-prime concerned at H, if $\forall s \in H$, $s < n$, the triplets $\hat{s}$, $\hat{n}$ are, respectively, $l_s, l_d, l, d, d_s, d_d, p, F, t$-relatively prime.*

Let $H = \{n! | n \in \mathbf{N}^*\}$. For the triplets $\hat{m}$, $m \in H$ there are particular properties.

Proposition 1 *Let $k < n$. The triplets $\widehat{(k!)}$, $\widehat{(n!)}$ are separated if* $n > \max\{2, k\}$.

Proof. Obviously, $n! - k! > 2$ if $n > 2$ and $k < n$, i.e. $n > \max\{2, k\}$.

Proposition 2 *Let* $n > \max\{2, k\}$ *and* $M_{kn} = \{m \in \mathrm{N} | k!+1 < m < n!-1\}$. *If* $k_1 < k_2$ *and* $n_1 > \max\{2, k_1\}$, $n_2 > \max\{2, k_2\}$, *then* $n_1 - k_1 \leq n_2 - k_2 \Rightarrow cardM_{k_1 n_1} < cardM_{k_2 n_2}$.

Proof. For $n > k \geq 2$ it is true that

$$n! - (n-1)! > k! - (k-1)! \tag{1}$$

Let $n > k \geq 2$, $1 \leq s \leq k$. Using (1) we can write:

$$\begin{gathered} n! - (n-1)! > k! - (k-1)! \\ (n-1)! - (n-2)! > (k-1)! - (k-2)! \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (n-s-1)! - (n-s)! > (k-s-1)! - (k-s)! \end{gathered}$$

By summing this inequalities, it results:

$$n! - (n-s)! > k! - (k-s)! \tag{2}$$

Let $2 \leq k_1 < n_1$, $2 \leq k_2 < n_2$, $k_1 < k_2$, $n_1 - k_1 \leq n_2 - k_2$. Then $n_2 - n_1 \geq k_2 - k_1 \geq 1$ and there is n_3 such that $n_2 > n_3 \geq n_1$ and $n_2 - n_3 = k_2 - k_1$.
Using (2) we can write:
$n_2! - n_3! > k_2! - k_1!$
Since $n_3! \geq n_1!$ we have:

$$n_2! - n_1! > k_2! - k_1! \tag{3}$$

According to $cardM_{k_1 n_1} = n_1! - 1 - (k_1! + 1)$,
$cardM_{k_2 n_2} = n_2! - 1 - (k_2! + 1)$, it results that:

$$cardM_{k_2 n_2} - cardM_{k_1 n_1} = n_2! - n_1! - (k_2! - k_1!)$$

That is, taking into account (3), $cardM_{k_1 n_1} < cardM_{k_2 n_2}$.

Definition 13 *Let* $k < n$. *The triplets* $\widehat{(k!)}$, $\widehat{(n!)}$ *are linked if* $k! - 1 = n$ *or* $k! + 1 = n$.

Proposition 3 *For* $k \in \mathrm{N}^*$ *there is p prime number, such that for any* $s \geq p$ *the triplets* $\widehat{(k!)}$, $\widehat{(s!)}$ *are not F-relatively prime.*

Proof. Obviously, for $k = 1$ and $k = 2$, the proposition is true.
If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_i^{\alpha_i}$ divide $k! - 1$ or $k! + 1$, then $p_j > k \geq 3$, for $j \in \{1, 2, \ldots, i\}$.
Let $\bar{n} = p_1 \cdot p_2 \cdots p_i$ and $p = \max_{1 \leq j \leq i} \{p_j\}$.
Obviously, $\bar{n} \geq 3$ because $p > k \geq 3$, $\bar{n}/k! - 1$ or $\bar{n}/k! + 1$.
For any $s \geq p$, $\bar{n}/s!$ and so, the triplets $\widehat{(k!)}$, $\widehat{(s!)}$ are not F-relatively prime.

Remark 1 **i)** *Let $k < n$. If $\widehat{(k!)}$, $\widehat{(n!)}$ are linked, then $n - k = k! - k \pm 1$. If $2 < k_1 < n_1$, $\widehat{(k_1!)}$ with $\widehat{(n_1!)}$ are linked and $k_2 < n_2$, $\widehat{(k_2!)}$ with $\widehat{(n_2!)}$ are linked, then $k_1 < k_2 \Rightarrow n_1 - k_1 < n_2 - k_2$ and in view of the proposition 2, results $card M_{k_1 n_1} < card M_{k_2 n_2}$.*
ii) *There are twin prime numbers with the triplet $\widehat{(n!)}$. For example 5 with 7 are from $\widehat{(3!)}$.*

Definition 14 *Considering the canonical decomposition of natural numbers $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, we define $\tilde{n} = \{p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_r^{\alpha_r}\}$, $\mathcal{M} = \{\tilde{n} | n \in \mathbf{N}^*\}$.*

Definition 15 *On $\mathcal{M}$ we consider the relation of order $\sqsubseteq$ defined by:*

$$\{p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_r^{\alpha_r}\} \sqsubseteq \{q_1^{\beta_1}, q_2^{\beta_2}, \ldots, q_t^{\beta_t}\}$$

if and only if $\{p_1, p_2, \ldots, p_r\} \subset \{q_1, q_2, \ldots, q_t\}$ and if $p_i = q_j$, then $\alpha_i \leq \beta_j$.

Remark 2 *For any triplet $\widehat{(n!)}$, $n \in \mathbf{N}^*$, we consider the sets:*
$A_n = \{k \in \mathbf{N}^* | \tilde{k} \sqsubseteq \widetilde{n!}\}$, $A_n^* = \{k \in A_n | k \notin A_h \text{ for } h < n\}$
$B_n = \{k \in \mathbf{N}^* | \tilde{k} \sqsubseteq \widetilde{n! - 1}\}$, $B_n^* = \{k \in B_n | k \notin B_h \text{ for } h < n\}$
$C_n = \{k \in \mathbf{N}^* | \tilde{k} \sqsubseteq \widetilde{n! + 1}\}$, $C_n^* = \{k \in C_n | k \notin C_h \text{ for } h < n\}$
$M_n = \{k \in \mathbf{N}^* | \tilde{k} \sqsubseteq \widetilde{n!} \text{ or } \tilde{k} \sqsubseteq \widetilde{n! - 1} \text{ or } \tilde{k} \sqsubseteq \widetilde{n! + 1}\}$
$M_n^* = \{k \in M_n | k \notin M_h \text{ for } h < n\}$.
It is obvious that:
$A_n^* = S^{-1}(n)$, $B_n^* = f_s^{-1}(n)$, $C_n^* = f_d^{-1}(n)$, $M_n^* = F^{-1}(n)$.
If $k \in A_n^*$, it is said that k has a factorial signature which is equivalent with the factorial signature of $n!$ (see [1]).
Let $k \in B_n^*$, $k = t_1^{r_1} \cdot t_2^{r_2} \cdots t_i^{r_i}$. Then $\{t_r\} \not\sqsubseteq \widetilde{n!}$ for $r = \overline{1, i}$ and for any $h < n$, there are $t_j^{r_j}$, $1 \leq j \leq i$, such that $\{t_j^{r_j}\} \not\sqsubseteq \widetilde{h! - 1}$.
Similarly, for $k \in C_n^*$: $\{t_r\} \not\sqsubseteq \widetilde{n!}$ for $r = \overline{1, i}$ and for any $h < n$, there are $t_j^{r_j}$, $1 \leq j \leq i$, such that $\{t_j^{r_j}\} \not\sqsubseteq \widetilde{h! + 1}$.

References

[1] Bălăcenoiu I., *The Factorial Signature of Natural Numbers*, Smarandache Notions Journal, vol. 9, 1998, p. 70-78.

[2] Bălăcenoiu I., Seleacu V., *Properties of the triplets* $\hat{p}$, Smarandache Notions Journal, vol. 8, 1997, p. 152-156.

[3] Smarandache F., *A function in the Number Theory*, An. Univ. Timişoara, Ser. Şt. Mat., vol. XXVIII, fasc. 1, 1980, p. 79-88.

Current address:

Department of Mathematics, University of Craiova
13, Al. I. Cuza st., Craiova 1100, Romania

EXPLORING SOME NEW IDEAS ON SMARANDACHE TYPE SETS, FUNCTIONS AND SEQUENCES

(Amarnath Murthy ,Superintending Engineer(E&T),Well Logging Section ,Oil And Natural Gas Corporation Ltd., Chandkheda, Ahmedabad, India-380005)

ABSTRACT: In this article I have defined a number of SMARANDACHE type sets ,sequences which I found very interesting. The problems and conjectures proposed would give food for thought and would pave ways for more work in this field.

(1)SMARANDACHE PATTERNED PERFECT SQUARE SEQUENCES.

Consider following sequence of numbers

13, 133, 1333, 13333, . . .--------(1)

The sequence formed by the square of the numbers is

169, 17689, 1776889, 177768889, . . . -------(2)

We define (1) as the root sequence

It is evident that the above sequence (2) follows a pattern.

i.e. The square of one followed by n three's is ,one followed by (n-1) seven's, followed by a six, followed by (n-1) eight's followed by a nine.

There are a finite number of such patterned perfect square sequences. Here we list the root sequences.

(I) 13, 133, 1333, 13333, . . .
(2) 16, 166, 1666, 16666, . . .
(3) 19, 199, 1999, 19999, . . .
(4) 23, 233, 2333, 23333, . . .
(5) 26, 266, 2666, 26666, . . .
(6) 29, 299, 2999, 29999, . . .

on similar lines we have the root sequences with the first terms as (7) 33, (8) 36, (9) 39,(10) 43,(11) 46,(12) 49, (13) 53,(14) 66, (15) 73, (16) 93,(17) 96,(18) 99.

There are some root sequences which start with a three digit number , like

799, 7999, 79999, . . .

The patterned perfect square sequence is

638401 , 63984001 , 6399840001 , 639998400001, . . .

(the nine's and zero's inserted are shown in darker print to identify the pattern.)

Open Problem : (1) Are there any patterned perfect cube sequences ?
(2) Are there any patterned higher perfect power sequences ?

(2) SMARANDACHE BREAKUP SQUARE SEQUENCES

4, 9, 284, 61209, . . .

the terms are such that we have
$4 = 2^2$
$49 = 7^2$
$49284 = 222^2$
$4928461209 = 70203^2$
T_n = the smallest number whose digits when placed adjacent to other terms of the sequence in the following manner

$T_1T_2 \ldots T_{n-1}T_n$ yields a perfect square.

$$\operatorname*{Lt}_{n\to\infty} \frac{(T_1T_2\ldots T_{n-1}T_n)^{1/2}}{10^k}$$ where k is the number of digits in the numerator for this kind of sequence can be analyzed. As it is evident that for large values of n the value of $\frac{(T_1T_2\ldots T_{n-1}T_n)^{1/2}}{10^k}$ is close to either 2.22.... or to 7.0203. . .

(3) SMARANDACHE BREAKUP CUBE SEQUENCES
On similar lines SMARANDACHE BREAKUP CUBE SEQUENCES can be defined. The same idea can be extended to define SMARANDACHE BREAKUP PERFECT POWER SEQUENCES

(4) SMARANDACHE BREAKUP INCREMENTED PERFECT POWER SEQUENCES

1, 6, 6375,

$1 = 1^1$, $16 = 4^2$, $166375 = 55^3$, etc.

T_n = the smallest number whose digits when placed adjacent to other terms of the sequence in the following manner

$T_1T_2 \ldots T_{n-1}T_n$ yields a perfect n^{th} power.

(5) SMARANDACHE BREAKUP PRIME SEQUENCE

2 , 3, 3, . . .

2, 23, 233 etc. are primes.

$T_1T_2...T_{n-1}T_n$ is a prime

(6) SMARANDACHE SYMMETRIC PERFECT SQUARE SEQUENCE
1, 4, 9, 121, 484, 14641, . . .

(7) SMARANDACHE SYMMETRIC PERFECT CUBE SEQUENCE

1, 8, 343, 1331 . . .

This can be extended to define
(8) SMARANDACHE SYMMETRIC PERFECT POWER SEQUENCE

(9) SMARANDACHE DIVISIBLE BY n SEQUENCE

1, 2, 3, 2, 5, 2, 5, 6, 1, 0, 8, 4. . .

the terms are the smallest numbers such that n divides $T_1T_2...T_{n-1}T_n$ the terms placed adjacent digit wise.
e.g. 1 divides 1 , 2 divides 12 , 3 divides 123 , 4 divides 1232 , 5 divides 12325 , 6 divides 123252 , 7 divides 1232535 , 8 divides 12325256 9 divides 123252561 , 10 divides 1232525610 , 11 divides 12325256108 ,
12 divides 123252561084 , etc.

(9) SMARANDACHE SEQUENCE OF NUMBERS WITH SUM OF THE DIGIT'S = PRIME

2,3,5,7,11,12,14,16,20,21,23,25,29, . . .

(10) SMARANDACHE SEQUENCE OF PRIMES WITH SUM OF THE DIGIT'S = PRIME

2,3,5,7,11,23, 29, 41,43, 47, 61, 67, 83, 89,. . .

(11) SMARANDACHE SEQUENCE OF PRIMES SUCH THAT 2P + 1 IS ALSO A PRIME

2, 3, 5, 11, 23, 29, 41, 53, . . .

(11) SMARANDACHE SEQUENCE OF PRIMES SUCH THAT 2P - 1 IS ALSO A PRIME

3, 7,19, 31, . . .

(13) SMARANDACHE SEQUENCE OF PRIMES SUCH THAT $P^2 + 2$ IS ALSO A PRIME

3, 17, . . .

(14) SMARANDACHE SEQUENCE OF SMALLEST PRIME WHICH DIFFER BY 2n FROM ITS PREDECESSOR

5, 17 , 29, 97, . . .

($T_1 = 5 = 3+ 2$, $T_2 = 17 = 13 + 4$, $T_3 = 29 = 23 + 6$,$T_4 = 97 = 89 + 8$ etc.)

(15)) SMARANDACHE SEQUENCE OF SMALLEST PRIME p FOR WHICH p + 2r IS A PRIME

3 , 13, 23, 89 ,. . .

3 + 2 X 1 = 5 is a prime , 13 + 2 X 2 = 17 is a prime , 23 + 2 X 3 = 29 , 89 + 2 X 4 = 97 is a prime etc.

(16) SMARANDACHE SEQUENCE OF THE SMALLEST NUMBER WHOSE SUM OF DIGITS IS n .

1, 2, 3, 4, 5, 6, 7, 8, 9, 19, 29, 39, 49, 59, 69, 79, 89, 99, 199, 299, 399, 499, 599, 699 , . . .

It is a sequence of the only numbers which have the following property.

$$N + 1 = \prod_{r=1}^{k} (a_r +1)$$

PROOF:

Let N be a k-digit number with a_r the r^{th} digit (a_1 = LSB) such that

$$N + 1= \prod_{r=1}^{k} (a_r +1) \qquad \text{---------------(1)}$$

to find all such k -digit numbers.

The largest k-digit number is $N = 10^k - 1$, with all the digits as 9 . It can be verified that this is a solution. Are there other solutions ?

Let the m^{th} digit be changed from 9 to a_m ($a_m < 9$) . Then the right member of (1) becomes $10^{(k-1)}(a_m + 1)$ .This amounts to the reduction in value by $10^{(k-1)}(9-a_m)$. The value of the k-digit number N goes down by $10^{(m-1)}(9-a_m)$. For the new number to be a solution these two values have to be equal which occurs only at m = k. This gives 8 more solutions. In all there are 9 solutions given by $a.10^k - 1$, for a = 1 to 9.
e.g. for k = 3 the solutions are

199 , 299 , 399 , 499, ,599 , 699 , 799 , 899 , 999 ,.

Are there infinitely many primes in this sequence.

(17) SMARANDACHE SEQUENCE OF NUMBERS SUCH THAT THE SUM OF THE DIGITS DIVIDES n

1,3,6,9,10,12,18,20,21,24,27,30,36,40,42,45,48,50,54,60,63,72,80,81,84,90, 100,102,108,110,112,114,120,126,132,133,135,140,144,150, . . .

(18)) SMARANDACHE SEQUENCE OF NUMBERS SUCH THAT EACH DIGIT DIVIDES n

1,2,3,4,5,6,7,8,9,10,11,12,15,20,22,24,30,33,36,40,44,50,55,60,66, . . .

(19)) SMARANDACHE POWER STACK SEQUENCE FOR n

SPSS(2)

1, 12, 124, 1248, 124816, 12481632 ,
The n^{th} term is obtained by placing the digits of the powers of 2 starting from 2^0 to 2^n from left to right.

SPSS(3)
1, 13, 139, 13927, 1392781, 1392781243, . . .

Problem : If n is an odd number not divisible by 5 how many of the above sequence SPSS(n) are prime ? (It is evident that n divides T_n iff $n \equiv 0 \mod (5)$).

(20) SMARANDACHE SELF POWER STACK SEQUENCE

SSPSS

1, 14, 1427, 1427256, 14272563125, 142725631257776, . . .

$T_r = T_{r-1}a_1a_2a_3\ldots a_k$ where , $r^r = a_1a_2a_3\ldots a_k$
(the digits are placed adjacent).

How many terms of the above sequence, SSPSS are prime ?

(21) SMARANDACHE PERFECT SQUARE COUNT PARTITION SEQUENCE

the rth term of SPSCPS (n) Is defined as

$T_r = O\{ x \mid x \text{ is a perfect square }, nr+1 \leq x \leq nr + n \}$

O stands for the order of the set

e.g. for n = 12 SPSCPS(12) is

3, 1, 2, 0, 1, 0 , 1 , 0, 0, 1, 0 , 1, 0, 1, 1 , 0, 0, 1, 0, 1

(number of perfect squares ≤ 12 is 3 (1, 4, and 9), number of perfect squares between 13 to 24 is 1 (only 16) etc.)

(21) SMARANDACHE PERFECT POWER COUNT PARTITION SEQUENCE

The r^{th} term of **SPPCPS (n,k)** is defined as

$T_r = O\{ x \mid x \text{ is a } k^{th} \text{ perfect power }, nr+1 \leq x \leq nr + n \}$
where O stands for the order of the set
By this definition we get

SPSCPS(12) = SPPCPS (12,2)

Another example, SPPCPS (100,3)is
4,1,1,1,0,1,0,1,0,1, 0,0,0,1, 0, 0, 0, 1, 0,0,0,1,0, 0, 0, 0, 0, 1, . . .

Problem: Does $\sum (T_r/(nr))$ converge as $n \to \infty$?

(22) SMARANDACHE BERTRAND PRIME SEQUENCE

According to Bertrand 's postulate there exists a prime between n and 2n. Starting from 2 let us form a sequence by taking the largest prime less than double of the previous prime in the sequence . We get

2, 3, 5, 7, 13, 23, 43, 83, 163, . . .

(23) SMARANDACHE SEMI- PERFECT NUMBER SEQUENCE

6, 12, 18, 20, 24, 30, 36, 40, . . .

A semi perfect number is defined as one which can be expressed as the sum of its (all or fewer) distinct divisors.

e.g. $12 = 2 + 4 + 6 = 1 + 2 + 3 + 6$
$20 = 1 + 4 + 5 + 10$
$30 = 2 + 3 + 10 + 15 = 5 + 10 + 15 = 1 + 3 + 5 + 6 + 15$ etc.

It is evident that every perfect number is also a semi perfect number.

THEOREM : There are infinitely many semi perfect numbers.
Proof: We shall prove that $N = 2^n p$ where p is a prime less than

$2^{n+1} - 1$, is a semi- perfect number.

The divisors of N are

row 1------ $1, 2, 2^2, 2^3, 2^4, \ldots 2^n$

row 2------ $p, 2p, 2^2p, 2^3p, 2^4p, \ldots 2^np$

we have $\sum_{r=0}^{n-1} 2^r p = p(1+2+2^2+2^3+\ldots 2^{n-1}) = p(2^n-1) = M$

M is short of N by p . The task ahead is to express p as the sum of divisors from the first row. It is an established fact that every number can be expressed as the sum of powers of 2 .i.e.

$$p = \sum_{r=0}^{n} a_r . 2^r \quad , \quad \text{where } a_r = 0 \text{ or } a_r = 1. \text{ iff } p \leq 2^{n+1} - 1 \text{ , the}$$

equality giving a perfect number.

(note:$a_1a_2a_3 \ldots a_n$ is the binary representation of p).

$N = M + p$ is expressible as the sum of its divisors.
Remark: This of-course is not exhaustive. There are many more such examples possible giving infinitely many semi perfect numbers. One can explore the possibility of more such expressions.

(24) SMARANDACHE CO-PRIME BUT NO PRIME SEQUENCE

4, 9, 10, 21, 22, 25, 26, 27, 28, 33,34 , 35 , 36 ,49, 50, 51, 52 , . . .

The n^{th} term T_n is defined as follows

$T_n = \{ x \mid (T_{n-1}, x) = 1$, x is not a prime and $(T_{n-1}, y) \neq 1$ for $T_{n-1} < y < x \}$

The smallest number which is not a prime but is relatively prime to the previous term in the sequence.

Open problem : Is it possible to as large as we want but finite increasing sequence k , k+1 , k+2 , k+3 , . . . included in the above sequence?

DEFINITION : We define a prime to be week , strong or balanced prime accordingly as p_r < = or > $(p_{r-1} + p_{r+1})/2$. where p_r is the r^{th} prime. e.g. 3 < (2+5)/2 3 is week prime . 5= (3 + 7)/2 is a balanced prime . 71 > (67 + 73)/2 is a strong prime .

(25) SMARANDACHE WEEK PRIME SEQUENCE :

3, 7 ,13, 19, 23,29, 31, 37, . . .

(26) SMARANDACHE STRONG PRIME SEQUENCE :

11 , 17, 41, . . .

(27) SMARANDACHE BALANCED PRIME SEQUENCE :

5 , 157, 173, 257, 263, 373, . . .

It is evident that for a balanced prime > 5 , $p_r = p_{r-1} + 6k$.

OPEN PROBLEM: Are there infinitly many terms in the SMARANDACHE BALANCED PRIME SEQUENCE ?

How big is N? One of the first estimates of its size was approximately [6]:

$$10^{6846168}$$

But this is a rather large number; to test all odd numbers up to this limit would take more time and computer power than we have. Recent work has improved the estimate of N. In 1989 J.R. Chen and T. Wang computed N to be approximately [7]:

$$10^{43000}$$

This new value for N is much smaller than the previous one, and suggests that some day soon we will be able to test all odd numbers up to this limit to see if they can be written as the sum of three primes.
Anyway assuming the truth of the generalized Riemann hypothesis [5], the number N has been reduced to 10^{20} by Zinoviev [9], Saouter [10] and Deshouillers. Effinger, te Riele and Zinoviev[11] have now successfully reduced N to 5.
Therefore the weak Goldbach conjecture is true, subject to the truth of the generalized Riemann hypothesis.
Let's now analyse the generalizations of Goldbach conjectures reported in [3] and [4]; six different conjectures for odd numbers and four conjectures for even numbers have been formulated. We will consider only the conjectures 1, 4 and 5 for the odd numbers and the conjectures 1, 2 and 3 for the even ones.

4.1 First Smarandache Goldbach conjecture on even numbers.

Every even integer n can be written as the difference of two odd primes, that is $n = p - q$ *with p and q two primes.*

This conjecture is equivalent to:

For each even integer n, we can find a prime q such that the sum of n and q is itself a prime p.

A program in Ubasic language to check this conjecture has been written.

for SLES and SLOS.

(2) SMARANDACHE DIVISOR SEQUENCES:

Define $A_n = \{ x | d(x) = n \}$
Then

$A_1 = \{ 1 \}$

$A_2 = \{ p \mid p \text{ is a prime} \}$

$A_3 = \{ x \mid x = p^2 , p \text{ is a prime} \}$

$A_4 = \{ x \mid x = p^3 \text{ or } x = p_1p_2 , p , p_1 ,p_2 \text{ are primes} \}$.

$A_4 \rightarrow$ 6 , 8 , 10 , 14 , 15 , 21 , 22 , 26 , 27 , . . .

We have

$\sum 1/T_n = 1$ for A_1

This limit does not exist for A_2

$\underset{n \rightarrow \infty}{\text{Lt}} \sum 1/T_n$ exists and is less than $\pi^2/6$ for A_3 as $\text{Lt} \sum 1/n^2 = \pi^2/6$.

The above limit does exist for A_p where p is a prime.

* Whether these limits exist for A_4, A_6 etc is to be explored.

DIVISOR SUB SEQUENCES

The sub sequences for A_4 A_5 etc can be defined as follows:

$B(r_1, r_2, r_3, \ldots r_k)$ is the sequence of numbers $p_1^{r_1} p_2^{r_2} p_3^{r_3} \ldots p_k^{r_k}$

in increasing order ,where $p_1, p_2, p_3, \ldots p_k$ are primes . All the

numbers having the same unique factorization structure.

DIVISOR MULTIPLE SEQUENCE

$SDMS = \{ n \mid n = k. d(n) \}$.

SDMS$\rightarrow$ 1, 2, 8 ,9, 12 , . . .

(3) SMARANDACHE QUAD PRIME SEQUENCE GENERATOR :

$SQPSG = \{ r \mid 90r+11 , 90r+13 , 90r+17, 90r+ 19 \text{ are all primes} \}$

SQPSG $\rightarrow$ 0, 1, 2, . . .

Are there infintely many terms in the above sequence ?

(4) SMARANDACHE PRIME LOCATION SEQUENCES

Define P_0 = sequence of primes .

P_1 = sequence of primeth primes

$P_1 \rightarrow 3, 5, 11, 17, \ldots$

P_2 = sequence of primeth , primeth prime .

$\downarrow$
$\downarrow$

P_r = sequence of primeth , primeth , . . . r times ,primes

* If T_n is the n^{th} term of P_r , then what is the minimum value of r for which

$$\lim_{n \to \infty} \sum 1/T_n \text{ exists ?.}$$

(5) SMARANDACHE PARTITION SEQUENCES

(i) PRIME PARTITION

Number of partitions into prime parts

$Sp_p(n) \rightarrow$ 0, 1, 1, 1 ,1 2, 2, 3, . . .

(ii) COMPOSITE PARTITION

Number of partitions into composite parts

$Sp_c(n) \rightarrow$ 1, 1, 1, 2, 1, 3, . . .

(iii) DIVISOR PARTITIONS

Number of partitions into parts which are the divisors of n.

$SP_d(n) \rightarrow$ 1, ,1 ,1 ,2 ,1,

On similar lines following two partition sequences can be defined.

(iv) CO-PRIME PARTITIONS : $SP_{cp}(n)$

Number of partitions into co-prime parts .

(v) NON- CO-PRIME PARTITIONS $SP_{ncp}(n)$

Number of partitions into non coprime parts.

(vi) PRIME SQUARE PARTITIONS

Partitions into prime square parts .

This idea could be generalised to define more such functions.

(6) SMARANDACHE COMBINATORIAL SEQUENCES.

(I) Let the first two terms of a sequence be 1 & 2. The $(n+1)^{th}$ term is defined as

T_{n+1} = sum of all the products of the previous terms of the sequence taking two at a time .

$T_1 = 1$, $T_2 = 2$, $\Rightarrow T_3 = 2$, and $T_4 = 8$,

SCS(2) = 1, 2, 2, 8 ,48, . . .

The above definition can be generalized as follows:

Let $T_k = k$ for $k = 1$ to n .

T_{n+1} = sum of all the products of the previous terms of the sequence taking r at a time. This defines SCS(r) .

Another generalization could be :

Let $T_k = k$ for $k = 1$ to n .

T_r = sum of all products of (r-1) terms of the sequence taking (r-2) at a time (r > n) . This defines SC_vS.

for n = 2 $T_1 = 1$, $T_2 = 2$, $T_3 = 3$, $T_4 = 17$ etc

$SC_vS \rightarrow$ 1 , 2 , 3, 17 , . . .

PROBLEM : (1) How many of the consecutive terms of SCS(r) are pairwise coprime ?

(2) How many of the terms of SC_vS are primes ?

(ii) SMARANDACHE PRIME PRODUCT SEQUENCES

SPPS(n)

T_n = sum of all the products of primes chosen from first n primes

taking (n-1) primes at a time.

SPPS(n) $\rightarrow$ 1, 5, 31, 247, 2927 . . .

$T_1 = 1$, $T_2 = 2 + 3$, $T_3 = 2*3 + 2*5 + 3*5 = 31$.

$T_4 = 2*3*5 + 2*3*7 + 2*5*7 + 3*5*7 = 247$ etc.

How many of these are primes ?

(7) SMARANDACHE ϕ -SEQUENCE

$(S\phi S) = \{ n \mid n = k * \phi(n) \}$

$S\phi S \rightarrow$ 1, 2 , 4 , 6 , 8 , 12 . . .

(8) SMARANDACHE PRIME DIVISIBILITY SEQUENCE

SPDS = $\{ n \mid n$ divides $p_n + 1$, p_n is the n^{th} prime. $\}$

SPDS$\rightarrow$ 1 , 2 , 3 , 4 , 10 , . . .

ON THE DIVISORS OF SMARANDACHE UNARY SEQUENCE

(Amarnath Murthy, Superintending Engineer(E&T),Well Logging Section Oil And Natural Gas Corporation Ltd., Chandkheda, Ahmedabad, India-380005)

ABSTRACT: Smarandache Unary Sequence is defined as follows:

$u(n) = 1111 \ldots$, p_n digits of "1" , where p_n is the n^{th} prime.

11, 111, 11111, 1111111 . . .
Are there an infinite number of primes in this sequence ? It is still an unsolved problem. The following property of a divisor of u(n) is established.
If 'd' is a divisor of u(n) then $d \equiv 1 \pmod{p_n}$. , for all $n > 3$ ---(1),

DESCRIPTION: Let $l(m) = 1111 ,\ldots$ m times $= (10^m - 1) / 9$

Then $u(n) = l(p_n)$.
Following proposition will be applied to establish (1).
Proposition : $l(p-1) \equiv 0 \pmod{p}$.-------(2)
PROOF: 9 divides $10^{p-1} -1$. From Fermat's little theorem if $p \geq 7$ is a prime then p divides $(10^{p-1} -1)/9$
as $(p, 9) = (p, 10) = 1$. Hence p divides $l(p-1)$

Coming back to the main proposition , let 'd' be a divisor of u(n).

Let $d = p^a q^b r^c \ldots$,where p , q, r, are prime factors of d .

p divides| 'd' $\Rightarrow$ p divides u(n) also p divides l(p-1) from proposition (2). in other words

p divides $(10^{p-1} -1)/9$ and p divides $(10^p - 1)/9$

p divides $(10^{A(p-1)} -1)/9$ and p divides $(^{10B.p} - 1)/9$

p divides $(10^{(A(p-1) - B.p})/9$

p divides $10^{B.p}\{(10^{A(p-1) - B.p} - 1)/9\}$.

p divides $(10^{A(p-1) - B.p} - 1)/9$. ---------(3)

There exist A and B such that

$A(p - 1) - B.p_n = (p - 1 , p_n)$. As p_n is a prime there are two possibilities :

(i). $(p - 1 , p_n) = 1$ or (ii). $(p - 1 , p_n) = p_n$.

In the first case , from (3) we get p divides (10 - 1)/9 or p

divides 1, which is absurd as $p > 1$. hence $(p - 1 , p_n) = p_n$;
or p_n divides $p - 1$

$$p \equiv 1 \pmod{p_n}$$

$$\Rightarrow \qquad p^a \equiv 1 \pmod{p_n}$$

on similar lines

$$q^b \equiv 1 \pmod{p_n}$$

hence $d = p^a q^b r^c \ldots \equiv 1 \pmod{p_n}$

This completes the proof.

COROLLARY : For any prime p there exists at least one prime q such that
$q \equiv 1 \pmod{p}$

Proof: As $u(n) \equiv 1 \pmod{p_n}$, and also every divisor of u(n) is

$\equiv 1 \pmod{p_n}$, the corollary stands proved. Also clearly such a 'q' is greater than p , this gives us a proof of the infinitude of the prime numbers as a by product.

REFERENCES;

1."Smarandache Partition Type And Other Sequences" , by
Eng. Dr. Fanel IACOBESCU , Bulletin of Pure and Applied Sciences. Vol. 16 E(No. 2) 1997; P.240 .

2. "On The Infinitude of Prime Numbers" . by Amarnath Murthy , The Mathematics Education , INDIA , Vol. XXIX, No. 1, March,1995 .

On Iterations That Alternate the Pseudo-Smarandache and Classic Functions of Number Theory

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
Hiawatha, IA 52233 USA

The Pseudo-Smarandache function was recently defined in a book by Kashihara[1].

Definition: For n > 0 and an integer, the value of the Pseudo-Smarandache function Z(n) is the smallest number m such that n evenly divides

$$\sum_{k=1}^{m} k$$

Note: It is well-known that this is equivalent to n evenly dividing $\frac{m(m+1)}{2}$.

The classic functions of number theory are also well-known and have the following definitions.

Let n be any integer greater than 1 with prime factorization

$$n = p_1^{\alpha 1} p_2^{\alpha 2} \ldots p_k^{\alpha k}$$

Definition: For n > 0 and an integer, the number of divisors function is denoted by d(n). It is well-known that $d(n) = (\alpha 1 + 1)(\alpha 2 + 1) \ldots (\alpha k + 1)$.

Definition: For n > 0 and an integer, the sum of the positive divisors of n is denoted by $\sigma(n)$. It is well-known that

$$\sigma(n) = (1 + p_1 + p_1^2 + \ldots + p_1^{\alpha 1}) \ldots (1 + p_k + p_k^2 + \ldots + p_k^{\alpha k})$$

Definition: For n > 0 and an integer, the Euler phi function $\phi(n)$, is the number of integers k, $1 \leq k < n$ that are relatively prime to n. It is well-known that

$$\phi(n) = p_1^{\alpha 1-1}(p_1 - 1) \ldots p_k^{\alpha k-1}(p_k - 1)$$

Choosing a number n and repeatedly iterating a function is something that has been done many times before. What we will do here now is alternate the iterations between two different functions.

Example:
Construct the sequence of function iterations by starting with the number of divisors function and then alternate it with the Pseudo-Smarandache function.

For example, if n = 3, then the iterations would be

$$d(3) = 2 \quad Z(2) = 3. \; d(3) = 2 \quad Z(2) = 3 \quad \ldots$$

Or notationally

$$Z(\ldots d(Z(d(Z(d(n)))))\ldots)$$

For reference purposes, we will refer to this is as the Z-d sequence.

Note that for n = 3, the sequence of numbers is a two-cycle. This is no accident and it is easy to prove that this is a general result. We do this in a roundabout way.

Theorem 1: Let p be a prime, then the Z-d sequence will always be the two cycle

$$2\ 3\ 2\ 3\ 2\ 3\ \ldots$$

Proof: Since d(p) = 2 for p any prime and Z(2) = 3, which is also a prime, the result is immediate.

This behavior is even more general.

Theorem 2: If $n = p_1p_2$, then the Z-d sequence will always be the two cycle

$$4\ 7\ 2\ 3\ 2\ 3\ \ldots$$

Proof: Since d(n) = 4 and Z(4) = 7 and d(7) = 2, which starts the repeated 2 3 2 3 . . . cycle.

Theorem 3: If $n = p^2$, the Z-d sequence will always be the two cycle

$$3\ 2\ 2\ 3\ 2\ 3\ 2\ 3\ 2\ \ldots$$

Proof: Since d(n) = 3, Z(3) = 2, d(2) = 2, Z(2) = 3 and d(3) = 2. the result follows.

This behavior is a general one that is easy to prove.

Theorem: If n is any integer greater than 1, then the Z-d sequence will always reduce to the 2 cycle

$$2\ 3\ 2\ 3\ 2\ 3\ 2\ 3\ \ldots$$

Proof: Explicitly testing all numbers n ≤ 50, the result holds. To complete the proof, we rely on two simple lemmas.

Lemma 1: If n > 50, then $\frac{n}{d(n)} > 3$.

Proof: By a double induction on the number of prime factors and their exponents.

Basis: If p > 50 is prime, then d(n) = 2.

Inductive 1: Assume that n = $p^k > 50$ and that $\frac{n}{d(n)} > 3$. Then the ratio of
$\frac{p^{k+1}}{d(p^{k+1})} = \frac{p*p^k}{k+1+1} > \frac{p^k}{k+1} > 3$.

Inductive 2: Assume that for n = $p_1^{\alpha 1} \ldots p_j^{\alpha j}$, $\frac{n}{d(n)} > 3$. Add an additional prime factor to n to some power. Since the additional prime factor is not necessarily larger than the others, we will call it q and append it to the end noting that the primes are not necessarily in ascending order.

$$\frac{n*q^m}{d(n*q^m)} = \frac{n*q^m}{d(n)*d(q^m)}$$

since d is multiplicative. Furthermore, it is well-known that d(n) $\leq$ n for all n > 1. Therefore, we have

$$\frac{n*q^m}{d(n*q^m)} = \frac{n*q^m}{d(n)*d(q^m)} \geq \frac{n}{d(n)} > 3.$$

and the proof is complete.

Lemma 2: If n > 1 is an integer, then the largest value that the ratio $\frac{Z(n)}{n}$ can have is 2.

Proof: It is well-known that if n = 2^k, then Z(n) = 2^{k+1} -1. For all other values of n, Z(n) is at most n.

Therefore, we have an alternating sequence of numbers where one at most doubles the previous value and the other always reduces it by at least a factor of three. Since the one that always reduces by a factor of three is done first this guarantees that the iterations will eventually reduce the value to a number less than 50.

Since the iteration of the Z-d sequence always goes to the same 2-cycle, the result will be the same if the order of the iterations is reversed to the d-Z sequence.

Another iterated sequence that can be constructed involves the Euler phi function and the Pseudo-Smarandache function.

Definition: For n $>$ 1, the Z-phi sequence is the alternating iteration of the Euler phi function followed by the Pseudo-Smarandache function.

$$Z(\ldots(\phi(Z(\phi(n))))\ldots)$$

The sequence is the rather boring

1 1 1 1 1 ...

for n=2.

This result is not universal, as for all numbers 3 $\leq$ n $\leq$ 14, the iterations move to the same

2 3 2 3 2 3

2-cycle. However, this is not a universal result, as when n = 15, the iteration creates the 2-cycle

8 15 8 15 8 15 8 15 ...

which is also the cycle for 16 and 17.

Examining the behavior of the Z-phi iteration for all numbers n $\leq$ 254, all create either the

2 3 2 3 2 3 ...

or

8 15 8 15 8 15 8 ...

2-cycles. However, when n =255, the iteration creates the new 2-cycle

128 255 128 255 128 ...

which is also the cycle for 256.

Creating and running a computer program to check for 2-cycles that are different from the previous three, no new 2-cycle is encountered until for n=65535

32768 65535 32768 65535 32768 ...

which is also the cycle for n=65536.

The pattern so far is clear and is summarized in the following chart

Pattern	First Instance
2 -3	$3 = 2^2 - 1$
8 - 15	$15 = 2^4 - 1$
128 - 255	$255 = 2^8 - 1$
32768 - 65535	$65535 = 2^{16} - 1$

which raises several questions.

1) Does the Z-phi sequence always reduce to a 2-cycle of the form $2^{2^k-1} - 2^{2^k} - 1$ for $k \geq 1$?

2) Does any additional patterns always appear first for $n = 2^{2^k} - 1$?

A computer search was conducted to test these questions.

Definition: For $n > 1$, the Z-sigma sequence is the alternating iteration of the sigma, sum of divisors function followed by the Pseudo-Smarandache function.

$$Z(\ldots(\sigma(Z(\sigma(n))))\ldots)$$

For n = 2, the Z-sigma sequence creates the 2-cycle

3 2 3 2 . . .

and for $3 \leq n \leq 15$, the Z-sigma sequence creates the 2-cycle

24 15 24 15 24 15 . . .

However, for n = 16, we get our first cycle that is not a 2-cycle, but is in fact a 12-cycle.

63 104 64 127 126 312 143 168 48 124 31 32 63 104 64 127 126 312 143 168 48 ...

The numbers $17 \leq n \leq 19$ all generate the 2-cycle 24 15 24 15 . . . , but n = 20 generates the 2-cycle

42 20 42 20 42 20 . . .

and n = 21 generates the 12-cycle

63 104 64 127 126 312 143 168 48 124 31 32 63 104 64 127 126 312 143 168 48 ...

The numbers 20 $\leq$ n $\leq$ 24 all generate the 2-cycle 24 15, but n = 25 generates the 12-cycle

63 104 64 127 126 312 143 168 48 124 31 32 63 104 64 127 126 312 143 168 48 ...

and n = 26 the 2-cycle 42 20 42 20 . . .

It is necessary to go up to n = 381 before we get the new cycle 1023 1536 1023 1536 . . . and a search up to n = 552,000 revealed no additional generated cycles. This leads to some obvious additional questions.

1) Is there another cycle generated by the Z-sigma sequence?

2) Is there an infinite number of numbers n that generate the 2-cycle 42 20?

3) Are there any other numbers n that generate the two cycle 2 3?

4) Is there a pattern to the first appearance of a new cycle?

In conclusion, the iterated sequences created by alternating a classic function of number theory with the Pseudo-Smarandache functions yields some interesting results that are only touched upon here. The author strongly encourages others to further explore these problems and is interested in hearing of any additional work in this area.

Reference

1. K. Kashihara, **Comments and Topics on Smarandache Notions and Problems**, Erhus University Press, Vail, AZ., 1996.

* This paper was presented at the April, 1999 meeting of the Iowa section of the Mathematical Association of America.

THE ALMOST PRESUMABLE MAXIMALITY OF SOME TOPOLOGICAL LEMMA

Florian Munteanu*, Octavian Mustafa†
Department of Mathematics, University of Craiova
13, Al. I. Cuza st., Craiova 1100, Romania

Abstract

Some splitting lemma of topological nature provides fundamental information when dealing with dynamics (see [1], pg.79). Because the set involved, namely $X \setminus \mathcal{P}_s$, is neither open nor closed, a natural question arise: can this set be modified in order to obtain aditional data ? Unfortunately, the answer is negative.

For a metric space X which is locally connected and locally compact and for some continuous mapping $f : X \to X$, the set ω-set of each element x of X is given by the formula

$$\omega(x) = \left\{ y \in X | y = \lim_{n \to +\infty} f^{k_n}(x) \, , \, \lim_{n \to +\infty} k_n = +\infty \right\}.$$

We also denote by $\omega_j(x)$, $1 \leq j \leq r$, the set

$$\omega_j(x) = \left\{ y \in X | y = \lim_{n \to +\infty} f^{m_n \cdot r + j}(x) \, , \, \lim_{n \to +\infty} m_n = +\infty \right\}.$$

Now, $\omega(x)$ can be splitted according to the following lemma.

Lemma 1 *a)* $\omega(x) = \bigcup_{j=1}^{r} \omega_j(x)$;

b) $f(\omega_j(x)) \subset \omega_{(j+1) \, mod \, r}$.

* *e-mail* address: munteanufm@hotmail.com
† *e-mail* address: Octawian@mail.yahoo.com

Its proof relies upon the properties of $\omega(x)$.

Lemma 2 *For some nonvoid subset S of X we consider C a component of $X \setminus S$. i.e. a maximal connected set (see [2], pg. 54). Then:*
a) $\overline{C}^X \subset C \cup \left(S \cap \partial^X S\right)$*;*
b) $\partial^X C \subset \left(C \cap \partial^X C\right) \cup \left(S \cap \partial^X S\right)$*.*
where $\overline{C}^X$ *signifies the closure of C under the topology of X while* $\partial^X C$ *is the boundary of C under the same topology.*

Remark 1 *For instance, if S is closed, then $\partial^X C \subset \partial^X S$ as the components of a locally connected space are open.*

Proof. a) First, let's show that $\overline{C}^X \subset C \cup S$. For $x \in X \setminus (C \cup S) = (X \setminus S) \setminus C$, as C is closed in $X \setminus S$, there will be some open $G \subset X$ such that

$$x \in G \cap (X \setminus S) \subset X \setminus (C \cup S).$$

Obviously,

$$[G \cap (X \setminus S)] \cap C = G \cap C = \emptyset$$

and so

$$x \notin \overline{C}^X .$$

Further on, let's assume that $x \in \overline{C}^X \cap S$. If $x \in X \setminus \partial^X S$, then $x \notin \overline{X \setminus S}^X$. There will be some open $W \subset X$ such that

$$x \in W \, ; \; W \cap \overline{X \setminus S}^X = \emptyset .$$

In particular, $W \cap C = \emptyset$ and so $x \notin \overline{C}^X$.
b) According to a), we have:

$$\begin{aligned} \overline{C}^X \cap \overline{X \setminus C}^X &= \partial^X C \subset \left(C \cap \overline{X \setminus C}^X\right) \cup \left[\left(S \cap \partial^X S\right) \cap \overline{X \setminus C}^X\right] \\ &= \left(C \cap \overline{X \setminus C}^X\right) \cup \left(S \cap \partial^X S\right) \end{aligned}$$

because of $S \cap \partial^X S \subset S \subset X \setminus C$.
Obviously,

$$C \cap \overline{X \setminus C}^X = \left(C \cap \overline{X \setminus C}^X\right) \cap \overline{C}^X = C \cap \partial^X C .$$

■

Remark 2 *It worths noticing that the sets* $(C \cap \partial^X C)$ *and* $(S \cap \partial^X S)$ *are disjoint; in other words,* $\partial^X C$ *is piecewise-made. Lemma 2 works equally well in any topological space.*

Lemma 3 *(Melbourne, Dellnitz, Golubitsky)*
For some nonvoid subset S *of* X, *we denote by* $\mathcal{P}_s$ *the union*

$$\mathcal{P}s(f) = \bigcup_{n=0}^{\infty} (f^n)^{-1}(S)$$

Let x *be some element of* S. *Then either* $\omega(x) \subset \overline{\mathcal{P}}_s^X$ *or the following are valid:*
a) $\omega(x) \backslash \mathcal{P}_s$ *is covered by finitely many (connected) components* $C_0, \ldots, C_{r-1}$ *of* $X \setminus \mathcal{P}_s$;
b) These components can be ordered so that
$f(C_i) \subset C_{(i+1) \bmod r}$;
c) $\omega(x) \subset \overline{C}_0^X \cup \ldots \cup \overline{C}_{r-1}^X$.

Remark 3 *Notice the splitting in relation with lemma 1. As we mentioned in the Abstract, it is quite natural to ask if* $X \backslash \mathcal{P}_s$ *can be replaced by the easier-to-work-with* $X \setminus \overline{\mathcal{P}}_s$. *The following lemma shows that this would imply no more the presence of finitely many components.*

Lemma 4 *Let* S *be some nonvoid subset of* X *which is not dense in* X, *i.e.* $\overline{S}^X \neq X$. *We consider* C *a component of* $X \setminus \overline{S}^X$ *and* D *a component of* $X \setminus S$ *such that* $C \subset D$. *Then any element* x *of* $D \setminus C$ *belongs either to* $\partial^X S$ *or to any other component of* $X \setminus \overline{S}^X$.

Proof. If $x \notin X \setminus \overline{S}$ then $x \in (X \setminus S) \cap \overline{S}^X \subset \partial^X S$. ∎
An example would be appropriate: in $\mathbf{R}^2$, we denote by $\mathcal{D}(0,r)$ the r-disk centered in 0. Now, for $X = \overline{\mathcal{D}(0,2)}^{\mathbf{R}^2}$, $S = \mathcal{D}(0,1) \cup (1,2] \cup [-2,-1)$, we have

$$\overline{S}^{\mathbf{R}^2} = \overline{\mathcal{D}(0,1)}^{\mathbf{R}^2} \cup [1,2] \cup [-2,-1]$$

$$D = X \setminus S,\ C \in \left\{ \left(X \setminus \overline{S}^{\mathbf{R}^2}\right) \cap (y > 0), \left(X \setminus \overline{S}^{\mathbf{R}^2}\right) \cap (y < 0) \right\}.$$

Further exemples can be architectured easily even to obtain infinitely many components of $X \setminus \overline{S}^X$.
In other words, finitely many components of $X \setminus S$ may include infinitely many components of $X \setminus \overline{S}^X$.

References

[1] I. Melbourne, M. Dellnitz, M. Golubitsky, *The structure of symmetric attractors*, Arch. Rational Mech. Anal., 123, pp. 75-98, 1993

[2] J. L. Kelley, *General topology*, D. Von Nostrand Comp., 1955

ON THE CONVERGENCE OF THE EULER HARMONIC SERIES

Sabin Tabirca* **Tatiana Tabirca***

Abstract: *The aim of this article is to study the convergence of the Euler harmonic series. Firstly, the results concerning the convergence of the Smarandache and Erdos harmonic functions are reviewed. Secondly, the Euler harmonic series is proved to be convergent for a>1, and divergent otherwise. Finally, the sums of the Euler harmonic series are given.*

Key words: ***series, convergence, Euler function.***

The purpose of this article is to introduce the Euler harmonic series and to study its convergence. This problem belongs to a new research direction in Number Theory that is represented by convergence properties of series made with the most used Number Theory functions.

1. Introduction

In this section, the important results concerning the harmonic series for the Smarandache and Erdos function are reviewed.

Definition 1. If $f : N \to N^*$ is a function, then the series $\sum_{n\geq 1} \frac{1}{f^a(n)}$ is the harmonic series associated to *f* and is shortly named the *f* harmonic series.

The convergence of this sort of series has been studied for the Smarandache and Erdos functions so far. Both are important functions in Number Theory being intensely studied. The definitions and main properties of these two important functions are presented in the following:

- The Smarandache function is $S: N^* \to N$ defined by

* Computer Science Department, Transilvania University of Brasov

$$S(n) = \min\{k \in N | k! = Mn\} \ (\forall n \in N^*). \tag{1}$$

- The Erdos function is $P : N^* \to N$ defined by

$$P(n) = \max\{p \in N \mid n = Mp \wedge p \text{ is prim}\} (\forall n \in N^* \setminus \{1\}),\ P(1) = 0. \tag{2}$$

The main properties of them are:

$$(\forall a,b \in N^*)\ (a,b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\},\ P(a \cdot b) = \max\{P(a), P(b)\}. \tag{3}$$

$$(\forall a \in N^*)\ P(a) \le S(a) \le a \text{ and the equalities occur } iif \text{ a is prim.} \tag{4}$$

Erdos [1995] found the relationship between these two functions that is given by

$$\lim_{n \to \infty} \frac{\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right|}{n} = 0. \tag{5}$$

The series $\sum_{n \ge 2} \frac{1}{S(n)}$ and $\sum_{n \ge 2} \frac{1}{P(n)}$ are obviously divergent from Equation (4).

The divergence of the series $\sum_{n \ge 2} \frac{1}{S^2(n)}$ was an open problem for more than ten years. Tabirca [1998] proved the its divergence using an analytical technique. Luca [1999] was able to prove the divergence of the series $\sum_{n \ge 2} \frac{1}{S^a(n)}$ refining Tabirca's technique. Thus, the Smarandache harmonic series $\sum_{n \ge 2} \frac{1}{S^a(n)}$, $a \in R$ is divergent. Based on this result and on Equation (5), Tabirca [1999] showed that the Erdos harmonic series $\sum_{n \ge 2} \frac{1}{P^a(n)}$, $a \in R$ is divergent too.

Unfortunately, this convergence property has not been studied for the Euler function. This function is defined as follow: $\varphi : N \to N$, $(\forall n \in N)\ \varphi(n) = \left|\left\{k = 1,2,\dots,n \mid (k,n) = 1\right\}\right|$. The main properties [Hardy & Wright, 1979] of this function are enumerated in the following:

$$(\forall a,b \in N)(a,b) = 1 \Rightarrow \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \text{ - the multiplicative property} \tag{6}$$

$$a = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_s^{m_s} \Rightarrow \varphi(a) = a \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_s}\right) \tag{7}$$

$$(\forall a \in N) \sum_{d|a} \varphi(d) = a. \tag{8}$$

More properties concerning this function can be found in [Hardy & Wright, 1979], [Jones & Jones, 1998] or [Rosen, 1993].

2. The Convergence of the Euler Harmonic Series

In this section, the problem of the convergence for the Euler harmonic series is totally solved. The Euler harmonic series $\sum_{n\geq 1}\frac{1}{\varphi^a(n)}$, $a\in R$ is proved to have the same behavior as the harmonic series $\sum_{n\geq 1}\frac{1}{n^a}$, $a\in R$.

Proposition 1. The series $\sum_{n\geq 1}\frac{1}{\varphi^a(n)}$ is divergent for $a\leq 1$.

Proof

The proof is based on the equation

$$\varphi(n)\leq n\ (\forall\ n\geq 1). \tag{9}$$

Since $\frac{1}{\varphi^a(n)}\geq\frac{1}{n^a}\ (\forall\ n\geq 1)$ and $\sum_{n\geq 1}\frac{1}{n^a}$ is divergent, it follows that $\sum_{n\geq 1}\frac{1}{\varphi^a(n)}$ is divergent too.

♦

The convergence of the series for $a>1$ is more difficult than the previous and is studied in the following.

Let us define the function $d: N^*\to N$ by $d(n)=|\{p \text{ prime} \mid n = Mp\}|$. The main properties of this function are given by the next proposition.

Proposition 2. The function d satisfies the following equation:

a) $d(1)=0$. (10a.)

b) $(\forall\ a,b\in N^*)(a,b)=1\Rightarrow d(a\cdot b)=d(a)+d(b)$. (10b.)

c) $(\forall\ n\in N^*)\ d(n)\leq\log_2(n)$. (10c.)

Proof

Equation (10a.) is obvious.

Let $a=p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s}$ and $b=q_1^{k_1}\cdot q_2^{k_2}\cdot\ldots\cdot q_t^{k_t}$ be the prime number decomposition of two relative prime numbers. Thus, $a\cdot b=p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s}\cdot q_1^{k_1}\cdot q_2^{k_2}\cdot\ldots\cdot q_t^{k_t}$ gives the prime number decomposition for ab. Since the equation $d(a\cdot b)=s+t$, $d(a)=s$ and $d(b)=t$ hold in the above hypothesises, Equation (10b) is true.

Let $n=p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s}$ be the prime number decomposition of n. Equation (10b) gives the following inequality

$$d(n)=d(p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s})=d(p_1^{m_1})+d(p_2^{m_2})+\ldots+d(p_s^{m_s})=1+1+\ldots+1\leq$$
$$\leq\log_2(p_1^{m_1})+\log_2(p_2^{m_2})+\ldots+\log_2(p_s^{m_s})=\log_2(p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s})=\log_2(n)$$

that proves Equation (10c). ♦

The following proposition proposes a new inequality concerning the Euler function.

Proposition 3. $(\forall\ n \geq 1)\ \varphi(n) \geq \dfrac{n}{1+\log_2 n}$.

Proof

Let $n = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_s^{m_s}$ be the prime number decomposition of n such that $p_1 < p_2 < \ldots < p_s$. Thus, $\varphi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_s}\right)$ holds. Using the order $p_1 < p_2 < \ldots < p_s$, it follows that $2 \leq p_1, 3 \leq p_2, \ldots, d(n)+1 < p_s$. These inequalities are used as follows:

$$\varphi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_s}\right) \geq$$

$$\geq n \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \ldots \cdot \left(1 - \frac{1}{d(n)+1}\right) = \frac{n}{d(n)+1}$$

Equation (10c) used in the last inequality gives $\varphi(n) \geq \dfrac{n}{1+\log_2 n}$. ♦

Proposition 4. If $a>1$, then the series $\sum\limits_{n\geq 1} \left(\dfrac{1+\log_2 n}{n}\right)^a$ is convergent.

Proof

The proof uses the following convergence test: "*if $(a_n)_{n>0}$ is a decreasing sequence, then the series $\sum\limits_{n>0} a_n$ and $\sum\limits_{n>0} 2^n \cdot a_{2^n}$ have the same convergence*".

Because the sequence $\left(\left(\dfrac{1+\log_2 n}{n}\right)^a\right)_{n>0}$ is decreasing, the above test can be applied. The condensed series is $\sum\limits_{n\geq 1} 2^n \cdot \left(\dfrac{1+\log_2 2^n}{2^n}\right)^a = \sum\limits_{n\geq 1} \dfrac{(1+n)^a}{2^{n\cdot(a-1)}} = 2^{a-1} \cdot \sum\limits_{n\geq 2} \dfrac{n^a}{2^{n\cdot(a-1)}}$ that is obviously convergent. ♦

Theorem 5. If $a>1$, then the series $\sum\limits_{n\geq 1} \dfrac{1}{\varphi^a(n)}$ is convergent.

Proof

According to Proposition 4, the series $\sum\limits_{n\geq 1} \left(\dfrac{1+\log_2 n}{n}\right)^a$ is convergent. Proposition 3 gives the inequality $\left(\dfrac{1+\log_2 n}{n}\right)^a \geq \dfrac{1}{\varphi^a(n)}$, thus the series $\sum\limits_{n\geq 1} \dfrac{1}{\varphi^a(n)}$ is convergent too.

♦

The interesting fact is that the Euler harmonic series has the same behaviour as the classical harmonic series. Therefore, both are convergent for $a>1$ and both are divergent for $a\leq 1$. The right question is to find information about the sum of the series in the convergence case. Let us denote $E(a)=\sum_{n\geq 1}\frac{1}{\varphi^{a}(n)}$ the sum of the Euler harmonic series for $a>1$. These constants can be computed by using a simple computation. They are presented in Table 1 for a=2,3, ...,7.

a	*E(a)*	*a*	*E(a)*
2	3.39049431	5	2.09837919
3	2.47619474	6	2.04796102
4	2.20815078	7	2.02369872

Table 1. The values for $E(a)$.

Unfortunately, none of the above constants are known. Moreover, a relationship between the classical constants (π, e, ...) and them are not obvious. Finding properties concerning the constants $E(a)$ still remains an open research problem.

References

1. Hardy, G. H. and Wright, E. M.: *Introduction to the Theory of Numbers*, (5th ed.), 1979, Oxford University Press, Oxford.
1. Jones, G. A. and Jones, M. J.: *Elementary Number Theory*,1998, Springer-Verlag, New York.
2. Luca, F.: On the Divergence of the Smarandache Harmonic Series, *Smarandache Notion Journal*, 1999, **10**, no.1-2-3, 116-119.
3. Rosen, K.: *Elementary Number Theory and its Application*, 1993, Addison-Wesley, New York.
4. Tabirca, S. and Tabirca, T.: The Convergence of the Smarandache Harmonic Series, *Smarandache Notion Journal*, 1998, **9**, no.1-2, 28-37.
5. Tabirca, S. Tabirca, T.: On the Erdos Harmonic Series, *Proceedings of the Conference "Paul Erdos and his Mathematics"*, 1999[to appear].

On two notes by M. Bencze

J. Sándor

Babeş-Bolyai University, 3400 Cluj-Napoca. Romania

In vol **10** of this Journal M. Bencze has published two notes on certain inequalities for the Smarandache function. In [2] it is proved that

$$S\left(\prod_{k=1}^{m} m_k\right) \le \sum_{k=1}^{m} S(m_k) \tag{1}$$

This, in other form is exactly inequality (2) from our paper [5], and follows easily from Le's inequality $S(ab) \le S(a) + S(b)$

In [1] it is proved that

$$S\left(\prod_{k=1}^{n} (a_k!)^{b_k}\right) \le \sum_{k=1}^{n} a_k b_k \tag{2}$$

The proof follows the method of the problem from [3], i.e.

$$S(m!^n) \le m \cdot n \tag{3}$$

This appears also in [4], [5]. We note here that relation (2) is a direct consequence of (1) and (3), since

$$S(a_1!^{b_1} \cdots a_n!^{b_n}) \le S(a_1!^{b_1}) + \ldots + S(a_n!^{b_n}) \le b_1 a_1 + \cdots + b_n a_n$$

References

1. M. Bencze, A new inequality for the Smarandache function, SNJ, 10 (1999), No. 1 – 2 - 3, p. 139
2. M. Bencze, An inequality for the Smarandache function, SNJ, 10 (1999), No. 1-2-3, p. 160
3. J. Sándor, Problem L : 87, Mat Lap (Cluj), No. 5/1997, p. 194
4. J. Sándor, On certain new inequalities and limits for the Smarandache function, SNJ, 9 (1998), 63 – 69
5. J. Sándor, On an inequality for the Smarandache function, SNJ 10 (1999), 125 – 127

On certain generalizations of the Smarandache function

J. Sándor

Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

1. The famous Smarandache function is defined by $S(n) := \min\{k \in \mathbf{N}^*: \ n|k!\}$, $n \geq 1$ positive integer. This arithmetical function is connected to the number of divisors of n, and other important number theoretic functions (see e.g. [6], [7], [9], [10]). A very natural generalization is the following one: Let $f : \mathbf{N}^* \to \mathbf{N}^*$ be an arithmetical function which satisfies the following property:

(P_1) For each $n \in \mathbf{N}^*$ there exists at least a $k \in \mathbf{N}^*$ such that $n|f(k)$.

Let $F_f : \mathbf{N}^* \to \mathbf{N}^*$ defined by

$$F_f(n) = \min\{k \in \mathbf{N} : \ n|f(k)\}. \tag{1}$$

Since every subset of natural numbers is well-ordered, the definition (1) is correct, and clearly $F_f(n) \geq 1$ for all $n \in \mathbf{N}^*$.

Examples. 1) Let $id(k) = k$ for all $k \geq 1$. Then clearly (P_1) is satisfied, and

$$F_{id}(n) = n. \tag{2}$$

2) Let $f(k) = k!$. Then $F_!(n) = S(n)$ - the Smarandache function.

3) Let $f(k) = p_k!$, where p_k denotes the kth prime number. Then

$$F_f(n) = \min\{k \in \mathbf{N}^* : \; n|p_k!\}. \tag{3}$$

Here (P_1) is satisfied, as we can take for each $n \geq 1$ the least prime greater than n.

4) Let $f(k) = \varphi(k)$, Euler's totient. First we prove that (P_1) is satisfied. Let $n \geq 1$ be given. By Dirichlet's theorem on arithmetical progressions ([1]) there exists a positive integer a such that $k = an + 1$ is prime (in fact for infinitely many a's). Then clearly $\varphi(k) = an$, which is divisible by n.

We shall denote this function by F_φ. (4)

5) Let $f(k) = \sigma(k)$, the sum of divisors of k. Let k be a prime of the form $an - 1$, where $n \geq 1$ is given. Then clearly $\sigma(n) = an$ divisible by n. Thus (P_1) is satisfied. One obtains the arithmetical function F_σ. (5)

2. Let $A \subset \mathbf{N}^*$, $A \neq \emptyset$ a nonvoid subset of $\mathbf{N}$, having the property:

(P_2) For each $n \geq 1$ there exists $k \in A$ such that $n|k!$.

Then the following arithmetical function may be introduced:

$$S_A(n) = \min\{k \in A : \; n|k!\}. \tag{6}$$

Examples. 1) Let $A = \mathbf{N}^*$. Then $S_{\mathbf{N}}(n) \equiv S(n)$ - the Smarandache function.

2) Let $A = \mathbf{N}_1$ = set of odd positive integers. Then clearly (P_2) is satisfied. (7)

3) Let $A = \mathbf{N}_2$ = set of even positive integers. One obtains a new Smarandache-type function. (8)

4) Let $A = P$ = set of prime numbers. Then $S_P(n) = \min\{k \in P : \; n|k!\}$. We shall

denote this function by $P(n)$, as we will consider more closely this function. (9)

3. Let $g : \mathbf{N}^* \to \mathbf{N}^*$ be a given arithmetical function. Suppose that g satisfies the following assumption:

(P_3) For each $n \geq 1$ there exists $k \geq 1$ such that $g(k)|n$. (10)

Let the function $G_g : \mathbf{N}^* \to \mathbf{N}^*$ be defined as follows:

$$G_g(n) = \max\{k \in \mathbf{N}^* : \; g(k)|n\}. \tag{11}$$

This is not a generalization of $S(n)$, but for $g(k) = k!$, in fact one obtains a "dual"-function of $S(n)$, namely

$$G_!(n) = \max\{k \in \mathbf{N}^* : \; k!|n\}. \tag{12}$$

Let us denote this function by $S_*(n)$.

There are many other particular cases, but we stop here, and study in more detail some of the above stated functions.

4. The function $P(n)$

This has been defined in (9) by: the least prime p such that $n|p!$. Some values are: $P(1) = 1$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, $P(5) = 5$, $P(6) = 3$, $P(7) = 7$, $P(8) = 5$, $P(9) = 7$, $P(10) = 5$, $P(11) = 11, \ldots$

Proposition 1. For each prime p one has $P(p) = p$, and if n is squarefree, then $P(n) =$ greatest prime divisor of n.

Proof. Since $p|p!$ and $p \nmid q!$ with $q < p$, clearly $P(p) = p$. If $n = p_1 p_2 \ldots p_r$ is squarefree, with $p_1, \ldots, p_r$ distinct primes, if $p_r = \max\{p_1, \ldots, p_r\}$, then $p_1 \ldots p_r | p_r!$. On the other hand, $p_1 \ldots p_r \nmid q!$ for $q < p_r$, since $p_r \nmid q!$. Thus p_r is the least prime with the required property.

The calculation of $P(p^2)$ is not so simple but we can state the following result:

Proposition 2. One has the inequality $P(p^2) \geq 2p+1$. If $2p+1 = q$ is prime, then $P(p^2) = q$. More generally, $P(p^m) \geq mp+1$ for all primes p and all integers m. There is equality, if $mp+1$ is prime.

Proof. From $p^2 | (1 \cdot 2 \dots p)(p+1) \dots (2p)$ we have $p^2 | (2p)!$. Thus $P(p^2) \geq 2p+1$. One has equality, if $2p+1$ is prime. By writing $p^m | \underbrace{1 \cdot 2 \dots p}\underbrace{(p+1) \dots 2p} \dots \underbrace{[(m-1)p+1] \dots mp}$, where each group of p consecutive terms contains a member divisible by p, one obtains $P(p^m) \geq mp+1$.

Remark. If $2p+1$ is not a prime, then clearly $P(p^2) \geq 2p+3$.

It is not known if there exist infinitely many primes p such that $2p+1$ is prime too (see [4]).

Proposition 3. The following double inequality is true:

$$2p+1 \leq P(p^2) \leq 3p-1 \tag{13}$$

$$mp+1 \leq P(p^m) \leq (m+1)p-1 \tag{14}$$

if $p \geq p_0$.

Proof. We use the known fact from the prime number theory ([1], [8]) tha for all $a \geq 2$ there exists at least a prime between $2a$ and $3a$. Thus between $2p$ and $3p$ there is at least a prime, implying $P(p^2) \leq 3p-1$. On the same lines, for sufficiently large p, there is a prime between mp and $(m+1)p$. This gives the inequality (14).

Proposition 4. For all $n, m \geq 1$ one has:

$$S(n) \leq P(n) \leq 2S(n)-1 \tag{15}$$

and

$$P(nm) \leq 2[P(n) + P(m)] - 1 \tag{16}$$

where $S(n)$ is the Smarandache function.

Proof. The left side of (15) is a consequence of definitions of $S(n)$ and $P(n)$, while the right-hand side follows from Chebyshev's theorem on the existence of a prime between a and $2a$ (where $a = S(n)$, when $2a$ is not a prime).

For the right side of (16) we use the inequality $S(mn) \leq S(n) + S(m)$ (see [5]): $P(nm) \leq 2S(nm) - 1 \leq 2[S(n) + S(m)] - 1 \leq 2[P(n) + P(m)] - 1$, by (15).

Corollary.

$$\lim_{n\to\infty} \sqrt[n]{P(n)} = 1. \tag{17}$$

This is an easy consequence of (15) and the fact that $\lim\limits_{n\to\infty} \sqrt[n]{S(n)} = 1$. (For other limits, see [6]).

5. The function $S_*(n)$

As we have seen in (12), $S_*(n)$ is in certain sense a dual of $S(n)$, and clearly $(S_*(n))!|n|(S(n))!$ which implies

$$1 \leq S_*(n) \leq S(n) \leq n \tag{18}$$

thus, as a consequence,

$$\lim_{n\to\infty} \sqrt[n]{\frac{S_*(n)}{S(n)}} = 1. \tag{19}$$

On the other hand, from known properties of S it follows that

$$\liminf_{n\to\infty} \frac{S_*(n)}{S(n)} = 0, \quad \limsup_{n\to\infty} \frac{S_*(n)}{S(n)} = 1. \tag{20}$$

For odd values n, we clearly have $S_*(n) = 1$.

Proposition 5. For $n \geq 3$ one has

$$S_*(n! + 2) = 2 \tag{21}$$

and more generally, if p is a prime, then for $n \geq p$ we have

$$S_*(n! + (p-1)!) = p - 1. \tag{22}$$

Proof. (21) is true, since $2|(n!+2)$ and if one assumes that $k!|(n!+2)$ with $k \geq 3$, then $3|(n!+2)$, impossible, since for $n \geq 3$, $3|n!$. So $k \leq 2$, and remains $k = 2$.

For the general case, let us remark that if $n \geq k+1$, then, since $k|(n!+k!)$, we have $S_*(n!+k!) \geq k$.

On the other hand, if for some $s \geq k+1$ we have $s!|(n!+k!)$, by $k+1 \leq n$ we get $(k+1)|(n!+k!)$ yielding $(k+1)|k!$, since $(k+1)|n!$. So, if $(k+1)|k!$ is not true, then we have

$$S_*(n! + k!) = k. \tag{23}$$

Particularly, for $k = p-1$ (p prime) we have $p \nmid (p-1)!$.

Corollary. For infinitely many m one has $S_*(m) = p - 1$, where p is a given prime.

Proposition 6. For all $k, m \geq 1$ we have

$$S_*(k!m) \geq k \tag{24}$$

and for all $a, b \geq 1$,

$$S_*(ab) \geq \max\{S_*(a), S_*(b)\}. \tag{25}$$

Proof. (24) trivially follows from $k!|(k!m)$, while (25) is a consequence of $(S_*(a))!|a \Rightarrow (S_*(a))!|(ab)$ so $S_*(ab) \geq S_*(a)$. This is true if a is replaced by b, so (25) follows.

Proposition 7. $S_*[x(x-1)\dots(x-a+1)] \geq a$ for all $x \geq a$ (x positive integer).(26)

Proof. This is a consequence of the known fact that the product of $\underline{a}$ consecutive integers is divisible by $a!$.

We now investigate certain properties of $S_*(a!b!)$. By (24) or (25) we have $S_*(a!b!) \geq \max\{a,b\}$. If the equation

$$a!b! = c! \tag{27}$$

is solvable, then clearly $S_*(a!b!) = c$. For example, since $3!\cdot 5! = 6!$, we have $S_*(3!\cdot 5!) = 6$. The equation (27) has a trivial solution $c = k!$, $a = k!-1$, $b = k$. Thus $S_*(k!(k!-1)!) = k$.

In general, the nontrivial solutions of (27) are not known (see e.g. [3], [1]).

We now prove:

Proposition 8. $S_*((2k)!(2k+2)!) = 2k+2$, if $2k+3$ is a prime; (28)

$S_*((2k)!(2k+2)!) \geq 2k+4$, if $2k+3$ is not a prime. (29)

Proof. If $2k+3 = p$ is a prime, (28) is obvious, since $(2k+2)!|(2k)!(2k+2)!$, but $(2k+3)! \nmid (2k)!(2k+2)!$. We shall prove first that if $2k+3$ is not prime, then

$$(2k+3)|(1\cdot 2\dots(2k)) \tag{$*$}$$

Indeed, let $2k+3 = ab$, with $a,b \geq 3$ odd numbers. If $a < b$, then $a < k$, and from $2k+3 \geq 3b$ we have $b \leq \dfrac{2}{3}k + 1 < k$. So $(2k)!$ is divisible by ab, since a,b are distinct numbers between 1 and k. If $a = b$, i.e. $2k+3 = a^2$, then $(*)$ is equivalent with $a^2|(1\cdot 2\dots a)(a+1)\dots(a^2-3)$. We show that there is a positive integer k such that $a+1 < ka \leq a^2-3$ or. Indeed, $a(a-3) = a^2-3a < a^2-3$ for $a > 3$ and $a(a-3) > a+1$ by $a^2 > 4a+1$, valid for $a \geq 5$. For $a = 3$ we can verify $(*)$ directly. Now $(*)$ gives

$$(2k+3)!|(2k)!(2k+2)!, \text{ if } 2k+3 \neq \text{prime} \tag{$**$}$$

implying inequality (29).

For consecutive odd numbers, the product of factorials gives for certain values

$$S_*(3!\cdot 5!) = 6, \quad S_*(5!\cdot 7!) = 8, \quad S_*(7!\cdot 9!) = 10,$$

$$S_*(9!\cdot 11!) = 12, \quad S_*(11!\cdot 13!) = 16, \quad S_*(13!\cdot 15!) = 16, \quad S_*(15!\cdot 17!) = 18,$$

$$S_*(17!\cdot 19!) = 22, \quad S_*(19!\cdot 21!) = 22, \quad S_*(21!\cdot 23!) = 28.$$

The following conjecture arises:

Conjecture. $S_*((2k-1)!(2k+1)!) = q_k - 1$, where q_k is the first prime following $2k+1$.

Corollary. From $(q_k - 1)!|(2k-1)!(2k+1)!$ it follows that $q_k > 2k+1$. On the other hand, by $(2k-1)!(2k+1)!|(4k)!$, we get $q_k \le 4k-3$. Thus between $2k+1$ and $4k+2$ there is at least a prime q_k. This means that the above conjecture, if true, is stronger than Bertrand's postulate (Chebyshev's theorem [1], [8]).

6. Finally, we make some remarks on the functions defined by (4), (5), other functions of this type, and certain other generalizations and analogous functions for further study, related to the Smarandache function.

First, consider the function F_φ of (4), defined by

$$F_\varphi = \min\{k \in \mathbf{N}^* : \; n|\varphi(k)\}.$$

First observe that if $n+1 =$ prime, then $n = \varphi(n+1)$, so $F_\varphi(n) = n+1$. Thus

$$n+1 = \text{prime} \;\Rightarrow\; F_\varphi(n) = n+1. \tag{30}$$

This is somewhat converse to the φ-function property

$$n+1 = \text{prime} \;\Rightarrow\; \varphi(n+1) = n.$$

Proposition 9. Let ϕ_n be the nth cyclotomic polynomial. Then for each $a \geq 2$ (integer) one has

$$F_\varphi(n) \leq \phi_n(a) \text{ for all } n. \tag{31}$$

Proof. The cyclotomic polynomial is the irreducible polynomial of grade $\varphi(n)$ with integer coefficients with the primitive roots of order n as zeros. It is known (see [2]) the following property:

$$n|\varphi(\phi_n(a)) \text{ for all } n \geq 1, \text{ all } a \geq 2. \tag{32}$$

The definition of F_φ gives immediately inequality (31).

Remark. We note that there exist in the literature a number of congruence properties of the function φ. E.g. it is known that $n|\varphi(a^n - 1)$ for all $n \geq 1$, $a \geq 2$. But this is a consequence of (32), since $\phi_n(a)|a^n - 1$, and $u|v \Rightarrow \varphi(u)|\varphi(v)$ implies (known property of φ) what we have stated.

The most famous congruence property of φ is the following

Conjecture. (D.H. Lehmer (see [4])) If $\varphi(n)|(n-1)$, then $n =$ prime.

Another congruence property of φ is contained in Euler's theorem: $m|(a^{\varphi(m)} - 1)$ for $(a, m) = 1$. In fact this implies

$$S_*[a^{\varphi(m!)} - 1] \geq m \text{ for } (a, m!) = 1 \tag{33}$$

and by the same procedure,

$$S_*(\varphi(a^{n!} - 1)] \geq n \text{ for all } n. \tag{34}$$

As a corollary of (34) we can state that

$$\limsup_{k \to \infty} S_*[\varphi(k)] = +\infty. \tag{35}$$

(It is sufficient to take $k = a^{n!} - 1 \to \infty$ as $n \to \infty$).

7. In a completely similar way one can define $F_d(n) = \min\{k : n|d(k)\}$, where $d(k)$ is the number of distinct divisors of k. Since $d(2^{n-1}) = n$, one has

$$F_d(n) \leq 2^{n-1}. \tag{36}$$

Let now $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ be the canonical factorization of the number n. Then Smarandache ([9]) proved that $S(n) = \max\{S(p_1^{\alpha_1}), \dots, S(p_r^{\alpha_r})\}$.

In the analogous way, we may define the functions $S_\varphi(n) = \max\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, $S_\sigma(n) = \max\{\sigma(p_1^{\alpha_1}), \dots, \sigma(p_r^{\alpha_r})\}$, etc.

But we can define $S_\varphi^1(n) = \min\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, $S^1(n) = \min\{\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})\}$, etc. For an arithmetical function f one can define

$$\Delta_f(n) = l.c.m.\{f(p_1^{\alpha_1}), \dots, f(p_r^{\alpha_r})\}$$

and

$$\delta_f(n) = g.c.d.\{f(p_1^{\alpha_1}), \dots, f(p_r^{\alpha_r})\}.$$

For the function $\Delta_\varphi(n)$ the following divisibility property is known (see [8], p.140, Problem 6).

If $(a, n) = 1$, then

$$n|[a^{\Delta_\varphi(n)} - 1]. \tag{37}$$

These functions and many related others may be studied in the near (or further) future.

References

[1] T.M. Apostol, *An introduction to analytic number theory*, Springer Verlag, 1976.

[2] M. Deaconescu and J. Sándor, *Variations on a theme by Hurwitz*, Gazeta Mat. (Perf. Met.) A, **8**(1987), No.4, 186-191.

[3] P. Erdös, *Quelques problèmes de théorie des nombres*, L'Enseignement Math. 1963, pp. 81-135.

[4] R.K. Guy, *Unsolved problems in number theory*, Springer Verlag, Second ed. 1994.

[5] M. Le, *An inequality concerning the Smarandache function*, Smarandache Notions J. **9**(1998), No.1-2, 124-125.

[6] J. Sándor, *On certain new inequalities and limits for the Smarandache function*, Smarandache Notion J. **9**(1998), 63-69.

[7] J. Sándor, *On values of arithmetical functions at factorials*, I (submitted).

[8] H.N. Shapiro, *Introduction to the theory of numbers*, Wiley, 1983.

[9] F. Smarandache, *A function in the number theory*, An. Univ. Timişoara, Ser. Mat., vol. **38**(1980), 79-88.

[10] *Smarandache Notion Journal* (collection).

This paper has appeared in *Notes Num. Theory Discr. Math.* (Bulgaria), 5(1999), 2, 41-51.

On the numerical function $S_{\min}^{-1}$

Vasile Seleacu
Department of Mathematics, University of Craiova
13, Al. I. Cuza st., Craiova 1100, Romania

In [1] on defines $S_{\min}^{-1} : \mathbf{N} \setminus \{1\} \mapsto \mathbf{N}$, $S_{\min}^{-1}(x) = \min\{S^{-1}(x)\}$, where $S^{-1}(x) = \{a \in \mathbf{N} \mid S(a) = x\}$, and S is the Smarandache function.
For example $S^{-1}(6) = \{2^4, 2^4 \cdot 3, 2^4 \cdot 3^2, 3^2, 3^2 \cdot 2, 3^2 \cdot 2^2, 3^2 \cdot 2^3, 2^4 \cdot 3 \cdot 5, 2^3 \cdot 3^2 \cdot 5, 2^4 \cdot 3^2 \cdot 5, 3^2 \cdot 5, 2^4 \cdot 5, 3^2 \cdot 5, 3^2 \cdot 2^4\}$ and $S_{\min}^{-1}(6) = 3^2$.
If $S(x) = n$ one knows that $card\,(S^{-1}(n)) = d(n!) - d((n-1)!)$ where d is the number of divisors of n.
If x is a prime number, then $card\,(S^{-1}(n)) = d((n-1)!)$.
We give below a table of the values of $S_{\min}^{-1}(n)$:

n	2	3	4	5	6	7	8	12	15
$S_{\min}^{-1}(n)$	2	3	4	5	3^2	7	2^5	3^5	5^3
n	16	21	24	27	36	40	52	56	60
$S_{\min}^{-1}(n)$	2^{12}	7^3	3^{10}	3^{11}	3^{16}	5^9	13^4	7^8	5^4

One knows [2] that if $p < q$ are two prime numbers, and $n > 1$ is a natural number such that $p \cdot q \mid n$, then $p^{l_p(n)} > q^{l_q(n)}$, where $l_p(n)$ is the exponent of p in the prime factors decomposition of $n!$.
According to the above properties we can deduce the calculus formula for function $S_{\min}^{-1}$:

$$S_{\min}^{-1}(n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = p_r^{l_{p_r}(n) - \alpha_r + 1} \tag{1}$$

where $p_1 < p_2 < \cdots < p_r$ are the prime numbers in the canonical decomposition of the number n.
We list a set of properties of the function $S_{\min}^{-1}$, which result directly from the definition and from formula (1):

1. $S_{\min}^{-1}(p) = p$ if p is a prime number.

2. $S^{-1}_{\min}(p \cdot q) = q^p$ if p and q are prime numbers and $p < q$.

3. $S\left(S^{-1}_{\min}(x)\right) = x$.

4. $S^{-1}_{\min}(q^p) = p \cdot q$ if p and q are prime numbers and $p < q$.

5. $S^{-1}_{\min}(x) < S^{-1}_{\min}(y)$ if x and y contain as the greatest prime factor p_r and $x < y$.

6. The equation $S^{-1}_{\min}(x) = S^{-1}_{\min}(x+1)$ has not solutions.

7. $S^{-1}_{\min}(S(x))$ is generally not equal to $S(x)$.

8. $\Lambda\left(S^{-1}_{\min}(x)\right) = \log p_r$, where Λ is the Mangoltd function.

It is open the problem to find other properties of the function $S^{-1}_{\min}$.

References

[1] Smarandache Function Journal, vol. 1, 1993.

[2] Florian Luca, *An inequality between prime powers dividing n!* (To appear in Smarandache Function Journal, vol. 9, 1998.)

[3] Ion Bălăcenoiu, *Remarkable Inequalities*, Proceedings of the first International Conference on Smarandache Type Notions in Number Theory, Craiova, Romania, 1997.

On Numbers Where the Values of the Pseudo-Smarandache Function Of It and The Reversal Are Identical

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
Hiawatha, IA 52233
e-mail 71603.522@compuserve.com

The Pseudo-Smarandache function was introduced by Kenichiro Kashihara in a book that is highly recommended[1].

Definition: For any n $\geq$ 1, the value of the Pseudo-Smarandache function is the smallest integer m such that n evenly divides

$$\sum_{k=1}^{m} k.$$

Definition: Let d = $a_1 a_2 \ldots a_k$ be a decimal integer. The **reversal** of d, Rev(d) is the number obtained by reversing the order of the digits of d.

Rev(d) = $a_k a_{k-1} \ldots a_2 a_1$.

If d contains trailing zeros, they are dropped when they become leading zeros.

In this paper, we will look for numbers n, such that Z(n) = Z(Rev(n)) and note some of the interesting properties of the solutions. If n is palindromic, then the above property is true by default. Therefore, we will restrict our set of interest to all non-palindromic numbers n such that Z(n) = Z(Rev(n)).

A computer program was written to search for all such n for 1 $\leq$ n $\leq$ 100,000 and the solutions are summarized below.

Z(180) = 80 = Z(81)
Z(990) = 44 = Z(99)
Z(1010) = 100 = Z(101)
Z(1089)=242 = Z(9801)
Z(1210) = 120 = Z(121)
Z(1313) = 403 = Z(3131)
Z(1572) = 392 = Z(2751)
Z(1810) = 180 = Z(181)
Z(1818) = 404 = Z(8181)
Z(2120) = 159 = Z(212)

$Z(2178) = 1088 = Z(8712)$
$Z(2420) = 120 = Z(242)$
$Z(2626) = 403 = Z(6262)$
$Z(2720) = 255 = Z(272)$
$Z(2997) = 1295 = Z(7992)$
$Z(3630) = 120 = Z(363)$
$Z(3636) = 504 = Z(6363)$
$Z(4240) = 159 = Z(424)$
$Z(4284) = 1071 = Z(4842)$
$Z(4545) = 404 = Z(5454)$
$Z(4640) = 319 = Z(464)$
$Z(5050) = 100 = Z(505)$
$Z(6360) = 159 = Z(636)$
$Z(7170) = 239 = Z(717)$
$Z(8780) = 439 = Z(878)$
$Z(9090) = 404 = Z(909)$
$Z(9490) = 364 = Z(949)$
$Z(9890) = 344 = Z(989)$
$Z(13332) = 1616 = Z(23331)$
$Z(15015) = 714 = Z(51051)$
$Z(16610) = 604 = Z(1661)$
$Z(21296) = 6655 = Z(69212)$
$Z(25520) = 319 = Z(2552)$
$Z(26664) = 1616 = Z(46662)$
$Z(27027) = 2079 = Z(72072)$
$Z(29970) = 1295 = Z(7992)$
$Z(32230) = 879 = Z(3223)$
$Z(37730) = 1715 = Z(3773)$
$Z(39960) = 1295 = Z(6993)$
$Z(45045) = 2079 = Z(54054)$
$Z(46662) = 1616 = Z(26664)$
$Z(49940) = 1815 = Z(4994)$
$Z(56650) = 824 = Z(5665)$
$Z(57057) = 2925 = Z(75075)$
$Z(63630) = 504 = Z(3636)$
$Z(64460) = 879 = Z(6446)$
$Z(80080) = 2079 = Z(8008)$
$Z(80640) = 4095 = Z(4608)$
$Z(81810) = 404 = Z(1818)$
$Z(92290) = 3355 = Z(9229)$
$Z(93390) = 1980 = Z(9339)$
$Z(96690) = 879 = Z(9669)$
$Z(97790) = 2540 = Z(9779)$

Several items to note from the previous list.

a) Of the 52 solutions discovered, 35 of the numbers have one trailing zero, where many of them are palindromes when the zeros are dropped. While no numbers with two trailing zeros were found, it seems likely that there are such numbers.

Unsolved Question: Given that Z(n) = Z(Rev(n)), what is the largest number of trailing zeros that n can have?

The previous question is directly related to the speed with which the Pseudo-Smarandache function grows.

d) Of the 17 remaining numbers, 9 exhibit the pattern $d_1d_2d_1d_2$ or $d_1d_20d_1d_2$.

Unsolved Question: Is this a pattern, in the sense that there is an infinite set of numbers n, such that $n = d_1d_20 \ldots 0d_1d_2$ and Z(n) = Z(Rev(n))?

e) Only three of the numbers contain unique nonzero digits and there are none with five digits.

Unsolved Question: What is the largest number of unique nonzero digits that a number n can have when Z(n) = Z(Rev(n)?

f) Three of the numbers exhibit the pattern $d_1d_2 \ldots d_2d_3$, with the largest interior pattern being three digits in length.

Unsolved Question: What is the largest interior pattern of repeating digits $d_2 \ldots d_2$ that can appear in a number $n = d_1d_2 \ldots d_2d_3$ such that Z(n) = Z(Rev(n))?

Reference

1. K. Kashihara, **Comments and Topics on Smarandache Notions and Problems**, Erhus University Press, Vail, AZ. 1996.

SMARANDACHE RECIPROCAL PARTITION OF UNITY SETS AND SEQUENCES

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRACT: Expression of unity as the sum of the reciprocals of natural numbers is explored . And in this connection Smarandache Reciprocal partition of unity sets and sequences are defined. Some results and Inequalities are derived and a few open problems are proposed.

DISCUSSION:

Define Smarandache Repeatable Reciprocal partition of unity set as follows:

$$SRRPS(n) = \{ x \mid x = (a_1, a_2, \ldots , a_n) \text{ where } \sum_{r=1}^{n} (1/a_r) = 1.\}$$

$f_{RP}(n)$ = order of the set SRRPS(n).

We have

SRRPS(1) = { (1) } , $f_{RP}(1) = 1$.

SRRPS(2) = { (2,2) } , $f_{RP}(2) = 1$.

SRRPS(3) = { (3,3,3),(2,3,6), (2,4,4) } , $f_{RP}(3) = 3.$,$1 = 1/2 + 1/3 + 1/6$ etc.

SRRPS(4) = { (4,4,4,4), (2,4,6,12), (2,3,7,42), (2,4,5,20), (2,6,6,6),(2,4,8,8,),(2,3,12,12), (4,4,3,6) , (3,3,6,6), (2,3,10,15) }

$f_{RP}(4) = 10$.

SMARANDACHE REPEATABLE RECIPROCAL PARTITION OF UNITY SEQUENCE is defined as

1 , 1 , 3 , 10 . . .

where the n^{th} term = $f_{RP}(n)$.

Define **SMARANDACHE DISTINCT RECIPROCAL PARTITION OF UNITY SET**

as follows

SDRPS(n) = { x | x = (a_1, a_2, . . ., a_n) where $\sum_{r=1}^{n} (1/a_r) = 1$ and $a_i = a_j \Leftrightarrow i = j$}

$f_{DP}(n)$ = order of SDRPS(n).

SDRPS(1) = { (1) } , $f_{DP}(1) = 1$.

SDRPS(2) = { } , $f_{RP}(2) = 0$.

SDRPS(3) = { (2,3,6) } , $f_{DP}(3) = 1$.

SRRPS(4) = {(2,4,6,12), (2,3,7,42), (2,4,5,20),(2,3,10,15)}

$f_{DP}(4) = 4$.

Smarandache Distinct Reciprocal partition of unity sequences

defined as follows
1 , 0 , 1 , 4 , 12 . . .

the n^{th} term is $f_{DP}(n)$.

Following Inequality regarding the function $f_{DP}(n)$ has been

established.

THEOREM(1.1)

$$f_{DP}(n) \geq \sum_{k=3}^{n-1} f_{DP}(k) + (n^2 - 5n + 8)/2 \ , n > 3 \qquad \text{----------(1.1)}$$

This inequality will be established in two steps.

Proposition (1.A)

For every n there exists a set of n natural numbers sum of whose reciprocals is 1.

Proof: This will be proved by induction. Let the proposition be true for $n = r$.

Let $a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n = k$ be r distinct natural numbers such that

$1/a_1 + 1/a_2 + 1/a_3 + \ldots + 1/a_r = 1$

We have , $1/k = 1/(k+1) + 1/(k(k+1))$, which gives us a set of r+1 distinct numbers $a_1 < a_2 < a_3 < \ldots < a_{r-1} < k+1 < k(k+1)$, sum of whose reciprocals is 1.

$P(r) \Rightarrow P(r+1)$, and as $P(3)$ is true i.e. $1/2 + 1/3 + 1/6 = 1$,

The proposition is true for all n.

This completes the proof of proposition (1.A).

Note: If $a_1, a_2, a_3, \ldots a_{n-1}$ are n-1 distinct natural numbers given by

$a_1 = 2.$
$a_2 = a_1 + 1.$
$a_3 = a_1 a_2 + 1$
.
.
.
$a_t = a_1 a_2 a_3 \ldots a_{t-1} + 1. \quad = a_{t-1}(a_{t-1} - 1) + 1$
.
.
.
$a_{n-2} = a_1 a_2 a_3 \ldots a_{n-3} + 1$
$a_{n-1} = a_1 a_2 a_3 \ldots a_{n-2}$

then these numbers form a set of (n - 1) distinct natural numbers such that

$$\sum_{t=1}^{n-1} 1/a_t = 1.$$

we have $a_t = a_{t-1}(a_{t-1} - 1) + 1$ except when $t = n - 1$ in which case

$a_{n-1} = a_{n-2}(a_{n-2} - 1)$

Let the above set be called **Principle Reciprocal Partition**.

*** It can easily be proved in the above set that

$a_{2t} \equiv 3 \bmod(10)$ and $a_{2t+1} \equiv 7 \bmod (10)$ for $t \geq 1$.

Consider the principle reciprocal partition for n-1 numbers . Each a_t contributes one to $f_{DP}(n)$ if broken into $a_t + 1$, $a_t(a_t + 1)$ except for $t = 1$. (as 2, if broken into 3 and 6, to give 1/2 =1/3 + 1/6, the number 3 is repeated and the condition of all distinct number is not fulfilled). There is a contribution of n - 2 from the principle set to $f_{DP}(n)$. The remaining $f_{DP}(n-1) -1$ members (excluding the principle partition) of SDRPS(n-1) would contribute at least one each to $f_{DP}(n)$ (breaking the largest number in each such set into two parts) . The contribution to $f_{DP}(n)$ thus is at least

$n-2 + f_{DP}(n-1) - 1 = f_{DP}(n-1) + n - 3$

$$\mathbf{f_{DP}(n) \geq f_{DP}(n-1) + n - 3 \quad \text{----------}(1.2)}$$

Also for each member $(b_1, b_2, \ldots, b_{n-1})$ of SDRPS(n-1) there exists a member of SDRPS(n) i.e. $(2, 2b_1, 2b_2, \ldots, 2b_{n-1})$ as we can see that

$1 = (1/2)(1 + 1/b_1 + 1/b_2 + \ldots + 1/b_{n-1}) = 1/2 + 1/2b_1 + \ldots + 1/2b_{n-1}$.

In this way there is a contribution of $f_{DP}(n-1)$ to $f_{DP}(n)$. -------(1.3)

Taking into account all these contributions to $f_{DP}(n)$ we get

$f_{DP}(n) \geq f_{DP}(n-1) + n - 3 + f_{DP}(n-1)$

$f_{DP}(n) \geq 2f_{DP}(n-1) + n - 3$

$f_{DP}(n) - f_{DP}(n-1) \geq f_{DP}(n-1) + n - 3$ -------------(1.4)

from (4) by replacing n by n-1 , n-2 , etc. we get

$f_{DP}(n-1) - f_{DP}(n-2) \geq f_{DP}(n-2) + n - 4$

$f_{DP}(n-2) - f_{DP}(n-3) \geq f_{DP}(n-3) + n - 5$

.

.

.

$f_{DP}(4) - f_{DP}(3) \geq f_{DP}(3) + 1$

summing up all the above inequalities we get

$$f_{DP}(n) - f_{DP}(3) \geq \sum_{k=3}^{n-1} f_{DP}(k) + \sum_{r=1}^{n-1} r$$

$$f_{DP}(n) \geq \sum_{k=3}^{n-1} f_{DP}(k) + (n-3)(n-2)/2 + 1$$

$$f_{DP}(n) \geq \sum_{k=3}^{n-1} f_{DP}(k) + (n^2 - 5n + 8)/2 \quad , \quad n > 3$$

Remarks : Readers can come up with stronger results as in my opinion the order of $f_{DP}(n)$ should be much more than what has been arrived at. This will be clear from the following theorem.

THEOREM(1.2):

If m is a member of an element of SRRPS(n) say,

$(a_1, a_2, a_3, \ldots, a_n)$.We have $a_k = m$ for some k and $\sum_{k=1}^{n} 1/a_k = 1$.

then m contributes $[\{d(m) + 1\}/2]$ elements to SRRPS(n+1), where the symbol [ ] stands for integer value and d(m) is the number of divisors of m.

Proof: For each divisor d of m there corresponds another divisor $m/d = d'$.

Case-I: m is not a perfect square . Then d(m) is even and there are d(m)/2 pairs of the type (d,d') such that dd' = m .

Consider the following identity

$1/(p.q) = 1/(p(p+q)) + 1/(q(p+q))$ ------------(1.5)

for each divisor pair (d,d') of m we have the following breakup

$1/(d.d') = 1/(d(d+d')) + 1/(d'(d+d'))$

Hence the contribution of m to SRRPS(n+1) is d(m)/2. As d(m) is even $d(m)/2 = [\{d(m)+1\}/2]$ Also.

Case-II m is a perfect square. In this case d(m) is odd and there is a divisor pair $d=d' = m^{1/2}$. This will contribute one to SRRPS(n+1) .The remaining {d(m) -1}/2 pairs of distinct divisors will contribute as many i.e. say ({d(m) -1}/2) .Hence the total contribution in this case would be

$\{d(m) -1\}/2 + 1 = \{d(m) +1\}/2 =[\{d(m) +1\}/2]$

Hence m contributes [{d(m) + 1}/2] elements to SRRPS(n+1)

This completes the proof.

Remarks:(1) The total contribution to SRRPS(n+1) by any element of SRRPS(n) is $\sum [\{d(a_k) + 1\}/2]$ ---------(1.6),

where each a_k is considered only once irrespective of its' repeated occurrence.

(2) In case of SDRPS(n+1) , the contribution by an element of SDRPS(n) is given by

$$\sum_{k=1}^{n} [\{d(a_k)\}/2] \qquad \text{-----------(1.7)}$$

because the divisor pair $d =d'= a_k^{\ 1/2}$ does not contribute.

Hence the total contribution of SDRP(n) to generate SDRPS(n+1) is the summation over all the elements of SDRPS(n) .

$$\sum_{f_{DP}(n)} \{ \sum_{k=1}^{n} [\{d(a_k)\}/2] \} \qquad \text{----------- (1.8)}$$

Generalizing the above approach.

The readers can further extend this work by considering the following identity

$$\frac{1}{pqr} = \frac{1}{pq(p+q+r)} + \frac{1}{qr(p+q+r)} + \frac{1}{rp(p+q+r)} \quad \text{----(1.9)}$$

which also suggests

$$\frac{1}{b_1 b_2 \ldots b_r} = \sum_{k=1}^{r} \{ (\prod_{t=1, t\neq k}^{r} b_t) (\sum_{s=1}^{r} b_s) \}^{-1} \quad \text{-------(1.10)}$$

The above identity can easily be established by just summing up the right hand member.

From (1.10) , the contribution of the elements of SDRPS(n) to SDRPS(n+r) can be evaluated if an answer to following tedious querries could be found.

OPEN PROBLEMS:

(1) In how many ways a number can be expressed as the product of 3 of its divisors?

(2) In general in how many ways a number can be expressed as the product of r of its' divisors?

(3) Finally in how many ways a number can be expressed as the product of its divisors?

An attempt to get the answers to the above querries leads to the need of the generalization of the theory of partition function.

REFERENCES:

[1] 'Smarandache Notion Journal' Vol. 10 ,No. 1-2-3, Spring 1999. Number Theory Association of the UNIVERSITY OF CRAIOVA.

[2] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

GENERALIZATION OF PARTITION FUNCTION, INTRODUCING SMARANDACHE FACTOR PARTITION

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

Partition function P(n) is defined as the number of ways that a positive integer can be expressed as the sum of positive integers. Two partitions are not considered to be different if they differ only in the order of their summands. A number of results concerning the partition function were discovered using analytic functions by Euler, Jacobi, Hardy , Ramanujan and others. Also a number of congruence properties of the function were derived. In the paper Ref.[1]

"SMARANDACHE RECIPROCAL PARTITION OF UNITY SETS AND SEQUENCES"

while dealing with the idea of Smarandache Reciprocal Partitions of unity we are confronted with the problem as to in how many ways a number can be expressed as the product of its divisors. Exploring this lead to the generalization of the theory of partitions.

DISCUSSION:

Definition : SMARANDACHE FACTOR PARTITION FUNCTION:

Let α_1 , α_2 , α_3 , . . . α_r be a set of r natural numbers and p_1 , p_2, p_3 ,. . .p_r be arbitrarily chosen distinct primes then

$F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha 1}\ p_2^{\alpha 2}\ p_3^{\alpha 3} \ldots p_r^{\alpha r}$$ could be expressed as the product of its' divisors.

Example: $F(1,2) = ?$,

Let $p_1 = 2$ and $p_2 = 3$, $N = p_1 . p_2^2 = 2.3^2 = 18$

N can be expressed as the product of its divisors in following 4 ways:

(**1**) N = 18, (**2**) N = 9 X 2

(**3**) N = 6 X 3 (**4**) N = 3 X 3 X 2. As per our definition F (1,2) = 4.

It is evident from the definition that $F(\alpha_1, \alpha_2) = F(\alpha_2, \alpha_1)$ or in general the order of α_i in $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_i \ldots \alpha_r)$ is immaterial. Also the primes $p_1, p_2, p_3, \ldots p_r$ are dummies and can be chosen arbitrarily.

We start with some elementry results to buildup the concept.

THEOREM(2.1) : F (α) = P(α)

where $P(\alpha)$ is the number of partitions of α.

PROOF: Let p be any prime and $N = p^{\alpha}$.

Let $\alpha = x_1 + x_2 + \ldots + x_m$ be a partition of α.

Then $N = (p^{x1})(p^{x2})(p^{x3})\ldots(p^{xn})$ is a **SFP** of N .i.e. each partition of α contributes one **SFP.** ---------(2.1)

Also let one of the **SFP** of N be

$N = (N_1).(N_2.)(N_3)\ldots(N_k)$. Each N_i has to be such that $N_i = p^{ai}$

Let $N_1 = p^{a1}$, $N_2 = p^{a2}$, etc. $N_k = p^{ak}$ then

$$N = (p^{a1})(p^{a2})\ldots(p^{ak})$$

$$N = p^{(a1+a2+a3+\ldots+an)}$$

$$\Rightarrow \quad \alpha = a_1 + a_2 + \ldots + a_k$$

which gives a partition of α. Obviously each **SFP** of N gives one unique partition of α . --------(2.2) .

from (2.1) and (2.2) we get

$$\mathbf{F(\alpha) = P(\alpha)}$$

THEOREM (2.2) $\quad \mathbf{F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k)}$

PROOF: Let $N = p_1^{\alpha} p_2$, where p_1, p_2 are arbitrarily chosen primes.

Case(1) Writing $N = (p_2)\, p_1^{\alpha}$ keeping p_2 as a separate entity (one of the factors in the factor partition of N) ,would yield $P(\alpha)$ Smarandache factor partitions .(from theorem (2.1)) .

Case(2) Writing $N = (p_1.p_2).\, p_1^{\alpha-1}$ keeping (p_1p2) as a separate entity (one of the factors in the SFP of N) ,would yield $P(\alpha-1)$ SFPs.

.
.
.

Case (r) In general writing $N = (p_1^r . p_2). p_1^{\alpha-r}$ and keeping $(p_1^r . p_2)$ as a separate entity would yield $P(\alpha-r)$ SFPs.

Contributions towards F(N) in each case (1), (2), . . . (r) are mutually disjoint as $p_1^r . p_2$ is unique for a given r . which ranges from zero to α. These are exhaustive also.

Hence

$$F(\alpha, 1) = \sum_{r=0}^{\alpha} P(\alpha-r)$$

Let $\alpha - r = k$ $\quad r = 0 \Rightarrow k = \alpha$

$\qquad r = \alpha \Rightarrow k = 0$

$$F(\alpha, 1) = \sum_{k=\alpha}^{0} P(k)$$

$$F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k)$$

This completes the proof of the theorem (2.2)

Some examples:

(1) F(3) = P(3) = 3 , Let p = 2 , $N = 2^3 = 8$

(I) N = 8 , (2) N = 4 X 2 , (3) N = 2 X 2 X 2 .

(2)

$$F(4, 1) = \sum_{k=0}^{4} P(k) = P(0) + P(1) + P(2) + P(3) + P(4)$$

$$= 1 + 1 + 2 + 3 + 5 = 12$$

Let $N = 2^4 \times 3 = 48$ here $p_1 = 2$, $p_2 = 3$.

The Smarandache factor partitions of 48 are

(1) N = 48
(2) N = 24 X 2
(3) N = 16 X 3
(4) N = 12 X 4
(5) N = 12 X 2 X 2
(6) N = 8 X 6
(7) N = 8 X 3 X 2
(8) N = 6 X 4 X 2
(9) N = 6 X 2 X 2 X 2
(10) N = 4 X 4 X 3
(11) N = 4 X 3 X 2 X 2
(12) N = 3 X 2 X 2 X 2 X 2

DEFINITIONS:
In what follows in the coming pages let us denote (for simplicity)

(1) $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N)$

where

$$N = p_1^{\alpha_1} \; p_2^{\alpha_2} \; p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 = 2$, $p_2 = 3$ etc.

(2) Also for the case (N is a square free number)

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1$$

Let us denote

$$\underbrace{F(1, 1, 1, 1, 1 \ldots)}_{\leftarrow \text{ n - ones } \rightarrow} = F(1\#n)$$

Examples: $F(1\#2) = F(1, 1) = F'(6) = 2$, $6 = 2 \times 3 = p_1 \times p_2$.

$F(1\#3) = F(1, 1, 1) = F'(2 \times 3 \times 5) = F'(30) = 5$.

(3) **Smarandache Star Function**

$$F^{'*}(N) = \sum_{d/N} F'(d_r) \qquad \text{where } d_r | N$$

$F^{'*}(N)$ = sum of $F'(d_r)$ over all the divisors of N.

e.g. N = 12 , divisors are 1, 2, 3, 4, 6, 12

$F^{'*}(12) = F'(1) + F'(2) + F'(3) + F'(4) + F'(6) + F'(12)$

$= 1 + 1 + 1 + 2 + 2 + 4 = 11$

THEOREM (2.3)

$$F^{'*}(N) = F'(Np) \quad , \quad (p,N) = 1, \text{ p is a prime.}$$

PROOF: We have by definition

$$F^{'*}(N) = \sum_{d/N} F'(d_r) \qquad \text{where } d_r | N$$

consider d_r a divisor of N . $Np = d_r\,(Np/d_r)$

let $(Np/d_r) = g(d_r)$, then $N = d_r \cdot g(d_r)$

for any divisor d_r of N , $g(d_r)$ is unique

$d_i = d_j \Leftrightarrow g(d_i) = g(d_j)$

Considering $g(d_r)$ as a single term (an entity , not further split into factors) in the SFP of **N.p** one gets $F'(d_r)$ SFPs .

Each $g(d_r)$ contributes $F'(d_r)$ factor partitions .

The condition p does not divide N , takes care that $g(d_i) \neq d_j$ for any divisor. because p divides $g(d_i)$ and p does not divide d_j.

This ensures that contribution towards $F'(Np)$ from each $g(d_r)$ is distinct and there is no repetition. Summing over all $g(d_r)$'s we get

$$F'(Np) = \sum_{d/N} F'(d_r)$$

or

$$F'^{*}(N) = F'(Np)$$

This completes the proof of the theorem (3) .

An application of theorem (2.3)

Theorem (2.2) follows from theorem (2.3)
To prove

$$\mathbf{F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k)}$$

Let $N = p^{\alpha}\ p_1$ then $F(\alpha,1) = F'(p^{\alpha} . p_1)$

from theorem (2.3)

$$F'(p^{\alpha} . p_1) = F'^{*}(p^{\alpha}) = \sum_{d/p^{\alpha}} F'(d_r)$$

The divisor of p^{α} are $p^0, p^1, p^2, \ldots p^{\alpha}$

hence

$$F'(p^{\alpha} . p_1) = F'(p^0) + F'(p^1) + \ldots + F'(p^{\alpha})$$

$$= P(0) + P(1) + P(2) + \ldots + P(\alpha - 1) + P(\alpha)$$

or

$$F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k)$$

THEOREM (2.4):

$$F(1\#(n+1)) = \sum_{r=0}^{n} {}^{n}C_r\, F(1\#r)$$

PROOF: From theorem (2.3) we have $F'(Np) = F'^{*}(N)$, p does not divide N. Consider the case $N = p_1 p_2 p_3 \ldots p_n$. We have , $F'(N)$

= F (1#n) and F'(Np) = F (1#(n+1)) as p does not divide N.

Finally we get

$$F\,(1\#(n+1)) = F'^{*}\,(N) \qquad \text{--------}(2.3)$$

The number of divisors of N of the type $p_1p_2p_3 \ldots p_r$. (containing exactly r primes is ${}^nC_r$. Each of the ${}^nC_r$ divisors of the type $p_1p_2p_3 \ldots p_r$ has the same number of SFPs F(1#r). Hence

$$F'^{*}\,(N) = \sum_{r=0}^{n} {}^nC_r\, F(1\#r) \qquad \text{-------------}(2.4)$$

From (2.3) and (2.4) we get

$$F\,(1\#\,(n+1)) = \sum_{r=0}^{n} {}^nC_r\, F(1\#r)$$

NOTE: It is to be noted that F(1#n) is the n^{th} Bell number.

Example: F(1#0) =F'(1) = 1.

F(1#1) = F'(p_1) = 1.

F(1#2) = F'(p_1 p_2) = 2.

F(1#2) = F'(p_1 p_2 p_3) = 5.

(i) $p_1\, p_2\, p_3$
(ii) $(p_1\, p_2) \times p_3$
(iii) $(p_1 p_3) \times p_2$
(iv) $(p_2 p_3) \times p_1$
(v) $p_1 \times p_2 \times p_3$

Let Theorem (4) be applied to obtain F (1#4)

$$F(1\#4) = \sum^{3} {}^nC_r\, F(1\#r)$$

$r=0$

$F(1\#4) = {}^3C_0\, F(1\#0) + {}^3C_1 F(1\#1) + {}^3C_2\, F(1\#2) + {}^3C_3\, F(1\#3)$

$= 1 \times 1 + 3 \times 1 + 3 \times 2 + 1 \times 5 = 15$

$F(1\#4) = F'(2 \times 3 \times 5 \times 7) = F'(210) = 15.$

(I) 210
(ii) 105 X 2
(iii) 70 X 3
(iv) 42 X 5
(v) 35 X 6
(vi) 35 X 3 X 2
(vii) 30 X 7
(viii) 21 X 10
(ix) 21 X 5 X 2
(x) 15 X 14
(xi) 15 X 7 X 2
(xii) 14 X 5 X 3
(xiii) 10 X 7 X 3
(ixv) 7 X 6 X 5
(xv) 7 X 5 X 3 X2

On similar lines one can obtain

$F(1\#5) = 52$, $F(1\#6) = 203$, $F(1\#7) = 877$, $F(1\#8) = 4140$.

$F(1\#9) = 21{,}147$.

DEFINITION:

$$\mathbf{F'^{**}(N) = \sum_{d_r/N} F'^{*}(d_r)}$$

d_r ranges over all the divisors of N.

If N is a square free number with n prime factors , let us denote

$$F'^{**}(N) = F^{**}(1\#n)$$

Example:

$$F'^{**}(p_1p_2p_3) = F^{**}(1\#3) = \sum_{d_r/N} F'(d_r)$$

$$= {}^3C_0\, F'^{*}(1) + {}^3C_1 F'^{*}(p_1) + {}^3C_2\, F'^{*}(p_1p_2) + {}^3C_3\, F'^{*}\ (p_1p_2p_3)$$

$$F'^{*}(1\#3) = 1 + [\ 3F'(1) + F'(p_1)] + 3[\ F'(1) + 2F'(p_1) + F'(p_1p_2)]$$

$$+ [\ F'(1) + 3F'(p_1) + 3F'(p_1p_2) + F'\ (p_1p_2p_3)\ ]$$

$$F'^{*}(1\#3) = 1 + 6 + 15 + 15 = 37.$$

An interesting observation is

(1) $F^{**}(1\#0) + F(1\#1) = F(1\#2)$

or

$F^{**}(1\#0) + F^{*}(1\#0) = F(1\#2)$

(2) $F^{**}(1\#1) + F(1\#2) = F(1\#3)$

or

$F^{**}(1\#1) + F^{*}(1\#1) = F(1\#3)$

(3) $F^{**}(1\#5) + F(1\#6) = F(1\#7)$

or

$F^{**}(1\#5) + F^{*}(1\#5) = F(1\#7)$

which suggests the possibility of

$$\mathbf{F^{**}(1\#n) + F^{*}(1\#n) = F(1\#(n+2))}$$

A stronger proposition

$F'(Np_1p_2) = F'^{*}(N) + F'^{**}(N)$

is established in theorem (2.5).

DEFINITION:

$$F'^{n*}(N) = \sum_{d_r/N} F'^{(n-1)*}(d_r) \qquad n > 1$$

where $\mathbf{F'^{*}(N)} = \sum_{d_r/N} \mathbf{F'(d_r)}$

and d_r ranges over all the divisors of N.

THEOREM(2.5) :

$$F'(Np_1p_2) = F'^*(N) + F'^{**}(N)$$

from theorem (3) we have

$F'(Np_1p_2) = F'^*(Np_1)$

Let $d_1, d_2, \ldots, d_n$ be all the divisors of N. The divisors of Np_1 would be

$d_1, \quad d_2, \quad \ldots, \quad d_n$

$d_1p_1, d_2p_1, \ldots, d_np_1$

$$\begin{aligned} F'^*(Np_1) &= [\, F'(d_1) + F'(d_2) + \ldots + F'(d_n)\,] + [\, F'(d_1p_1) + F'(d_2p_1) + \\ &\quad \ldots + F'(d_np_1)] \\ &= F'^*(N) + [\, F'^*(d_1) + F'^*(d_2) + \ldots + F'^*(d_n)\,] \end{aligned}$$

$$\begin{aligned} F'^*(Np_1) &= F'^*(N) + F'^{**}(N) \qquad (\text{ by definition}) \\ &= F'^*(N) + F'^{2*}(N) \end{aligned}$$

This completes the proof of theorem (2.5).

THEOREM(2.6):

$$\mathbf{F'(Np_1p_2p_3) = F'^*(N) + 3F'^{2*}(N) + F'^{3*}(N)}$$

PROOF:

From theorem (2.3) we have

$F'(Np_1p_2p_3) = F'^*(Np_1p_2)$.

Also If $d_1, \quad d_2, \quad \ldots, \quad d_n$ be all the divisors of N. Then the divisors of Np_1p_2 would be

$$d_1, \quad d_2, \quad \ldots, \quad d_n$$

$$d_1p_1 \quad d_2p_1, \quad \ldots, \quad d_np_1$$

$$d_1p_2, \quad d_2p_2, \quad \ldots, \quad d_np_2$$

$$d_1p_1p_2, \quad d_2p_1p_2, \quad \ldots, \quad d_np_1p_2$$

Hence

$$\begin{aligned} F'^*(Np_1p_2) = {} & [F'(d_1) + F'(d_2) + \ldots + F'(d_n)] + \\ & [F'(d_1p_1) + F'(d_2p_1) + \ldots F'(d_np_1)] + \\ & [F'(d_1p_2) + F'(d_2p_2) + \ldots + F'(d_np_2)] + \\ & [F'(d_1p_1p_2) + F'(d_2p_1p_2) + \ldots + F'(d_np_1p_2)] \end{aligned}$$

$$= F'^*(N) + 2[F'^*(d_1) + F'^*(d_2) + \ldots + F'^*(d_n)] + S \qquad \text{----(2.5)}$$

where $S = [F'(d_1p_1p_2) + F'(d_2p_1p_2) + \ldots + F'(d_np_1p_2)$

Now from theorem (2.5) we get,

$$F'(d_1p1p2) = F'^*(d_1) + F'^{**}(d_1) \qquad \text{----(1)}$$

$$F'(d_2p_1p_2) = F'^*(d_2) + F'^{**}(d_2) \qquad \text{----(2)}$$

.

.

.

$$F'(d_np1p2) = F'^*(d_n) + F'^{**}(d_n) \qquad \text{----(n)}$$

on summing up (1), (2) . . . upto (n) we get

$$S = F'^{2*}(N) + F'^{3*}(N) \qquad \text{----(2.6)}$$

substituting the value of S in (A) and also taking

$F'^*(d_1) + F'^*(d_2) + \ldots + F'^*(d_n) = F'^{2*}(N)$

we get.,

$$F'(Np_1p_2p_3) = F'^{*}(N) + 2F'^{2*}(N) + F'^{2*}(N) + F'^{3*}(N)$$

$$F'(Np_1p_2p_3) = F'^{*}(N) + 3F'^{2*}(N) + F'^{3*}(N)$$

This completes the proof of theorem (2.6).The above result which has been observed to follow a beautiful pattern can further be generalized.

REFERENCES:

[1] "Amarnath Murthy" , 'Smarandache Reciprocal Partition Of Unity Sets And Sequences', SNJ, Vol. 11, No. 1-2-3, 2000.

[2] 'Smarandache Notion Journal' Vol. 10 ,No. 1-2-3, Spring 1999. Number Theory Association of the UNIVERSITY OF CRAIOVA.

[3] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

A GENERAL RESULT ON THE SMARANDACHE STAR FUNCTION

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRCT: In this paper ,the result (theorem-2.6) Derived in REF. [2], the paper "Generalization Of Partition Function, Introducing 'Smarandache Factor Partition' which has been observed to follow a beautiful pattern has been generalized.

DEFINITIONS In [2] we define SMARANDACHE FACTOR PARTITION FUNCTION , as follows:

Let α_1 , α_2 , α_3 , . . . α_r be a set of r natural numbers and p_1 , p_2, p_3 ,. . .p_r be arbitrarily chosen distinct primes then $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha 1} \; p_2^{\alpha 2} \; p_3^{\alpha 3} \; \ldots \; p_r^{\alpha r}$$ could be expressed as the product of its' divisors. For simplicity , we denote $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r) = F'(N)$,where

$$N = p_1^{\alpha_1} \; p_2^{\alpha_2} \; p_3^{\alpha_3} \; \ldots \; p_r^{\alpha_r} \; \ldots \; p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 = 2$, $p_2 = 3$ etc.

Also for the case

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1$$

we denote

$$F(\underbrace{1, 1, 1, 1, 1 \ldots}_{\leftarrow \text{ n - ones } \rightarrow}) = F(1\#n)$$

Smarandache Star Function

(1) $F^{'*}(N) = \sum_{d/N} F'(d_r)$ where $d_r | N$

(2) $F'^{**}(N) = \sum_{d_r/N} F'^{*}(d_r)$

d_r ranges over all the divisors of N.

If N is a square free number with n prime factors, let us denote

$$F'^{**}(N) = F^{**}(1\#n)$$

Here we generalise the above idea by the following definition

Smarandache Generalised Star Function

(3) $F'^{n*}(N) = \sum_{d_r/N} F'^{(n-1)*}(d_r)$ $n > 1$

and d_r ranges over all the divisors of N.

For simplicity we denote

$$F'(Np_1p_2 \ldots p_n) = F'(N@1\#n)$$, where

$(N, p_i) = 1$ for $i = 1$ to n and each p_i is a prime.

$F'(N@1\#n)$ is nothing but the Smarandache factor partition of (a number N multiplied by n primes which are coprime to N).

In [3] a proof of the following result is given:

$$F'(Np_1p_2p_3) = F'^{*}(N) + 3F'^{2*}(N) + F'^{3*}(N)$$

The present paper aims at generalising the abve result.

DISCUSSION:

THEOREM(3.1)

$$F'(N@1\#n) = F'(Np_1p_2...p_n) = \sum_{m=0}^{n} [\ a_{(n,m)} F'^{m*}(N)]$$

where

$$a_{(n,m)} = (1/m!) \sum_{k=1}^{m} (-1)^{m-k} . {}^{m}C_k . k^n$$

PROOF:

Let the divisors of N be

$d_1, \quad d_2, \quad ..., \quad d_k$

Consider the divisors of $(Np_1p_2...p_n)$ arranged as follows

$d_1, \quad d_2, \quad ..., \quad d_k$ -------say type (0)

$d_1p_i, \quad d_2p_i, \quad ..., \quad d_kp_i$ -------say type (1)

$d_1p_ip_j, \quad d_2p_ip_j, \quad ..., \quad d_kp_ip_j$ -------say type (2)

$d_1p_ip_j..., d_2p_ip_j..., ..., d_kp_ip_j...$ -------say type (t)

(there are t primes in the term $d_1p_ip_j...$ apart from d_1)

$d_1p_1p_2...p_n, \quad d_2p_1p_2...p_n, \quad d_np_1p_2...p_n,$ -------say type (n)

There are ${}^{n}C_0$ divisors sets of the type (0)

There are ${}^{n}C_1$ divisors sets of the type (1)

There are ${}^{n}C_2$ divisors sets of the type (2) and so on

There are ${}^{n}C_t$ divisors sets of the type (t)

There are ${}^{n}C_n$ divisors sets of the type (n)

Let $Np_1p_2...p_n = M$ .Then

$F^*(M) = {}^{n}C_0$[sum of the factor partitions of all the divisors of row (0)]

+ ${}^{n}C_1$[sum of the factor partitions of all the divisors of row (1)]

+ ${}^{n}C_2$[sum of the factor partitions of all the divisors of row (2)]

+ . . .

+ ${}^{n}C_t$[sum of the factor partitions of all the divisors of row (t)]

+ . . .

+ ${}^{n}C_n$[sum of the factor partitions of all the divisors of row (n)]

Let us consider the contributions of divisor sets one by one.

Row (0) or type (0) contributes

$$F'(d_1) + F'(d_2) + F'(d_3) + ... + F'(d_n) = F'^*(N)$$

Row (1) or type (1) contributes

$$[F'(d_1p_1) + F'(d_2p_1) + ... F'(d_kp_1)]$$

$$= [F'^*(d_1) + F'^*(d_2) + ... + F'^*(d_k)]$$

$$= F'^{2*}(N)$$

Row (2) or type (2) contributes

$$[F'(d_1p_1p_2) + F'(d_2p_1p_2) + ... + F'(d_kp_1p_2)$$

Applying theorem (5) on each of the terms

$$F'(d_1p1p2) = F'^*(d_1) + F'^{**}(d_1) \quad \text{----(1)}$$

$$F'(d_2p_1p_2) = F'^*(d_2) + F'^{**}(d_2) \quad \text{----(2)}$$

.

.

.

$$F'(d_kp1p2) = F'^*(d_k) + F'^{**}(d_k) \quad \text{----(k)}$$

on summing up (1) , (2) ... upto (n) we get

$$F'^{2*}(N) + F'^{3*}(N)$$

At this stage let us denote the coefficients as $a_{(n,r)}$ etc. say

$F'(N@1\#r) = a_{(r,1)}F'^{*}(N) + a_{(r,2)}F'^{2*}(N) + \ldots + a_{(r,t)}F'^{t*}(N) + \ldots +$

$a_{(r,r)}F'^{r*}(N)$

Consider row (t) , one divisor set is

$d_1p_1p_2\ldots p_t$, $d_2p_1p_2\ldots p_t$, . . . $d_kp_1p_2\ldots p_t$,

and we have

$F'(d_1@1\#t) = a_{(t,1)}F'^{*}(d_1) + a_{(t,2)}F'^{2*}(d_1) + \ldots + a_{(t,t)}F'^{t*}(d_1)$

$F'(d_2@1\#t) = a_{(t,1)}F'^{*}(d_2) + a_{(t,2)}F'^{2*}(d_2) + \ldots + a_{(t,t)}F'^{t*}(d_2)$

.

.

.

$F'(d_k@1\#t) = a_{(t,1)}F'^{*}(d_k) + a_{(t,2)}F'^{2*}(d_k) + \ldots + a_{(t,t)}F'^{t*}(d_k)$

summing up both the sides columnwise we get for row (t) or divisors of type (t) one of the ${}^{n}C_t$ divisor sets contributes

$a_{(t,1)}F'^{2*}(N) + a_{(t,2)}F'^{3*}(N) + \ldots + a_{(t,t)}F'^{(t+1)*}(N)$

similarly for row (n) we get

$a_{(n,1)}F'^{2*}(N) + a_{(n,2)}F'^{3*}(N) + \ldots + a_{(n,n)}F'^{(n+1)*}(N)$

All the divisor sets of type (0) contribute

${}^{n}C_0\ a_{(0,0)}F'^{*}(N)$ factor partitions.

All the divisor sets of type (1) contribute

${}^{n}C_1\ a_{(1,1)}F'^{2*}(N)$ factor partitions.

All the divisor sets of type (2) contribute

${}^{n}C_2\ \{a_{(2,1)}F'^{2*}(N) + a_{(2,2)}F'^{3*}(N)\}$ factor partitions.

All the divisor sets of type (3) contribute

${}^{n}C_3\{a_{(3,1)}F'^{2*}(N) + a_{(3,2)}F'^{3*}(N) + a_{(3,3)}F'^{4*}(N\ \}$ factor partitions.

All the divisor sets of row (t) or type (t) contribute

$^nC_t \{ a_{(t,1)}F'^{2}*(N) + a_{(t,2)}F'^{3}*(N) +\ldots+ a_{(t,t)}F'^{(t+1)}*(N)\}$

.

.

.

All the divisor sets of row (n) or type (n) contribute

$^nC_n\{a_{(n,1)}F'^{2}*(N) + a_{(n,2)}F'^{3}*(N) +\ldots+ a_{(n,n)}F'^{(n+1)}*(N)\}$

Summing up the contributions from the divisor sets of all the types and considering the coefficient of $F'^{m}*(N)$ for $m = 1$ to $(n+1)$ we get, coefficient of $F'*(N) = a_{(0,0)} = 1 = \mathbf{a_{(n+1,1)}}$

coefficient of $F'^{2}*(N)$

$= {}^nC_1\, a_{(1,1)} + {}^nC_2\, a_{(2,1)} + {}^nC_3\, a_{(3,1)} + \ldots\ {}^nC_t\, a_{(t,1)} + \ldots + {}^nC_n\, a_{(n,1)}$

$= \mathbf{a_{(n+1,2)}}$

coefficient of $F'^{3}*(N)$

$= {}^nC_2\, a_{(2,2)} + {}^nC_3\, a_{(3,2)} + {}^nC_4\, a_{(4,2)} + \ldots\ {}^nC_t\, a_{(t,2)} + \ldots + {}^nC_n\, a_{(n,2)}$

$= \mathbf{a_{(n+1,3)}}$

coefficient of $F'^{m}*(N) =$

$a_{(n+1,m)} = {}^nC_{m-1}.a_{(m-1,m-1)} + {}^nC_m.a_{(m,m-1)} + \ldots + {}^nC_n.a_{(n,m-1)}$

coefficient of $F'^{(n+1)}*(N) =$

$a_{(n+1,n+1)} = {}^nC_n.a_{(n,n)} = {}^nC_n.{}^{n-1}C_{n-1}\,.a_{(n-1,n-1)} = {}^nC_n.{}^{n-1}C_{n-1}.\ldots$

${}^2C_2.a_{(1,1)}$

$= 1$

Consider $a_{(n+1,2)}$

$= {}^nC_1\, a_{(1,1)} + {}^nC_2\, a_{(2,1)} +\ldots\ {}^nC_t\, a_{(t,1)} + \ldots + {}^nC_n\, a_{(n,1)}$

$$= {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$$

$$= 2^n - 1$$

$$= (2^{n+1} - 2)/2 \,.$$

Consider $a_{(n+1,3)}$

$$= {}^nC_2\, a_{(2,2)} + {}^nC_3\, a_{(3,2)} + {}^nC_4\, a_{(4,2)} + \ldots {}^nC_t\, a_{(t,2)} + \ldots + {}^nC_n\, a_{(n,2)}$$

$$= {}^nC_2(2^1-1) + {}^nC_3(2^2-1) + {}^nC_4(2^3-1) + \ldots + {}^nC_n(2^{n-1} - 1)$$

$$= {}^nC_2 2^1 + {}^nC_3 2^2 + \ldots + {}^nC_n\, 2^{n-1} - \{ {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n \}$$

$$=(1/2)\{ {}^nC_2 2^2 + {}^nC_3 2^3 + \ldots + {}^nC_n\, 2^n\} - \{ \sum_{r=0}^{n} {}^nC_r - {}^nC_1 - {}^nC_0 \}$$

$$= (1/2)\{ \sum_{r=0}^{n} {}^nC_r\, 2^r - {}^nC_1.2^1 - {}^nC_0.2^0 \} - \{ 2^n - n - 1 \}$$

$$= (1/2)\{ 3^n - 2n - 1\} - 2^n + n + 1$$

$$= (1/2)\{ 3^n - 2^{n+1} + 1\} \qquad \text{----------(3.1)}$$

$$= \{1/3!\}\{ (1).3^{n+1} - (3).\, 2^{n+1} + (3).\,(1)^{n+1} - (1)\,(0)^{n+1} \}$$

Evaluating $a_{(n+1,4)}$

$$a_{(n+1,4)} = {}^nC_3\, a_{(3,3)} + {}^nC_4\, a_{(4,3)} + \ldots + {}^nC_n\, a_{(n,3)}$$

$$= {}^nC_3\{3^2 + 1 - 2^3\}/2 + {}^nC_4\{ 3^3 + 1 - 2^4 \}/2 + \ldots + {}^nC_n\{ 3^{n-1} + 1 - 2^n \}/2$$

$$= (1/2)[\{ 3^2.{}^nC_3 + 3^3\, {}^nC_4 + \ldots + 3^{n-1}.{}^nC_n \} + \{ {}^nC_3 + {}^nC_4 + \ldots + {}^nC_n \}$$

$$- \{ {}^nC_3 2^3 + {}^nC_4 2^4 + \ldots + {}^nC_n\, 2^n\}]$$

$$= (1/2)[\, (1/3)\ \{ \sum_{r=0}^{n} {}^nC_r\, 3^r - 3^2\, {}^nC_2 - 3\, {}^nC_1 - {}^nC_0\} + \{ \sum_{r=0}^{n} {}^nC_r - {}^nC_2 - {}^nC_1$$

$$- {}^nC_0 \} - \{ \sum_{r=0}^{n} {}^nC_r.2^r - 2^2\, {}^nC_2 - 2\, {}^nC_1 - {}^nC_0\}]$$

$$= (1/2)\, [(1/3)\{ 4^n - 9n(n-1)/2 - 3n - 1 \} + \{ 2^n - n(n-1)/2 - n - 1 \}$$

$- \{ 3^n - 4n(n-1)/2 - 2n - 1\}$

$$a_{(n+1,4)} = (1/4!)\ [\ (1)\ 4^{n+1} - (4)\ 3^{n+1} + (6)\ 2^{n+1} - (4)\ 1^{n+1} + 1(0)^{n+1}\ ]$$

Observing the pattern we can explore the possibility of

$$\mathbf{a_{(n,r)} = (1/r!) \sum_{k=0}^{r} (-1)^{r-k} \,.\, {}^rC_k \,.\, k^n \quad \text{-------}(3.2)}$$

which is yet to be established. Now we shall apply induction.

Let the following proposition (3.3) be true for r and all n > r.

$$a_{(n+1,r)} = (1/r!) \sum_{k=1}^{r} (-1)^{r-k} \,.\, {}^rC_k \,.\, k^{n+1} \quad \text{------}(3.3)$$

Given (3.3) our aim is to prove that

$$a_{(n+1,r+1)} = (1/(r+1)!) \sum_{k=1}^{r+1} [\ (-1)^{(r+1)-k} \,.\, {}^{r+1}C_k\ (k)^{n+1}\ ]$$

we have

$$a_{(n+1,r+1)} = {}^nC_r\ a_{(r,r)} + {}^nC_{r+1}\ a_{(r+1,r)} + {}^nC_{r+2}\ a_{(r+2,r)} + \ldots + {}^nC_n\ a_{(n,r)}$$

$$a_{(n+1,r+1)} = {}^nC_r \{(1/r!) \sum_{k=0}^{r} (-1)^{r-k} \,.\, {}^rC_k \,.\, k^r \} + {}^nC_{r+1} \{(1/r!) \sum_{k=0}^{r} (-1)^{r-k} \,.\, {}^rC_k\ k^{r+1}$$

$$+ \ldots + {}^nC_n \{(1/r!) \sum_{k=0}^{r} (-1)^{r-k} \,.\, {}^rC_k \,.\, k^n \}$$

$$= (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} \,.\, {}^rC_k \{ {}^nC_r\ k^r + {}^nC_{r+1}\ k^{r+1} + \ldots + {}^nC_n\ k^n \}]$$

$$= (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} \,.\, {}^rC_k \{ \sum_{q=0}^{n} {}^nC_q\ k^q - \sum_{q=0}^{r-1} {}^nC_q\ k^q \}]$$

$$= (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} \,.\, {}^rC_k\ (1+k)^n\ ] - (1/r!) \sum_{q=0}^{r-1} [(-1)^{r-k} \,.\, {}^rC_k \{\sum {}^nC_q\ k^q \}]$$

If we denote the I[st] and the second term as T_1 and T_2, we have

$$a_{(n+1,r+1)} = T_1 - T_2 \qquad \text{----------(3.4)}$$

consider $T_1 = (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} \cdot {}^rC_k \, (1+k)^n]$

$$= (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} \{ r!/((k!)(r-k)!)\} (1+k)^n]$$

$$= (1/(r+1)!) \sum_{k=0}^{r} [(-1)^{r-k} \{(r+1)!/((k+1)!(r-k)!)\}(1+k)^{n+1}]$$

$$= (1/(r+1)!) \sum_{k=0}^{r} [(-1)^{r-k} \cdot {}^{r+1}C_{k+1} \, (1+k)^{n+1}]$$

$$= (1/(r+1)!) \sum_{k=0}^{r} [(-1)^{(r+1)-(k+1)} \cdot {}^{r+1}C_{k+1} \, (1+k)^{n+1}]$$

Let $k + 1 = s$, we get, $s = 1$ at $k = 0$ and $s = r + 1$ at $k = r$

$$= (1/(r+1)!) \sum_{s=1}^{r+1} [(-1)^{(r+1)-s} \cdot {}^{r+1}C_s \, (s)^{n+1}].$$

replacing s by k we get

$$= (1/(r+1)!) \sum_{k=1}^{r+1} [(-1)^{(r+1)-k} \cdot {}^{r+1}C_k \, (k)^{n+1}]$$

in this if we include $k = 0$ case we get

$$\mathbf{T_1 = (1/(r+1)!) \sum_{k=0}^{r+1} [(-1)^{(r+1)-k} \cdot {}^{r+1}C_k \, (k)^{n+1}]} \qquad \text{----(3.5)}$$

T_1 is nothing but the right hand side member of (3.3).

To prove (3.3) we have to prove $a_{(n+1,r+1)} = T_1$

In view of (3.4) our next step is to prove that $T_2 = 0$

$$T_2 = (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} .{}^rC_k \{ \sum_{q=0}^{r-1} {}^nC_q\, k^q \}]$$

$$= (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} .{}^rC_k \{ {}^nC_0\, k^0 + {}^nC_1\, k^1 + {}^nC_2\, k^2 + . . . + {}^nC_{r-1}\, k^{r-1} \}]$$

$$= (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} .{}^rC_k] + {}^nC_1[(1/r!) \sum_{k=0}^{r} \{(-1)^{r-k} .{}^rC_k\, k \}] +$$

$${}^nC_2[(1/r!) \sum_{k=0}^{r}\{(-1)^{r-k}\, {}^rC_k\, k^2\}] + . . . + {}^nC_{r-1}[(1/r!) \sum_{k=0}^{r} \{(-1)^{r-k} .{}^rC_k\, k^{r-1} \}]$$

$$= (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} .{}^rC_k] + {}^nC_1[(1/r!) \sum_{k=0}^{r} \{(-1)^{r-k} .{}^rC_k\, k \}] +$$

$$[{}^nC_2 .a_{(2,r)} + {}^nC_3 .a_{(3,r)} .+ . . . + {}^nC_{r-1} .a_{(r-1,r)}]$$

$=$ X + Y + Z say where

$$X = (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} .{}^rC_k] \ , \quad Y = {}^nC_1[(1/r!) \sum_{k=0}^{r} \{(-1)^{r-k} .{}^rC_k\, k \}]$$

$$Z = [{}^nC_2 .a_{(2,r)} + {}^nC_3 .a_{(3,r)} + . . . + {}^nC_{r-1} .a_{(r-1,r)}]$$

We shall prove that $X = 0$, $Y = 0$, $Z = 0$ seperately.

(1) $$X = (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} .{}^rC_k]$$

$$= (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} .{}^rC_{r-k}]$$

let $r - k = w$ then we get at $k = 0$ $w = r$ and at $k = r$ $w = 0$.

$$= (1/r!) \sum_{w=r}^{0} [(-1)^{w} .{}^rC_w]$$

$$= (1/r!) \sum_{w=0}^{r} [(-1)^w . {}^rC_w]$$

$$=(1 - 1)^r /r!$$

$$= 0$$

We have proved that $X = 0$

(2)

$$Y = {}^nC_1[(1/r!) \sum_{k=0}^{r} \{(-1)^{r-k} . {}^rC_k\ k \}]$$

$$= {}^nC_1[(1/(r-1)!) \sum_{k=1}^{r} \{(-1)^{r-1-(k-1)} . {}^{r-1}C_{k-1} \}]$$

$$= {}^nC_1[(1/(r-1)!) \sum_{k-1=0}^{r-1} \{(-1)^{r-1-(k-1)} . {}^{r-1}C_{k-1} \}]$$

$$= {}^nC_1[(1/(r-1)!)(1 - 1)^{r-1}$$

$$= 0$$

We have proved that $Y = 0$

(3) To prove

$$Z = [{}^nC_2 .a_{(2,r)} + {}^nC_3 .a_{(3,r)} + . . . + {}^nC_{r-1} .a_{(r-1,r)}] = 0 \quad \text{----(3.6)}$$

Proof:

Refer the matrix

$\underline{a_{(1,1)}}$	$a_{(1,2)}$	$a_{(1,3)}$	$a_{(1,4)}$		. . .	$a_{(1,r)}$
$a_{(2,1)}$	$\underline{a_{(2,2)}}$	$a_{(2,3)}$	$a_{(2,4)}$		. . .	$a_{(2,r)}$
$a_{(3,1)}$	$a_{(3,2)}$	$\underline{a_{(3,3)}}$	$a_{(3,4)}$		. . .	$a_{(3,r)}$
$a_{(4,1)}$	$a_{(4,2)}$	$a_{(4,3)}$	$\underline{a_{(4,4)}}$	$a_{(4,5)}$	. . .	$a_{(4,r)}$

.....

$$\begin{array}{ccccccc} \ldots & \ldots & \ldots & \ldots & \underline{a_{(r-1,r-1)}} & \mathbf{a_{(r-1,r)}} & \ldots \\ a_{(r,1)} & a_{(r,2)} & a_{(r,3)} & \ldots & a_{(r,r-1)} & \underline{a_{(r,r)}} & \end{array}$$

The Diagonal elements are underlined . And the the elements above the leading diagonal are shown with bold face.

We have

$$a_{(1,r)} = [(1/r!) \sum_{k=0}^{r} \{(-1)^{r-k} . {}^{r}C_k \; k \}] = Y/ {}^{n}C_1 = 0 \text{ for } r > 1$$

All the elements of the first row except $a_{(1,1)}$ (the one on the leading diagonal) are zero.
Also

$$a_{(n+1,r)} = a_{(n,r-1)} + r \,.\, a_{(n,r)} \qquad \text{--------(3.7)}$$

(This can be easily established by simplifying the right hand side.)

(7) gives us

$$a_{(2,r)} = a_{(1,r-1)} + r \,.\, a_{(1,r)} = 0 \text{ for } r > 2$$

i.e. $a_{(2,r)}$ can be expresssed as a linear combination of two elements of the first row (except the one on the leading diagonal)

$\Rightarrow a_{(2,r)} = 0 \quad r > 2$

Similarly $a_{(3,r)}$ can be expresssed as a linear combination of two elements of the second row of the type $a_{(2,r)}$ with $r > 3$

$\Rightarrow a_{(2,r)} = 0 \quad r > 3$

and so on $a_{(r-1,r)} = 0$

substituting

$a_{(2,r)} =.a_{(3,r)} = \ldots = a_{(r-1,r)} = 0$ in (6)

we get $Z = 0$

With $X = Y = Z = 0$ we get $T_2 = 0$
or $a_{(n+1,r+1)} = T_1 - T_2 = T_1$

from (5) we have

$$T_1 = (1/(r+1)!) \sum_{k=0}^{r+1} [(-1)^{(r+1)-k} \; {}^{r+1}C_k \, (k)^{n+1}]$$

which gives

$$a_{(n+1,r+1)} = (1/(r+1)!) \sum_{k=0}^{r+1} [(-1)^{(r+1)-k} \; {}^{r+1}C_k \, (k)^{n+1}]$$

We have proved ,if the propposition (3.3) is true for r it is true for (r+1) as well .We have already verified it for 1, 2, 3 etc. Hence by induction (3.3) is true for all n.

This completes the proof of theorem (3.1) .

Remarks: This proof is quite lengthy , clumsy and heavy in algebra. The readers can try some analytic , combinatorial approach.

REFERENCES:

[1] V.Krishnamurthy, "COMBINATORICS Theory and Applications" East West Press Private Ltd. ,1985.

[2] "Amarnath Murthy" , 'Generalization Of Partition Function, Introducing 'Smarandache Factor Partition', SNJ, Vol. 11, No. 1-2-3, 2000.

[3] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

[4] 'Smarandache Notion Journal' Vol. 10 ,No. 1-2-3, Spring 1999. Number Theory Association of the UNIVERSITY OF CRAIOVA .

MORE RESULTS AND APPLICATIONS OF THE GENERALIZED SMARANDACHE STAR FUNCTION

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRCT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION , as follows:

Let α_1 , α_2 , α_3 , . . . α_r be a set of r natural numbers and p_1 , p_2, p_3 ,. . .p_r be arbitrarily chosen distinct primes then $F(\alpha_1 , \alpha_2 , \alpha_3 , . . . \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1 , \alpha_2 , \alpha_3 , . . . \alpha_r)$ is defined as the number of ways in which the number

$N = p_1^{\alpha 1} \; p_2^{\alpha 2} \; p_3^{\alpha 3} \; . . . \; p_r^{\alpha r}$ could be expressed as the product of its' divisors. For simplicity , we denote $F(\alpha_1 , \alpha_2 , \alpha_3 , . . . \alpha_r) = F'(N)$,where

$$N = p_1^{\alpha_1} \; p_2^{\alpha_2} \; p_3^{\alpha_3} \; . . . \; p_r^{\alpha_r} \; . . . \; p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 =2$, $p_2 = 3$ etc.

Also for the case

$$\alpha_1 = \alpha_2 = \alpha_3 =. . . = \alpha_r =. . . = \alpha_n = 1$$

Let us denote

$$\underbrace{F (1 , 1, 1, 1, 1. . .)}_{\leftarrow \text{ n - ones } \rightarrow} = F (1\#n)$$

In [2] we define **The Generalized Smarandache Star**

Function as follows:

Smarandache Star Function

(1) $F'^{*}(N) = \sum_{d/N} F'(d_r)$ where $d_r | N$

(2) $F'^{**}(N) = \sum_{d_r/N} F'^{*}(d_r)$

d_r ranges over all the divisors of N.

If N is a square free number with n prime factors , let us denote

$$F'^{**}(N) = F^{**}(1\#n)$$

Smarandache Generalised Star Function

(3) $F'^{n*}(N) = \sum_{d_r/N} F'^{(n-1)*}(d_r)$ $n > 1$

and d_r ranges over all the divisors of N.

For simplicity we denote

$$F'(Np_1p_2...p_n) = F'(N@1\#n) \quad , \text{where}$$

$$(N,p_i) = 1 \text{ for } i = 1 \text{ to } n \text{ and each } p_i \text{ is a prime.}$$

F'(N@1#n) is nothing but the Smarandache factor partition of (a number N multiplied by n primes which are coprime to N).

In [3] I had derived a general result on the Smarandache Generalised Star Function. In the present note some more results and applications of Smarandache Generalised Star Function are explored and derived.

DISCUSSION:

THEOREM(4.1) :

$$F'^{n*}(p^{\alpha}) \quad = \quad \sum_{k=0}^{\alpha} {}^{n+k-1}C_{n-1}\, P(\alpha-k) \quad \text{------}(4.1)$$

Following proposition shall be applied in the proof of this

$$\sum_{k=0}^{\alpha} {}^{r+k-1}C_{r-1} = {}^{\alpha+r}C_r \qquad \text{-------}(4.2)$$

Let the proposition (4.1) be true for $n = r$ to $n = 1$.

$$F'^{r*}(p^{\alpha}) \quad = \quad \sum_{k=0}^{\alpha} {}^{r+k-1}C_{r-1}\, P(\alpha-k) \qquad \text{-------}(4.3)$$

$$F'^{(r+1)*}(p^{\alpha}) = \sum_{t=0}^{\alpha} F'^{r*}(p^{t})$$

(p ranges over all the divisors of p^{α} for $t = 0$ to α)

RHS = $F'^{r*}(p^{\alpha}) + F'^{r*}(p^{\alpha-1}) + F'^{r*}(p^{\alpha-2}) + \ldots + F'^{r*}(p) + F'^{r*}(1)$

from the proposition (4.3) we have

$$F'^{r*}(p^{\alpha}) \quad = \quad \sum_{k=0}^{\alpha} {}^{r+k-1}C_{r-1}\, P(\alpha-k)$$

expanding RHS from $k = 0$ to α

$$F'^{r*}(p^{\alpha}) = {}^{r+\alpha-1}C_{r-1}\, P(0) + {}^{r+\alpha-2}C_{r-1}\, P(1) + \ldots + {}^{r-1}C_{r-1}\, P(\alpha)$$

similarly

$$F'^{r*}(p^{\alpha-1}) = {}^{r+\alpha-2}C_{r-1}\, P(0) + {}^{r+\alpha-3}C_{r-1}\, P(1) + \ldots + {}^{r-1}C_{r-1}\, P(\alpha-1)$$

$$F'^{r*}(p^{\alpha-2}) = {}^{r+\alpha-3}C_{r-1}\, P(0) + {}^{r+\alpha-4}C_{r-1}\, P(1) + \ldots + {}^{r-1}C_{r-1}\, P(\alpha-2)$$

.

.

$F'^{r*}(p) = {}^{r}C_{r-1}\, P(0) + {}^{r-1}C_{r-1}\, P(1)$

$F'^{r*}(1) = {}^{r-1}C_{r-1}\, P(0)$

summing up left and right sides separately we find that the

$$LHS = F'^{(r+1)*}(p^{\alpha})$$

The RHS contains $\alpha + 1$ terms in which P(0) occurs , α terms in which P(1) occurs etc.

$$RHS = [\sum_{k=0}^{\alpha} {}^{r+k-1}C_{r-1}].P(0) + \sum_{k=0}^{\alpha-1} {}^{r+k-1}C_{r-1}\, P(1) + \ldots + \sum_{k=0}^{1} {}^{r+k-1}C_{r-1}\, P(\alpha-1)$$

$$+ \sum_{k=0}^{0} {}^{r+k-1}C_{r-1}\, P(\alpha)$$

Applying proposition (4.2) to each of the $\sum$ we get

$$RHS = {}^{r+\alpha}C_r\, P(0) + {}^{r+\alpha-1}C_r\, P(1) + {}^{r+\alpha-2}C_r\, P(2) + \ldots + {}^{r}C_r\, P(\alpha)$$

$$= \sum_{k=0}^{\alpha} {}^{r+k}C_r\, P(\alpha-k)$$

or

$$F'^{(r+1)*}(p^{\alpha}) = \sum_{k=0}^{\alpha} {}^{r+k}C_r\, P(\alpha-k)$$

The proposition is true for $n = r+1$, as we have

$$F'^{*}(p^{\alpha}) = \sum_{k=0}^{\alpha} P(\alpha-k) = \sum_{k=0}^{\alpha} {}^{k}C_0\, P(\alpha-k) = \sum_{k=0}^{\alpha} {}^{k+1-1}C_{1-1}\, P(\alpha-k)$$

The proposition is true for n = 1

Hence by induction the proposition is true for all n.

This completes the proof of theorem (4.1).

Following theorem shall be applied in the proof of theorem (4.3)

THEOREM (4.2)

n-r

$$\sum_{k=0} {}^{n}C_{r+k}\ {}^{r+k}C_r\ m^k = {}^{n}C_r\ (1+m)^{(n-r)}$$

PROOF:

$$LHS = \sum_{k=0}^{n-r} {}^{n}C_{r+k}\ {}^{r+k}C_r\ m^k$$

$$= \sum_{k=0}^{n-r} (n!)/\{(r+k)!.(n-r-k)!\}\ .(r+k)!/\{(k)!.(r)!\}\ .\ m^k$$

$$= \sum_{k=0}^{n-r} (n!)/\{(r)!.(n-r)!\}\ .(n-r)!/\{(k)!.(n-r-k)!\}\ .\ m^k$$

$$= {}^{n}C_r \sum_{k=0}^{n-r} {}^{n-r}C_k\ m^k$$

$$= {}^{n}C_r\ (1+m)^{(n-r)}$$

This completes the proof of theorem (4.2)

THEOREM(4.3):

$$F^{m*}(1\#n) = \sum_{r=0}^{n} {}^{n}C_r\ m^{n-r}\ F(1\#r)$$

Proof:

From theorem (2.4) (ref.[1] ne have

$$F^{*}(1\#n) = F\ (\ 1\#\ (n+1)) = \sum_{r=0}^{n} {}^{n}C_r\ F(1\#r) = \sum_{r=0}^{n} {}^{n}C_r\ (1)^{n-r}\ F(1\#r)$$

hence the proposition is true for m = 1.

Let the proposition be true for m = s. Then we have

$$F^{s*}(1\#n) = \sum_{r=0}^{n} {}^{n}C_r\ s^{n-r}\ F(1\#r)$$

or

$$F^{s*}(1\#0) = \sum_{r=0}^{0} {}^{n}C_0\ s^{0-r}\ F(1\#0) \qquad F^{s*}(1\#1) = \sum_{r=0}^{1} {}^{n}C_1\ s^{1-r}\ F(1\#1)$$

$$F^{s*}(1\#2) = \sum_{r=0}^{2} {}^{n}C_2\ s^{2-r}\ F(1\#1) \qquad F^{s*}(1\#3) = \sum_{r=0}^{3} {}^{n}C_1\ s^{3-r}\ F(1\#3)$$

$F^{s*}(1\#0) = {}^0C_0\, F(1\#0)$ ----(0)

$F^{s*}(1\#1) = {}^1C_0\, s^1\, F(1\#0) + {}^1C_1\, s^0 F(1\#1)$ ----(1)

$F^{s*}(1\#2) = {}^2C_0\, s^2\, F(1\#0) + {}^2C_1\, s^1 F(1\#1) + {}^2C_2\, s^0 F(1\#2)$ ----(2)

.

.

.

$F^{s*}(1\#r) = {}^rC_0\, s^r\, F(1\#0) + {}^rC_1\, s^1 F(1\#1) + \ldots + {}^rC_r\, s^0 F(1\#r)$ ----(r)

.

.

.

$F^{s*}(1\#n) = {}^nC_0\, s^r\, F(1\#0) + {}^nC_1\, s^1 F(1\#1) + \ldots + {}^nC_n\, s^0 F(1\#r)$ ----(n)

multiplying the r^{th} equation with ${}^nC_r$ and then summing up we get the RHS as

$=[{}^nC_0{}^0C_0\, s^0 + {}^nC_1{}^1C_0\, s^1 + {}^nC_2{}^2C_0\, s^2 + \ldots + {}^nC_k{}^kC_0\, s^k + \ldots + {}^nC_n{}^nC_0\, s^n]F(1\#0)$

$[{}^nC_1{}^1C_1\, s^0 + {}^nC_2{}^2C_1\, s^1 + {}^nC_3{}^3C_1\, s^2 + \ldots + {}^nC_k{}^kC_1\, s^k + \ldots + {}^nC_n{}^nC_1\, s^n]F(1\#1)$

. . .

$[{}^nC_r{}^rC_r\, s^0 + {}^nC_{r+1}{}^{r+1}C_r\, s^1 + \ldots + {}^nC_{r+k}{}^{r+k}C_r\, s^k + \ldots + {}^nC_n{}^nC_r\, s^n]F(1\#r)$

$+ {}^nC_n{}^nC_n\, s^0]F(1\#n)$

$$= \sum_{r=0}^{n} \left\{ \sum_{k=0}^{n-r} {}^nC_{r+k}\, {}^{r+k}C_r\, s^k \right\} F(1\#r)$$

$$= \sum_{r=0}^{n} {}^nC_r\, (1+s)^{n-r}\, F(1\#n) \qquad \text{, by theorem (4.2)}$$

$$\text{LHS} = \sum_{r=0}^{n} {}^nC_r\, F^{s*}(1\#r)$$

Let $N = p_1p_2p_3 \ldots p_n$. Then there are ${}^nC_r$ divisors of N containing exactly r primes . Then LHS = the sum of the s^{th} Smarandache star functions of all the divisors of N. $= F^{'(s+1)*}(N) = F^{(s+1)*}(1\#n)$.

Hence we have

$$F^{(s+1)*}(1\#n) = \sum_{r=0}^{n} {}^nC_r\, (1+s)^{n-r}\, F(1\#n)$$

$F^{s*}(1\#0)= {}^0C_0\, F(1\#0)$ ----(0)

$F^{s*}(1\#1) = {}^1C_0\, s^1\, F(1\#0) + {}^1C_1\, s^0F(1\#1)$ ----(1)

$F^{s*}(1\#2) = {}^2C_0\, s^2\, F(1\#0) + {}^2C_1\, s^1F(1\#1) + {}^2C_2\, s^0F(1\#2)$ ----(2)

.

.

.

$F^{s*}(1\#r) = {}^rC_0\, s^r\, F(1\#0) + {}^rC_1\, s^1F(1\#1) + \ldots + {}^rC_r\, s^0F(1\#r)$ ----(r)

.

.

.

$F^{s*}(1\#n) = {}^nC_0\, s^r\, F(1\#0) + {}^nC_1\, s^1F(1\#1) + \ldots + {}^nC_n\, s^0F(1\#r)$ ----(n)

multiplying the r^{th} equation with ${}^nC_r$ and then summing up we get the RHS as

$=[{}^nC_0{}^0C_0\, s^0 + {}^nC_1{}^1C_0\, s^1 + {}^nC_2{}^2C_0\, s^2 + \ldots + {}^nC_k{}^kC_0\, s^k + \ldots + {}^nC_n{}^nC_0\, s^n]F(1\#0)$

$[{}^nC_1{}^1C_1\, s^0 + {}^nC_2{}^2C_1\, s^1 + {}^nC_3{}^3C_1\, s^2 + \ldots + {}^nC_k{}^kC_1\, s^k + \ldots + {}^nC_n{}^nC_1\, s^n]F(1\#1)$

. . .

$[{}^nC_r{}^rC_r\, s^0 + {}^nC_{r+1}{}^{r+1}C_r\, s^1 + \ldots + {}^nC_{r+k}{}^{r+k}C_r\, s^k + \ldots + {}^nC_n{}^nC_r\, s^n]F(1\#r)$

$+ {}^nC_n{}^nC_n\, s^0]F(1\#n)$

$$= \sum_{r=0}^{n} \left\{ \sum_{k=0}^{n-r} {}^nC_{r+k}\, {}^{r+k}C_r\, s^k \right\} F(1\#r)$$

$$= \sum_{r=0}^{n} {}^nC_r\, (1+s)^{n-r}\, F(1\#n) \quad \text{, by theorem (4.2)}$$

$$LHS = \sum_{r=0}^{n} {}^nC_r\, F^{s*}(1\#r)$$

Let $N = p_1p_2p_3 \ldots p_n$. Then there are ${}^nC_r$ divisors of N containing exactly r primes . Then LHS = the sum of the s^{th} Smarandache star functions of all the divisors of N. $= F'^{(s+1)*}(N) = F^{(s+1)*}(1\#n)$.

Hence we have

$$F^{(s+1)*}(1\#n) = \sum_{r=0}^{n} {}^nC_r\, (1+s)^{n-r}\, F(1\#n)$$

which takes the same format

$P(s) \Rightarrow P(s+1)$

and it has been verified that the proposition is true for m = 1

hence by induction the proposition is true for all m.

$$\mathbf{F^{m*}(1\#n) = \sum_{r=0}^{n} {}^{n}C_r\, m^{n-r}\, F(1\#r)}$$

This completes the proof of theorem (4.3)

NOTE:

From theorem (3.1) we have

$$F'(N@1\#n) = F'(Np_1p_2...p_n) = \sum_{m=0}^{n} a_{(n,m)}\, F'^{m*}(N)$$

where

$$a_{(n,m)} = (1/m!) \sum_{k=1}^{m} (-1)^{m-k}\, .{}^{m}C_k\, .k^n$$

If $N = p_1p_2...p_k$ Then we get

$$F(1\#(k+n) = \sum_{m=0}^{n} [\, a_{(n,m)} \sum_{t=0}^{k} {}^{k}C_t\, m^{k-t}\, F(1\#t)\,] \qquad \text{------(4.4)}$$

The above result provides us with a formula to express B_n in terms of smaller Bell numbers. It is in a way generalisation of theorem (2.4) in Ref [5].

THEOREM(4.4):

$$F(\alpha,1\#(n+1)) = \sum_{k=0}^{\alpha} \sum_{r=0}^{n} {}^{n}C_r\, F(k,1\#r)$$

PROOF: LHS = $F(\alpha,1\#(n+1)) = F'(p^{\alpha}\, p_1p_2p_3...p_{n+1}) = F'^{*}(p^{\alpha}p_1p_2p_3...p_n) + \sum F'$(all the divisors containing only p^0) + $\sum F'$ (all the divisors containing only p^1) + $\sum F'$ (all the divisors containing only p^2) +. . .+ $\sum F'$ (all the divisors containing only p^r) +. . .+ $\sum F'$ (all the divisors containing only p^α)

$$= \sum_{r=0}^{n} {}^nC_r\, F(0,1\#r) + \sum_{r=0}^{n} {}^nC_r\, F(1,1\#r) + \sum_{r=0}^{n} {}^nC_r\, F(2,1\#r) + \sum_{r=0}^{n} {}^nC_r\, F(3,1\#r)$$

$$+\ldots+ \sum_{r=0}^{n} {}^nC_r\, F(k,1\#r) +\ldots+ \sum_{r=0}^{n} {}^nC_r\, F(\alpha,1\#r)$$

$$= \sum_{k=0}^{\alpha} \sum_{r=0}^{n} {}^nC_r\, F(k,1\#r)$$

This is a reduction formula for $F(\alpha,1\#(n+1))$

A Result of significance

From theorem (3.1) of Ref.: [2] , we have

$$F'(p^\alpha@1\#(n+1)) = \mathbf{F(\alpha,1\#(n+1))} = \sum_{m=0}^{n} a_{(n+1,m)}\, F'^{m*}(N)$$

where

$$a_{(n+1,m)} = (1/m!) \sum_{k=1}^{m} (-1)^{m-k} \,.{}^mC_k\, .k^{n+1}$$

and

$$F'^{m*}(p^\alpha) = \sum_{k=0}^{\alpha} {}^{m+k-1}C_{m-1}\, P(\alpha-k)$$

This is the first result of some substance , giving a formula for evaluating the number of Smarandache Factor Partitions of numbers representable in a (one of the most simple) particular canonical form. The complexity is evident. The challenging task ahead for the readers is to derive similar expressions for other canonical forms.

REFERENCE

[1] "Amarnath Murthy" , 'Generalization Of Partition Function, Introducing 'Smarandache Factor Partition', SNJ, Vol. 11, No. 1-2-3, 2000.

[2] "Amarnath Murthy" , 'A General Result On The "Smarandache Star Function" , SNJ, Vol. 11, No. 1-2-3, 2000.

[3] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

[4] 'Smarandache Notion Journal' Vol. 10 ,No. 1-2-3, Spring 1999. Number Theory Association of the UNIVERSITY OF CRAIOVA.

PROPERTIES OF SMARANDACHE STAR TRIANGLE

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRCT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION , as follows: Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r$ be a set of r natural numbers and $p_1, p_2, p_3, \ldots p_r$ be arbitrarily chosen distinct primes then $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha 1} p_2^{\alpha 2} p_3^{\alpha 3} \ldots p_r^{\alpha r}$$ could be expressed as the

product of its' divisors. For simplicity , we denote $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N)$,where

$$N = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 = 2$, $p_2 = 3$ etc.

Also for the case

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1$$

Let us denote

$$\underbrace{F(1, 1, 1, 1, 1 \ldots)}_{\leftarrow \text{ n - ones } \rightarrow} = F(1\#n)$$

In [2] we define **The Generalized Smarandache Star Function** as follows:

Smarandache Star Function

(1) $F'^{*}(N) = \sum_{d/N} F'(d_r)$ where $d_r \mid N$

(2) $F'^{**}(N) = \sum_{d_r/N} F'^{*}(d_r)$

d_r ranges over all the divisors of N.

If N is a square free number with n prime factors, let us denote

$$F'^{**}(N) = F^{**}(1\#n)$$

Smarandache Generalised Star Function

(3) $F'^{n*}(N) = \sum_{d_r/N} F'^{(n-1)*}(d_r)$ $n > 1$

and d_r ranges over all the divisors of N.

For simplicity we denote

$F'(Np_1p_2...p_n) = F'(N@1\#n)$, where

$(N,p_i) = 1$ for $i = 1$ to n and each p_i is a prime.

$F'(N@1\#n)$ is nothing but the Smarandache factor partition of (a number N multiplied by n primes which are coprime to N).

In [2] I had derived a general result on the Smarandache Generalised Star Function. In the present note we define **SMARANDACHE STAR TRIANGLE'** (SST) and derive some properties of SST.

DISCUSSION:

DEFINITION : 'SMARANDACHE STAR TRIANGLE' (SST)

As established in [2]

$$a_{(n,m)} = (1/m!) \sum_{k=1}^{m} (-1)^{m-k} \cdot {}^{m}C_{k} \cdot k^{n} \text{ ---------- } (1)$$

we have $a_{(n,n)} = a_{(n,1)} = 1$ and $a_{(n,m)} = 0$ for $m > n$. Now if one arranges these elements as follows

$a_{(1,1)}$

$a_{(2,1)}$ $a_{(2,2)}$

$a_{(3,1)}$ $a_{(3,2)}$ $a_{(3,3)}$

.

.

.

$a_{(n,1)}$ $a_{(n,2)}$ ... $a_{(n,n-1)}$ $a_{(n,n)}$

we get the following triangle which we call as the '**SMARANDACHE STAR TRIANGLE**' in which $a_{(r,m)}$ is the m^{th} element of the r^{th} row and is given by (A) above. It is to be noted here that the elements are the Stirling numbers of the first kind.

1

1 1

1 3 1

1 7 6 1

1 15 25 10 1

...

Some propoerties of the SST.

(1) The elements of the first column and the last element of each row is unity.

(2) The elements of the second column are $2^{n-1} - 1$, where n is the row number.

(3) Sum of all the elements of the n^{th} row is the n^{th} Bell.

PROOF:

From theorem(3.1) of Ref; [2] we have

$$F'(N@1\#n) = F'(Np_1p_2...p_n) = \sum_{m=0}^{n} a_{(n,m)} F'^{m*}(N)$$

if $N = 1$ we get $F'^{m*}(1) = F'^{(m-1)*}(1) = F'^{(m-2)*}(1) = ... = F'(1) = 1$

hence

$$F'(p_1p_2...p_n) = \sum_{r=0}^{n} a_{(n,m)}$$

(**4**)The elements of a row can be obtained by the following reduction formula

$$\mathbf{a_{(n+1,m+1)} = a_{(n,m)} + (m+1) . a_{(n+1,m+1)}}$$

instead of having to use the formula (4.5).

(5) If $N = p$ in theorem (3.1) Ref;[2] we get $F'^{m*}(p) = m + 1$. Hence

$$F'(pp_1p_2...p_n) = \sum_{m=1}^{n} a_{(n,m)} F'^{m*}(N)$$

or
$$B_{n+1} = \sum_{m=1}^{n} (m+1)\ a_{(n,m)}$$

(6) Elements of second leading diagonal are triangular numbers in their natural order.

(7) If p is a prime, p divides all the elements of the p^{th} row except the I^{st} and the last, which are unity. This has been established in the following theorem.

THEOREM(1.1):

$$\mathbf{a_{(p,r)} \equiv 0 \ (\text{mod } p) \text{ if } p \text{ is a prime and } 1 < r < p}$$

Proof:

$$a_{(p,r)} = (1/r!) \sum_{k=1}^{m} (-1)^{r-k} \cdot {}^{r}C_k \cdot k^p$$

Also

$$a_{(p,r)} = (1/(r-1)!) \sum_{k=0}^{r-1} (-1)^{r-1-k} \cdot {}^{r-1}C_k \cdot (k+1)^{p-1}$$

$$a_{(p,r)} = (1/(r-1)!) \sum_{k=0}^{r-1} [(-1)^{r-1-k} \cdot {}^{r-1}C_k \cdot \{(k+1)^{p-1} - 1\}] +$$

$$(1/(r-1)!) \sum_{k=0}^{r-1} (-1)^{r-1-k} \cdot {}^{r-1}C_k$$

applying Fermat's little theorem, we get

$$a_{(p,r)} = \text{a multiple of } p + 0$$

$$\Rightarrow \quad \mathbf{a_{(p,r)} \equiv 0 \ (\text{mod } p)}$$

COROLLARY: (1.1)

$$F(1\#p) \equiv 2 \ (\text{mod } p)$$

$$a_{(p,1)} = a_{(p,p)} = 1$$

$$F(1\#p) = \sum_{k=0}^{p} a_{(p,k)} = \sum_{k=2}^{p-1} a_{(p,k)} + 2$$

$$F(1\#p) \equiv 2 \pmod p$$

(8) The coefficient of the r^{th} term $b_{(n,r)}$ in the expansion of x^n as

$$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,r)}\, {}^xP_r + \ldots + b_{(n,n)}\, {}^xP_n$$

is equal to $a_{(n,r)}$.

THEOREM(1.2): B_{3n+2} is even else B_k is odd.

From theorem (2.5) in REF. [1] we have

$F'(Nq_1q_2) = F'^*(N) + F'^{**}(N)$ where q_1 and q_2 are prime.

and $(N,q_1) = (N,q_2) = 1$

let $N = p_1p_2p_3\ldots p_n$ then one can write

$$F'(p_1p_2p_3\ldots p_nq_1q_2) = F'^*(p_1p_2p_3\ldots p_n) + F'^{**}(p_1p_2p_3\ldots p_n)$$

or $F(1\#(n+2)) = F(1\#(n+1)) + F^{**}(1\#n)$

but

$$F^{**}(1\#n) = \sum_{r=0}^{n} {}^nC_r\, 2^{n-r} F(1\#r)$$

$$F^{**}(1\#n) = \sum_{r=0}^{n-1} \{{}^nC_r\, 2^{n-r} F(1\#r)\} + F(1\#n)$$

the first term is an even number say $= E$,This gives us

$F(1\#(n+2)) - F(1\#(n+1)) - F(1\#n) = E$, an even number. ---(1.1)

Case- I: $F(1\#n)$ is even and $F(1\#(n+1))$ is also even $\Rightarrow$

$F(1\#(n+2))$ is even.

Case -II: $F(1\#n)$ is even and $F(1\#(n+1))$ is odd $\Rightarrow F(1\#(n+2))$ is odd.

again by (1.1) we get

$F(1\#(n+3)) - F(1\#(n+2)) - F(1\#(n+1)) = E$, $\Rightarrow F(1\#(n+3))$ is even. Finally we get

$F(1\#n)$ is even $\Leftrightarrow F(1\#(n+3))$ is even

we know that $F(1\#2) = 2 \Rightarrow F(1\#2)$, $F(1\#5)$, $F(1\#8)$, . . .are even

$\Rightarrow B_{3n+2}$ **is even else B_k is odd**

This completes the proof.

REFERENCES:

[1] "Amarnath Murthy", 'Generalization Of Partition Function, Introducing 'Smarandache Factor Partition', SNJ, Vol. 11, No. 1-2-3, 2000.

[2] "Amarnath Murthy", 'A General Result On The " Smarandache Star Function", SNJ, Vol. 11, No. 1-2-3, 2000.

[3] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

[4] 'Smarandache Notion Journal' Vol. 10 ,No. 1-2-3, Spring 1999. Number Theory Association of the UNIVERSITY OF CRAIOVA .

SMARANDACHE FACTOR PARTITIONS OF A TYPICAL CANONICAL FORM.

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION , as follows:

Let $\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r$ be a set of r natural numbers and $p_1 , p_2, p_3 ,\ldots p_r$ be arbitrarily chosen distinct primes then $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha 1} \; p_2^{\alpha 2} \; p_3^{\alpha 3} \ldots p_r^{\alpha r}$$ could be expressed as the product of its' divisors. For simplicity , we denote $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r) = F'(N)$,where

$$N = p_1^{\alpha_1} \; p_2^{\alpha_2} \; p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 =2$, $p_2 = 3$ etc.

In the present note we derive a formula forr the case $N = p_1^{\alpha}p_2^{2}$

DISCUSSION:

Theorem(5.1):

$$F'(p_1^{\alpha}p_2^{2}) = F(\alpha,2) = \sum_{k=0}^{\alpha} P(k) + \sum_{j=0}^{r} \sum_{i=0}^{\alpha-2j} P(i)$$

where $r = [\alpha/2]$ $\alpha =2r$ or $\alpha = 2r +1$

PROOF: Following are the distinct mutually exclusive and exhaustive cases. Only the numbers in the bracket [ ] are to be further decomposed.

Case I: $(p_2)\,[p_1^{\alpha}p_2^{2}]$ gives $F'^{*}(p_1^{\alpha}) = \sum_{k=0}^{\alpha} P(i)$

Case II: $\{A_1\} \rightarrow \quad (p_2^2)\,[p_1^{\alpha}] \quad \text{-------}\!\!\rightarrow P(\alpha)$

$\{A_2\} \rightarrow \quad (p_2^2\, p_1)\,[p_1^{\alpha-1}] \text{-------}\!\!\rightarrow P(\alpha-1)$

.

.

.

$\{A_{\alpha}\} \rightarrow \quad (p_2^2\, p_1^{\alpha})\,[p_1^{\alpha-\alpha}] \quad \text{-------}\!\!\rightarrow P(\alpha-\alpha) = P(0)$

Hence Case II contributes $\sum_{i=0}^{\alpha} P(i)$

Case III: $\{B_1\} \rightarrow (p_1p_2)(p_1p_2)\,[p_1^{\alpha-2}] \quad \text{-------}\!\!\rightarrow P(\alpha-2)$

$\{B_2\} \rightarrow (p_1p_2)\,(p_1^2\, p_2)\,[p_1^{\alpha-3}] \text{-------}\!\!\rightarrow P(\alpha-3)$

.

.

.

$\{B_{\alpha-2}\} \rightarrow (p_1p_2)\quad (p_1^{\alpha-1}\, p_2)\,[p_1^{\alpha-\alpha}] \quad \text{-------}\!\!\rightarrow P(\alpha-\alpha) = P(0)$

Hence Case III contributes $\sum_{i=0}^{\alpha-2} P(i)$

Case IV: $\{C_1\} \rightarrow (p_1^2\, p_2)\,(p_1^2\, p_2)\,[p_1^{\alpha-4}] \quad \text{-------}\!\!\rightarrow P(\alpha-4)$

$\{C_2\} \rightarrow (p_1^2\, p_2)(p_1^3\, p_2)\,[p_1^{\alpha-5}] \quad \text{-------}\!\!\rightarrow P(\alpha-5)$

.

.

.

$\{C_{\alpha-4}\} \rightarrow (p_1^2\, p_2)(p_1^{\alpha-2}\, p_2)\,[p_1^{\alpha-\alpha}] \quad \text{-------}\!\!\rightarrow P(\alpha-\alpha) = P(0)$

Hence Case IV contributes $\sum_{i=0}^{\alpha-4} P(i)$

{ **NOTE:** The factor partition $(p_1^2 p_2)(p_1 p_2)[p_1^{\alpha-3}]$ has already been covered in case III hence is omitted in case IV. The same logic is extended to remaining (following) cases also.}

Case V: $\{D_1\} \rightarrow (p_1^3 p_2)(p_1^3 p_2)[p_1^{\alpha-6}]$ $\text{------}\rightarrow P(\alpha-4)$

$\{D_2\} \rightarrow (p_1^3 p_2)(p_1^4 p_2)[p_1^{\alpha-7}]$ $\text{------}\rightarrow P(\alpha-5)$

.

.

.

$\{D_{\alpha-4}\} \rightarrow (p_1^3 p_2)(p_1^{\alpha-3} p_2)[p_1^{\alpha-\alpha}]$ $\text{------}\rightarrow P(\alpha-\alpha) = P(0)$

Hence Case V contributes $\sum_{i=0}^{\alpha-6} P(i)$

On similar lines case VI contributes $\sum_{i=0}^{\alpha-8} P(i)$

we get contributions upto $\sum_{i=0}^{\alpha-2r} P(i)$

where $2r < \alpha < 2r+1$ or $r = [\alpha/2]$

summing up all the cases we get

$$F'(p_1^{\alpha} p_2^{2}) = F(\alpha,2) = \sum_{k=0}^{\alpha} P(k) + \sum_{j=0}^{r} \sum_{i=0}^{\alpha-2j} P(i)$$

where $\mathbf{r = [\alpha/2]}$ $\quad \boldsymbol{\alpha = 2r}$ **or** $\boldsymbol{\alpha = 2r+1}$

This completes the proof of theorem (5.1).

COROLLARY:(5.1)

$$F'(p_1^{\alpha}p_2^{2}) = \sum_{k=0}^{r} (k+2)\,[\,P(\alpha-2k) + P(\alpha-2k-1)] \qquad \text{-----(5.1)}$$

Proof: In theorem (5.1) consider the case $\alpha = 2r$, we have

$$F'(p_1^{2r}p_2^{2}) = F(\alpha,2) = \sum_{k=0}^{2r} P(k) + \sum_{j=0}^{r} \sum_{i=0}^{\alpha-2j} P(i) \qquad \text{-------(5.2)}$$

Second term on the RHS can be expanded as follows

$$\begin{array}{l}
P(\alpha) + P(\alpha-1) + P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0) \\
\qquad\qquad P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0) \\
. \qquad\qquad\qquad\qquad P(\alpha-4) + \ldots P(2) + P(1) + P(0) \\
. \\
. \\
\qquad\qquad\qquad\qquad\qquad\qquad P(2) + P(1) + P(0) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P(0)
\end{array}$$

summing up column wise

$$= [P(\alpha) + P(\alpha-1)] + 2\,[P(\alpha-2) + P(\alpha-3)] + 3\,[\,P(\alpha-4) + P(\alpha-5)] + \ldots$$

$$+ (r-1)\,[P(2) + P(1)] + r\,P(0).$$

$$= \sum_{k=0}^{r} (k+1)\,[\,P(\alpha-2k) + P(\alpha-2k-1)]$$

{Here $P(-1) = 0$ has been defined.}

hence

$$F'(p_1^{\alpha}p_2^{2}) = \sum_{k=0}^{r} P(k) + \sum_{k=0}^{r} (k+1)\,[\,P(\alpha-2k) + P(\alpha-2k-1)]$$

or

$$F'(p_1^{\alpha}p_2^{2}) = \sum_{k=0}^{r} (k+2)\,[\,P(\alpha-2k) + P(\alpha-2k-1)]$$

Consider the case $\alpha = 2r+1$, the second term in the expression (5.2) can be expanded as

$$
\begin{array}{r}
P(\alpha) + P(\alpha-1) + P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0) \\
P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0) \\
. \qquad P(\alpha-4) + \ldots P(2) + P(1) + P(0) \\
. \\
. \\
P(3) + P(2) + P(1) + P(0) \\
P(1) + P(0)
\end{array}
$$

summing up column wise we get

$= [P(\alpha) + P(\alpha-1)] + 2\,[P(\alpha-2) + P(\alpha-3)] + 3\,[P(\alpha-4) + P(\alpha-5)] + \ldots$

$+ (r-1)\,[P(3) + P(2)] + r[P(1) + P(0)].$

$$= \sum_{k=0}^{r} (k+1)\,[P(\alpha-2k) + P(\alpha-2k-1)], \quad \alpha = 2r+1$$

on adding the first term , we get

$$F'(p_1^{\alpha}p_2^{2}) = \sum_{k=0}^{r} (k+2)\,[P(\alpha-2k) + P(\alpha-2k-1)]$$

{**Note** here P(-1) shall not appear.}
Hence for all values of α we have

$$\mathbf{F'(p_1^{\alpha}p_2^{2}) = \sum_{k=0}^{[\alpha/2]} (k+2)\,[P(\alpha-2k) + P(\alpha-2k-1)]}$$

This completes the proof of the Corollary (5.1).

REFERENCES:

[1] "Amarnath Murthy" , 'Generalization Of Partition Function, Introducing 'Smarandache Factor Partition', SNJ, Vol. 11, No. 1-2-3, 2000.

[2] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

[3] 'Smarandache Notion Journal' Vol. 10 ,No. 1-2-3, Spring 1999. Number Theory Association of the UNIVERSITY OF CRAIOVA.

LENGTH / EXTENT OF SMARANDACHE FACTOR PARTITIONS

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION (**SFP**) , as follows:

Let $\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r$ be a set of r natural numbers and $p_1 , p_2, p_3 , \ldots p_r$ be arbitrarily chosen distinct primes then $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha 1} \; p_2^{\alpha 2} \; p_3^{\alpha 3} \ldots p_r^{\alpha r}$$ could be expressed as the product of its' divisors. For simplicity , we denote $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r) = F'(N)$,where

$$N = p_1^{\alpha_1} \; p_2^{\alpha_2} \; p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 =2$, $p_2 = 3$ etc.

Also for the case

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1$$

we denote

$$F(\underbrace{1 , 1, 1, 1, 1 \ldots}_{\leftarrow \text{ n - ones } \rightarrow}) = F(1\#n)$$

In the present note we define two interesting parameters the

length and **extent** of an **SFP** and study the interesting properties they exhibit for square free numbers.

DISCUSSION:

DEFINITION: Let $F'(N) = R$

LENGTH : If we denote each SFP of N , say like F_1, F_2, . . . F_R arbtrarily and let F_k be the SFP representation of N as the product of its divisors as follows:

F_k ---- $N = (h_1)(h_2)(h_3)(h_4)...(h_t)$, where each h_i ($1 < i < t$) is an entity in the SFP ' F_k ' of N. Then $T(F_k) = t$ is defined as the '**length**' of the factor partition F_k.

e.g. say $60 = 15$ **X** 2 **X** 2 , is a factor partition F_k of 60. Then

$$T(F_k) = 3.$$

$T(F_k)$ can also be defined as one more than the number of product signs in the factor partition.

EXTENT : The extent of a number is defined as the sum of the lengths of all the SFPs.

Consider F(1#3)

$N = p_1p_2p_3 = 2 \times 3 \times 5 = 30.$

SN	Factor Partition	length
1	30	1
2	15 X 2	2
3	10 X 3	2
4	6 X 5	2
5	5 X 3 X 2	3

Extent (30) = $\sum$ length = 10

We observe that

F(1#4) - F(1#3) = 10. = Extent { F(1#4)}

Consider F(1#4)

N = 2 X 3 X 5 X 7 = 210

SN	Factor Partition	Length
1	210	1
2	105 X 2	2
3	70 X 3	2
4	42 X 5	2
5	35 X 6	2
6	35 X 3 X 2	3
7	30 X 7	2
8	21 X 10	2
9	21 X 5 X 2	3
10	15 X 14	2
11	15 X 7 X 2	3
12	14 X 5 X 2	3
13	10 X 7 X 3	3
14	7 X 6 X 5	3
15	7 X 5 X 3 X 2	4

Extent(210) = $\sum$ length = 37

We observe that

F(1#5) - F(1#4) = 37. =Extent { F(1#4)}

Similarly it has been verified that

F(1#6) - F(1#5) = Extent { F(1#5)}

CONJECTURE (6.1)

F(1#(n+1)) - F(1#n) = Extent { F(1#n)}

CONJECTURE (6.2)

$$F(1\#(n+1)) = \sum_{r=0}^{n} \text{Extent } \{F(1\#r)\}$$

Motivation for this conjecture:

If conjecture (1) is true then we would have

F(1#2) - F(1#1) = Extent { F(1#1)}

F(1#3) - F(1#2) = Extent { F(1#2)}

F(1#4) - F(1#3) = Extent { F(1#3)}

.

.

.

F(1#(n+1)) - F(1#n) = Extent { F(1#n)}

Summing up we would get

$$F(1\#(n+1)) - F(1\#1) = \sum_{r=1}^{n} \text{Extent } \{F(1\#r)$$

F(1#1) = 1 = Extent {F(1#0) can be taken , hence we get

$$\mathbf{F(1\#(n+1)) = \sum_{r=0}^{n} Extent \ \{F(1\#r)}$$

Another Interesting Observation:

Given below is the chart of r versus w where w is the number of

SFPs having same length r.

F(1#0) = 1 , Σ r. w = 1

r	1
w	1

F(1#1) = 1 , Σ r. w = 1

r	1
w	1

F(1#2) = 2 , Σ r. w = 3

r	1	2
w	1	1

F(1#3) = 5 , Σ r. w = 10

r	1	2	3
w	1	3	1

F(1#4) = 15 , $\sum$ r. w = 37

r	1	2	3	4
w	1	7	6	1

F(1#5) = 52, $\sum$ r. w = 151

r	1	2	3	4	5
w	1	15	25	10	1

The interesting observation is the row of w is the same as the n^{th} row of the **SMARANDACHE STAR TRIANGLE. (ref.: [4])**

CONJECTURE (6.3)

$$w_r = a_{(n,r)} = (1/r!) \sum_{k=0}^{r} (-1)^{r-k} \,.\, {}^rC_k \,.\, k^n$$

where w_r is the number of SFPs of F(1#n) having length r.

Further Scope: One can study the length and contents of other cases (other than the square-free numbers.) explore for patterns if any.

REFERENCES:

[1] "Amarnath Murthy" , 'Generalization Of Partition Function, Introducing 'Smarandache Factor Partition', SNJ, Vol. 11, No. 1-2-3, 2000.

[2] "Amarnath Murthy" , 'A General Result On The "Smarandache Star Function" , SNJ, Vol. 11, No. 1-2-3, 2000.

[3] "Amarnath Murthy" , 'More Results And Applications Of The Generalized Smarandache Star Function' SNJ,.1999.

[4] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

SOME MORE IDEAS ON SMARANDACHE FACTOR PARTITIONS

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION (**SFP**) , as follows:

Let α_1 , α_2 , α_3 , . . . α_r be a set of r natural numbers and p_1 , p_2, p_3 ,. . .p_r be arbitrarily chosen distinct primes then $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha 1} \; p_2^{\alpha 2} \; p_3^{\alpha 3} \ldots p_r^{\alpha r}$$ could be expressed as the product of its' divisors. For simplicity , we denote $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r) = F'(N)$,where

$$N = p_1^{\alpha_1} \; p_2^{\alpha_2} \; p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 =2$, $p_2 = 3$ etc.

In this note another result pertaining to SFPs has been derived.

DISCUSSION:

Let

$$N = p_1^{\alpha_1} \; p_2^{\alpha_2} \; p_3^{\alpha_3} \ldots p_r^{\alpha_r}$$

(1) L(N) = length of that factor partition of N which contains the maximum number of terms. In this case we have

$$L(N) = \sum_{i=1}^{r} \alpha_i$$

(2)

$A_{L(N)}$ = A set of L(N) distinct primes.

(3) $B(N) = \{ p: p \mid N , p \text{ is a prime.} \}$

$B(N) = \{ p_1 ; p_2 , \ldots p_r \}$

(4) $\Psi[N, A_{L(N)}] = \{ x \mid d(x) = N \text{ and } B(x) \subseteq A_{L(N)} \}$, where d(x) is the number of divisors of x.

To derive an expression for the order of the set $\Psi[N, A_{L(N)}]$ defined above.

There are F'(N) factor partitions of N. Let F_1 be one of them.

$F_1 \longrightarrow N = s_1 \times s_2 \times s_3 \times \ldots \times s_t.$

if

$$\theta = p_1^{s_1-1} p_2^{s_2-1} p_3^{s_3-1} \ldots p_t^{s_t-1} p_{t+1}^{0} p_{t+2}^{0} \ldots p_{L(N)}^{0}$$

where $p_t \in A_{L(N)}$, then $\theta \in \Psi[N, A_{L(N)}]$ for

$d(\theta) = s_1 \times s_2 \times s_3 \times \ldots \times s_t \times 1 \times 1 \times 1 \ldots = N$

Thus each factor partition of N generates a few elements of Ψ .

Let $E(F_1)$ denote the number of elements generated by F_1

$F_1 \longrightarrow N = s_1 \times s_2 \times s_3 \times \ldots \times s_t.$

multiplying the right member with unity as many times as required to make the number of terms in the product equal to L(N) .

$$N = \prod_{k=1}^{L(N)} s_k$$

where $s_{t+1} = s_{t+2} = s_{t+3} = \ldots = s_{L(N)} = 1$

Let x_1 s's are equal

x_2 s's are equal

.

.

.

x_m s's are equal

such that $x_1 + x_2 + x_3 + \ldots + x_m = L(N)$. Where any x_i can be unity also.

Then we get

$E(F_1) = \{L(N)\}! / \{(x_1)!(x_2)!(x_3)! \ldots (x_m)!\}$

summing over all the factor partitions we get

$$\mathbf{O(\Psi[\,N,\,A_{L(N)}]) = \sum_{k=1}^{F'(N)} E(F_k)} \quad \text{-----------}\mathbf{(7.1)}$$

Example:

$N = 12 = 2^2.3$, $L(N) = 3$, $F'(N) = 4$

Let $A_{L(N)} = \{2,3,5\}$

$F_1 \text{-----}\rightarrow N = 12 = 12 \times 1 \times 1$, $x_1 = 2$, $x_2 = 1$

$E(F_1) = 3! / \{(2!)(1!)\} = 3$

$F_2 \text{-----}\rightarrow N = 12 = 6 \times 2 \times 1$, $x_1 = 1$, $x_2 = 1$, $x_3 = 1$

$E(F_2) = 3! / \{(1!)(1!)(1!)\} = 6$

$F_3 \text{-----}\rightarrow N = 12 = 4 \times 3 \times 1$, $x_1 = 1$, $x_2 = 1$, $x_3 = 1$

$E(F_3) = 3! / \{(1!)(1!)(1!)\} = 6$

$F_4 \text{-----}\rightarrow N = 12 = 3 \times 2 \times 2$, $x_1 = 1$, $x_2 = 2$

$E(F_4) = 3! / \{(2!)(1!)\} = 3$

$$O(\Psi[N, A_{L(N)}]) = \sum_{k=1}^{F'(N)} E(F_k) = 3 + 6 + 6 + 3 = 18$$

To verify we have

$\Psi[N, A_{L(N)}] = \{ 2^{11}, 3^{11}, 5^{11}, 2^5 X 3, 2^5 X 3, 3^5 X 2, 3^5 X 5, 5^5 X 2,$
$5^5 X 3, 2^3 X 3^2, 2^3 X 5^2, 3^3 X 2^2, 3^3 X 5^2, 5^3 X 2^2, 5^3 X 3^2, 2^2 X 3 X 5,$
$3^2 X 2 X 5, 5^2 X 2 X 3, \}$

REFERENCES:

[1] "Amarnath Murthy" , 'Generalization Of Partition Function, Introducing 'Smarandache Factor Partition', SNJ, Vol. 11, No. 1-2-3, 2000.

[2] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.

A NOTE ON THE SMARANDACHE DIVISOR SEQUENCES

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION (**SFP**) , as follows:

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r$ be a set of r natural numbers and $p_1, p_2, p_3, \ldots p_r$ be arbitrarily chosen distinct primes then $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ is defined as the number of ways in which the number

$N = p_1^{\alpha 1}\ p_2^{\alpha 2}\ p_3^{\alpha 3} \ldots p_r^{\alpha r}$ could be expressed as the product of its' divisors. For simplicity , we denote $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N)$,where

$$N = p_1^{\alpha_1}\ p_2^{\alpha_2}\ p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 = 2$, $p_2 = 3$ etc.

In [2] we have defined SMARANDACHE DIVISOR SEQUENCES as follows

$\mathbf{P_n = \{ x \mid d(x) = n \}}$, $d(x)$ = number of divisors of n.

$P_1 = \{1\}$

$P_2 = \{ x \mid x \text{ is a prime} \}$

$P_3 = \{ x \mid x = p^2 ,\ p \text{ is a prime} \}$

$P_4 = \{ x \mid x = p^3 \text{ or } x = p_1 p_2 ,\ p, p_1, p_2 \text{ are primes} \}$

Let F_1 be a SFP of N . Let $\Psi_{F1} = \{ y | d(y) = N \}$, generated by the SFP F_1 of N. It has been shown in Ref. [3] that each SFP generates some elements of Ψ or P_n. Here each SFP generates infinitely many elements of P_n . Similarly Ψ_{F1} , Ψ_{F2} , Ψ_{F3} ,. . . $\Psi_{F'(N)}$, are defined. It is evident that all these F_k's are disjoint and also

$P_N = \cup \Psi_{Fk} \quad 1 \leq k \leq F'(N)$.

THEOREM(7.1) There are F'(N) disjoint and exhaustive subsets in which P_N can be decomposed.

PROOF: Let $\theta \in P_N$, and let it be expressed in canonical form as follows

$$\theta = p_1^{\alpha_1} \, p_2^{\alpha_2} . \, p_3^{\alpha_3} \ldots p_r^{\alpha_r}$$

Then $\quad d(\theta) = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \ldots (\alpha_r+1)$

Hence $\theta \in \Psi_{Fk}$ for some k where F_k is given by

$$N = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \ldots (\alpha_r+1)$$

Again if $\theta \in \Psi_{Fs}$, and $\theta \in \Psi_{Ft}$ then from unique factorisation theorem F_s and F_t are identical SFPs of N.

REFERENCES:

[1] "Amarnath Murthy" , 'Generalization Of Partition Function, Introducing 'Smarandache Factor Partition', SNJ, Vol. 11, No. 1-2-3, 2000.

[2] "Amarnath Murthy" , 'Some New Smarandache Sequences, Functions And Partitions ', SNJ, Vol. 11, No. 1-2-3, 2000.

[3] "Amarnath Murthy" , ' Some more Ideas on SFPS. SNJ, Vol. 11, No. 1-2-3, 2000.

[4] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.

ALGORITHM FOR LISTING OF SMARANDACHE FACTOR PARTITIONS

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION (**SFP**) , as follows:

Let $\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r$ be a set of r natural numbers and $p_1 , p_2, p_3 ,\ldots p_r$ be arbitrarily chosen distinct primes then $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha 1}\, p_2^{\alpha 2}\, p_3^{\alpha 3} \ldots p_r^{\alpha r}$$ could be expressed as the product of its' divisors. For simplicity , we denote $F(\alpha_1 , \alpha_2 , \alpha_3 , \ldots \alpha_r) = F'(N)$,where

$$N = p_1^{\alpha_1}\, p_2^{\alpha_2}\, p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 =2$, $p_2 = 3$ etc.

In this note an algorithm to list out all the SFPs of a number without missing any is developed.

DISCUSSION:

DEFINITION: $F'_x(y)$ is defined as the number of those SFPs of y which involve terms not greater than x.

If F_1 be a factor partition of y :

$F_1 \longrightarrow x_1 \text{ X } x_2 \text{ X } x_3 \text{ X } \ldots x_r$, then F_1 is included in $F_x'(y)$ iff

$x_i \leq x$ for $1 \leq I \leq r$

clearly $F'_x(y) \leq F'(y)$, The equality holds good iff $x \geq y$.

Example: $F'_8(24) = 5$. Out of 7 only the last 5 are included in $F'_8(24)$.

(1) 24
(2) 12 X 2
(3) 8 X 3
(4) 6 X 4
(5) 6 X 2 X 2
(6) 4 X 3 X 2
(7) 3 X 2 X 2 X 2.

ALGORITHM: Let $d_1, d_2, d_3, \ldots d_r$ be the divisors of N in descending order. For listing the factor partitions following are the steps:

(A) (1) Start with $d_1 = N$.

(2) Write all the factor partitions involving d_2 and so on.

(B) While listing care should be taken that the terms from left to right should be written in descending order.

** At $d_k \geq N^{1/2} \geq d_{k+1}$, and onwards , step (B) will ensure that there is no repeatition.

Example: N = 36 , Divisors are 36 , 18, 12, 9 , 6, 4, 3 , 2 , 1.

36 ---→ 36
18 ---→ 18 X 2
12 ---→ 12 X 3
9 ---→ 9 X 4
9 X 2 X 2
6 ---→ 6 X 6
6 ---→ 6 X 3X 2
--------------------------------**$d_k = N^{1/2}$**
4 ---→ 4 X 3 X 3

3 --→3 X 3 X 2 X 2
2 --→NIL
1 --→NIL

FORMULA FOR F'(N)

$$F'(N) = \sum_{d_r/N} F'_{d_r}(N/d_r) \qquad \text{-------(8.1)}$$

Example:

$N = 216 = 2^3 3^3$

(1)	216	--→$F_{216}(1) = 1$
(2)	108 X 2	--→$F_{108}(2) = 1$
(3)	72 X 3	--→$F_{72}(3) = 1$
(4)	54 X 4	--→$F_{54}(4) = 2$
(5)	54 X 2 X 2	
(6)	36 X 6	--→$F_{36}(6) = 2$
(7)	36 X 3 X 2	
(8)	27 X 8	--→$F_{27}(8) = 3$
(9)	27 X 4 X 2	
(10)	27 X 2 X 2 X 2	
(11)	24 X 9	--→$F_{24}(9) = 2$
(12)	24 X 3 X 3	
(13)	18 X 12	--→$F_{18}(12) = 4$
(14)	18 X 6 X 2	
(15)	18 X 4 X 3	
(16)	18 X 3 X 2 X 2	
(17)	12 X 9 X 2	--→$F_{12}(18) = 3$
(18)	12 X 6 X 3	
(19)	12 X 3 X 3 X 2	
(20)	9 X 8 X 3	--→$F_9(24) = 5$
(21)	9 X 6 X 4	
(22)	9 X 6 X 2 X 2	
(23)	9 X 4 X 3 X 2	
(24)	9 X 3 X 2 X 2	
(25)	8 X 3 X 3 X 3	--→$F_8(27) = 1$
(26)	6 X 6 X 6	--→$F_6(36) = 4$
(27)	6 X 6 X 3 X 2	
(28)	6 X 4 X 3 X 3	
(29)	6 X 3 X 3 X 2 X 2	
(30)	4 X 3 X 3 X 3 X 2 X 2	--→$F_4(54) = 1$
(31)	3 X 3 X 3 X 2 X 2 X 2	--→$F_3(72) = 1$
		--→$F_2(108) = 0$
		--→$F_1(216) = 0$

$$F'(216) = \sum_{d_r/N} F'_{d_r}(216/d_r) = 31$$

Remarks: This algorithm would be quite helpfull in developing a computer program for the listing of SFPs.

REFERENCES:

[1] "Amarnath Murthy" , 'Generalization Of Partition Function, Introducing 'Smarandache Factor Partition', SNJ, Vol. 11, No. 1-2-3, 2000.

[2] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.

EXPANSION OF x^n IN SMARANDACHE TERMS OF PERMUTATIONS

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRACT:

DEFINITION of SMARANDACHE TERM

Consider the expansion of x^n as follows

$$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,n)}\, {}^xP_n \quad ---(9.1)$$

We define $b_{(n,r)}\, x(x-1)(x-2)\ldots(x-r+1)(x-r)$ as the r^{th} **SMARANDACHE TERM** in the above expansion of x^n .

In the present note we study the coefficients $b_{(n,r)}$.of the the r^{th} **SMARANDACHE TERM** in such an expansion. We are encountered with fascinating coincidences.

DISCUSSION:

Let us examine the coefficients $b_{(n,r)}$.of the the r^{th} **SMARANDACHE TERM** in such an expansion.

Taking $x = 1$ gives $b_{(n,1)} = 1$

Taking $x = 2$ gives $b_{(n,2)} = (2^n - 2)/2$

Taking $x = 3$ gives $b_{(n,3)} = \{3^n - 3 - 6(2^n - 2)/2\}/6$

$$= \{1/3!\}\{ (1).3^n - (3).\ 2^n + (3).\ (1)^n - (1)\ (0)^n \}$$

Taking $x = 4$ gives

$$b_{(n,4)} = (1/4!)\ [\ (1)\ 4^n - (4)\ 3^n + (6)\ 2^n - (4)\ 1^n + 1(0)^n\ ]$$

This suggests the possibility of

$$b_{(n,r)} = (1/r!) \sum_{k=1}^{r} (-1)^{r-k} . {}^rC_k . k^n = a_{(n,r)}$$

THEOREM (9.1)

$$b_{(n,r)} = (1/r!) \sum_{k=1}^{r} (-1)^{r-k} . {}^rC_k . k^n = a_{(n,r)}$$

First Proof:

This will be proved in two parts. First we shall prove the following proposition.

$$b_{(n+1,r)} = b_{(n,r-1)} + r.b_{(n,r)}$$

we have

$x^n = b_{(n,1)} x + b_{(n,2)} x(x-1) + b_{(n,3)} x(x-1)(x-2) +. . .+ b_{(n,n)} {}^xP_n$

$x = r$, gives,

$r^n = b_{(n,1)} r + b_{(n,2)} r(r-1) + b_{(n,3)} r(r-1)(r-2) +. . .+ b_{(n,n)} {}^rP_n$

multiplying both the sides by r ,

$r^{n+1} = b_{(n,1)} r.r + b_{(n,2)} r(r-1) + b_{(n,3)} r.r(r-1)(r-2) +. . .+ b_{(n,r)} r. {}^rP_r +$ terms equal to zero.

$r^{n+1} = b_{(n,1)} r.{}^rP_1 + b_{(n,2)} r .{}^rP_2 + b_{(n,3)} r. {}^rP_3 +. . .+ b_{(n,r)} r. {}^rP_r$

Using the identity $\mathbf{r. {}^rP_k = {}^rP_{k+1} + k. {}^rP_k}$ we can write

$r^{n+1} = b_{(n,1)} \{.{}^rP_2 + 1{}^rP_1\} + b_{(n,2)} \{{}^rP_3 + 2 .{}^rP_2\} +. . .+ b_{(n,r)} \{ {}^rP_r + r. {}^rP_{r-1} \}$

$r^{n+1} = b_{(n,1)} {}^rP_1 + \{ b_{(n,1)} + 2. b_{(n,2)}\} {}^rP_2 + \{ b_{(n,2)} + 3. b_{(n,3)}\} {}^rP_3 + . . . +$

$\{ b_{(n,r-1)} + r.\ b_{(n,r)}\}\ {}^rP_r$ ----------(9.2)

Otherwise also we have

$r^{n+1} = b_{(n+1,1)}\ {}^rP_1 + b_{(n+1,2)}\ .{}^rP_2 + b_{(n+1,3)}\ .\ {}^rP_3 + \ldots + b_{(n+1,r)}\ .\ {}^rP_r$

The coefficients of ${}^rP_t$ ($t < r$) are independent of r hence these can seperately be equated giving us

$$\mathbf{b_{(n+1,r)} = b_{(n,r-1)} + r.b_{(n,r)}}$$

Now we shall proceed by induction. Let

$$b_{(n,r)} = (1/r!) \sum_{k=0}^{r} (-1)^{r-k}\ .{}^rC_k\ .k^n$$

$$b_{(n,r-1)} = (1/(r-1)!) \sum_{k=0}^{r-1} (-1)^{r-1-k}\ .{}^{r-1}C_k\ .k^n$$

be true. Then the sum $b_{(n,r-1)} + r.b_{(n,r)}$ equals

$$(1/(r-1)!) \sum_{k=0}^{r-1} (-1)^{r-1-k}\ .{}^{r-1}C_k\ .k^n + r.\ (1/r!) \sum_{k=0}^{r} (-1)^{r-k}\ .{}^rC_k\ .k^n$$

$$= ((-1)^{r-1}/r!) \left[\sum_{k=0}^{r-1}(-1)^{-k}\ r\ \{ {}^{r-1}C_k - {}^rC_k\}k^n \right] + r^{n+1}/r!$$

$$= ((-1)^{r-1}/r!) \left[\sum_{k=0}^{r-1}(-1)^{-k}\ \{ -k.\ {}^rC_k\}k^n \right] + r^{n+1}/r!$$

$$=(1/r!) \sum_{k=0}^{r-1} (-1)^{r-k}\ \ {}^rC_k\ k^{n+1}$$

which gives us

$$b_{(n+1,r)} = (1/r!) \sum_{k=0}^{r-1} (-1)^{r-k}\ \ {}^rC_k\ k^{n+1}$$

$b_{(n+1,r)}$ also takes the same form. Hence by induction the proof is complete.

Second Proof: This proof is totally based on a combinatorial approach . **This method also provides us with a proof of the Conecture (6.3) of ref. [3] as a by product.**

If n objects no two alike are to be distributed in x boxes, no two alike, the number of ways this can be done is $\mathbf{x}^n$ since there are k alternatives for disposals of the first object, k alternatives for the disposal of the second, and so on.

Alternately let us proceed with a different approach. Let us consider the number of distributions in which exactly a given set of r boxes is filled (rest are empty.). Let it be called f(n,r).

We derive a formula for f(n,r) by using the inclusion exclusion principle.The method is illustrated by the computation of f(n,5). Consider the total number of arrangements, 5^n of n different objects in 5 different boxes. Say that such an arrangement has property 'a'. In case the first box is empty, property 'b' incase the second box is empty, and similar property 'c', 'd', and 'e' . for the other three boxes respectively. To find the number of distributions with no box empty, we simply count the number of distributions having none of the properties 'a', 'b', 'c', . . .etc. We can apply the following formula.

$$\mathbf{N - {}^rC_1.N(a) + {}^rC_2.N(a,b) - {}^rC_3.N(a,b,c) +. . . \text{------}(9.3)}$$

Here $N = 5^n$ is the total number of distributions. By N(a) we mean the number of distributions with the first box empty.and so $N(a) = 4^n$. Similarly N(a,b) is the number of distributions with the first two boxes empty. But this is the same as the number of distributions into 3 boxes and $N(a,b) = 3^n$. Thus we can write

$N = 5^n$, $N(a) = 4^n$, $N(a,b) = 3^n$ etc. $N(a,b,c,d,e) = 0$.

Applying formula (9.3) we get

$$f(n,5) = 5^n - {}^5C_1.4^n + {}^5C_2.3^n - {}^5C_3.2^n + {}^5C_4.1^n - {}^5C_5.0^n$$

by the direct generalization of this with r in place of 5 , we see that

$$\mathbf{f(n,r) = r^n - {}^rC_1.(r-1)^n + {}^rC_2.(r-2)^n - {}^rC_3.(r-3)^n + \ldots}$$

$$\mathbf{f(n,r) = \sum_{k=0}^{r} (-1)^k\, {}^rC_k\, (r-k)^n}$$

$\mathbf{f(n,r) = r!\ .\ a_{(n,r)}}$ **, from theorem (3.1). of ref. [1]**

Now these r boxes out of the given x boxes can be chosen in ${}^xC_r$ ways. Hence the total number of ways in which n distinct objects distributed in x distinct boxes occupying exactly r of them (with the rest x-r boxes empty) , defined as **d(n,r/x)** , is given by

$$\mathbf{d(n,r/x) = r!\ .\ a_{(n,r)}\, {}^xC_r}$$

$$\mathbf{d(n,r/x) = a_{(n,r)}.\, {}^xP_r}$$

Summing up all the cases for r =0 to r = x , the total number of ways in which n distinct objects can be distributed in x distinct boxes is given by

$$\sum_{r=0}^{x} d(n,r/x)) = \sum_{r=0}^{x} {}^{x}P_r\, a_{(n,r)} \qquad \text{-------}(9.4)$$

equating the two results obtained by two different approaches we get

$$x^n = \sum_{r=0}^{n} {}^{x}P_r\, a_{(n,r)}$$

REMARKS:

If n distinct objects are to be distributed in x distinct boxes with no box empty , then $n < x$ is mandetory for a possible distribution.e.g. 5 objects can not be placed in 7 boxes with no empty boxes (a sort of converse of peigon hole principle) Hence we get the following result

$$f(n,r) = 0, \quad \text{for } n < k.$$

$$f(n,r) = \sum_{k=0}^{r} (-1)^k\, {}^{r}C_k\, (r-k)^n = 0 \ \text{ if } \ n < r.$$

Further Generalisation:

(1) One can go ahead with the following generalisation of expansion of x^n as follows

$$x^n = g_{(n/k,1)}\, x + g_{(n/k,2)}\, x(x-k) + g_{(n/k,3)}\, x(x-k)(x-2k) + \ldots + g_{(n/k,n)}\ x(x-k)(x-2k)\ldots(x-(n-1)k)(x-nk+k)$$

$g_{(n/k,r)} = b_{(n,r)} = a_{(n,r)}$ for $k = 1$ has been dealt with in this note. One can explore for beautiful patterns for $k = 2, 3$ etc.

We can call (define) $g_{(n/k,r)}\, x(x-k)(x-2k)\ldots(x-(n-1)k)(x-rk+k)$ as the r^{th} **Smarandache Term of the** k^{th} **kind** in such an expansion.

(2) Another generalisation could be

$x^{n!} = c_{(n/k,1)}(x-k) + c_{(n/k,2)}(x-k)(x^2-k) + c_{(n/k,3)}(x-k)(x^2-k)(x^3-k) + \ldots + \ldots + c_{(n/k,n)}(x-k)(x^2-k)(x^3-k)\ldots(x^n-k)$

For $k = 1$ if we denote $c_{(n/k,r)} = c_{(n,r)}$ for simplicity we get

$x^{n!} = c_{(n,1)}(x-1) + c_{(n,2)}(x-1)(x^2-1) + c_{(n,3)}(x-1)(x^2-1)(x^3-1) + \ldots + \ldots + c_{(n,n)}(x-1)(x^2-1)(x^3-1)\ldots(x^n-1)$

We can define $c_{(n/k,r)}.(x-k)(x^2-k)(x^3-k)\ldots(x^r-k)$ as the **r^{th} Smarandache Factorial Term of the k^{th} kind** in the expansion of $x^{n!}$. One can again explore for patterns for the coefficient $c_{(n/k,r)}$.

REFERENCES:

[1] "Amarnath Murthy" , 'Generalization Of Partition Function, Introducing 'Smarandache Factor Partition', SNJ, Vol. 11, No. 1-2-3, 2000.

[2] "Amarnath Murthy" , 'A General Result On The "Smarandache Star Function" , SNJ, Vol. 11, No. 1-2-3, 2000.

[3] "Amarnath Murthy" , 'More Results And Applications Of The Generalized Smarandache Star Function' SNJ, Vol. 11, No. 1-2-3, 2000.

[4] "Amarnath Murthy" , ' Length / Extent of Smarandache Factor Partitions. , SNJ, Vol. 11, No. 1-2-3, 2000.

[5] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

MISCELLANEOUS RESULTS AND THEOREMS ON SMARANDACHE TERMS AND FACTOR PARTITIONS

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRACT: In **[1]** we define SMARANDACHE FACTOR PARTITION FUNCTION (**SFP**) , as follows:

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r$ be a set of r natural numbers and $p_1, p_2, p_3, \ldots p_r$ be arbitrarily chosen distinct primes then $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ is defined as the number of ways in which the number

$N = p_1^{\alpha 1} \; p_2^{\alpha 2} \; p_3^{\alpha 3} \ldots p_r^{\alpha r}$ could be expressed as the product of its' divisors. For simplicity , we denote $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N)$,where

$$N = p_1^{\alpha_1} \; p_2^{\alpha_2} \; p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}$$

and p_r is the r^{th} prime. $p_1 = 2$, $p_2 = 3$ etc.

Also for the case

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1$$

we denote

$$\underbrace{F(1, 1, 1, 1, 1 \ldots)}_{\leftarrow \text{ n - ones } \rightarrow} = F(1\#n)$$

In **[2]** we define $\mathbf{b_{(n,r)}\, x(x-1)(x-2)\ldots(x-r+1)(x-r)}$ as the $\mathbf{r^{th}}$ **SMARANDACHE TERM** in the expansion of

$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,n)}\, {}^{x}P_n$

In this note some more results depicting how closely the coefficients of the **SMARANDACHE TERM** and **SFPs** are related. are derived.

DISCUSSION:

Result on the $[i^j]$ matrix:

Theorem (9.1) in **[2]** gives us the following result

$$x^n = \sum_{r=0}^{n} {}^{x}P_r\, a_{(n,r)}$$ which leads us to the following

beautiful result.

$$\sum_{k=1}^{x} k^n = \sum_{k=1}^{x} \sum_{r=1}^{k} {}^{k}P_r\, a_{(n,r)}$$

In matrix notation the same can be written as follows for $x = 4 = n$.

$$\begin{bmatrix} {}^1P_1 & 0 & 0 & 0 \\ {}^2P_1 & {}^2P_2 & 0 & 0 \\ {}^3P_1 & {}^3P_2 & {}^3P_3 & 0 \\ {}^4P_1 & {}^4P_2 & {}^4P_3 & {}^4P_4 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1^1 & 1^2 & 1^3 & 1^4 \\ 2^1 & 2^2 & 2^3 & 2^4 \\ 3^1 & 3^2 & 3^3 & 3^4 \\ 4^1 & 4^2 & 4^3 & 4^4 \end{bmatrix}$$

In gerneral

$$\mathbf{P} * \mathbf{A'} = \mathbf{Q} \quad \text{where } \mathbf{P} = \left[{}^{i}P_j\right]_{nXn} \quad \text{-----(10.1)}$$

$$A = \left[a_{(i,j)}\right]_{nXn} \quad \text{and } \mathbf{Q} = \left[i^j\right]_{nXn}$$

(**A'** is the transpose of **A**)

Consider the expansion of x^n, again

$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,n)}\, {}^{x}P_n$

for $x = 3$ we get

$x^3 = b_{(3,1)}\, x + b_{(3,2)}\, x(x-1) + b_{(3,3)}\, x(x-1)(x-2)$

comparing the coefficient of powers of x on both sides we get

$$b_{(3,1)} - b_{(3,2)} + 2\, b_{(3,3)} = 0$$

$$b_{(3,2)} - 3\, b_{(3,3)} = 0$$

$$b_{(3,3)} = 1$$

In matrix form

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} b_{(3,1)} \\ b_{(3,2)} \\ b_{(3,3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_3 * A_3 = B_3$$

$$A_3 = C_3^{-1} . B_3$$

$$C_3^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(C_3^{-1})' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

similarly it has been observed that

$$(C_4^{-1})' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix}$$

The above observation leads to the following theorem.

THEOREM (10.1)

In the expansion of x^n as

$$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,n)}\, {}^xP_n$$

If C_n be the coefficient matrix of equations obtained by equating the coefficient of powers of x on both sides then

$$(C_n^{1})' = \left[a_{(i,j)}\right]_{nXn} = \text{star matrix of order n}$$

PROOF: It is evident that C_{pq} the element of the p^{th} row and q^{th} column of C_n is the coefficient of x^p in ${}^xP_q$. And also C_{pq} is independent of n. The coefficient of x^p on the RHS is

coefficient of $x^p = \sum_{q=1}^{n} b_{(n,q)}\, C_{pq}$, also

coefficient of $x^p = 1$ if $p = n$

coefficient of $x^p = \;=0$ if $p \neq n$.

in matrix notation

$$\text{coefficient of } x^p = \left[\sum_{q=1}^{n} b_{(n,q)} C_{pq}\right]$$

$$= \left[\sum_{q=1}^{n} b_{(n,q)} C'_{qp}\right]$$

$= i_{np}$ where $i_{np} = 1$, if $n = p$ and $i_{np} = 0$, if $n \neq p$.

$= I_n$ (identity matrix of order n.)

$$\left[b_{(n,q)}\right]\left[C_{p,q}\right]' = I_n$$

$$\left[a_{(n,q)}\right]\left[C_{p,q}\right]' = I_n \qquad \text{as } b_{(n,q)} = a_{(n,q)}$$

$$A_n . C_n' = I_n$$

$$A_n . = I_n [C_n']^{-1}$$

$$A_n . = [C_n']^{-1}$$

This completes the proof of theorem (10.1).

THEOREM (10.2)

If $C_{k,n}$ is the coefficient of x^k in the expansion of ${}^xP_n$, then

$$\sum_{k=1}^{n} F(1\#k)\, C_{k,n} = 1$$

PROOF: In property (3) of the STAR TRIANGLE following proposition has been established.

$F'(1\#n) = \sum_{m=1}^{n} a_{(n,m)} = B_n$, in matrix notation the same can be expressed as follows for $n = 4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix}$$

In general

$$\begin{bmatrix} 1 \end{bmatrix}_{1 \times n} * \underset{(c_n^{-1})'}{\begin{bmatrix} a_{(i,j)} \end{bmatrix}'_{n \times n}} = \begin{bmatrix} B_i \end{bmatrix}_{1 \times n}$$

$$\begin{bmatrix} B_i \end{bmatrix}_{1 \times n} * \underset{(c_n)}{\begin{bmatrix} \quad \end{bmatrix}_{n \times n}} = \begin{bmatrix} 1 \end{bmatrix}_{1 \times n}$$

In $C_{n,n}$, $C_{p,q}$ the p^{th} row and q^{th} column is the coefficient of x^p in ${}^xP_q$. Hence we have

$$\sum_{k=1}^{n} F(1\#k)\, C_{k,n} = 1 = \sum_{k=1}^{n} B_k\, C_{k,n}$$

THEOREM(10.3)

$$\sum_{k=1}^{n} F(1\#(k+1))\, C_{k,n} = n + 1 = \sum_{k=1}^{n} B_{k+1}\, C_{k,n}$$

PROOF:
It has already been established that

$$B_{n+1} = \sum_{m=1}^{n} (m+1)\; a_{(n,m)}$$

In matrix notation

$$\begin{bmatrix} j+1 \end{bmatrix}_{1 \times n} * \underset{(c_n^{-1})}{\begin{bmatrix} a_{(i,j)} \end{bmatrix}'_{n\times n}} = \begin{bmatrix} B_{j+1} \end{bmatrix}_{1 \times n}$$

$$\begin{bmatrix} j+1 \end{bmatrix}_{1 \times n} = \begin{bmatrix} B_{j+1} \end{bmatrix}_{1 \times n} * \underset{(c_n)}{\begin{bmatrix} \quad \end{bmatrix}_{n\times n}}$$

$$\sum_{k=1}^{n} B_{k+1}\, C_{k,n} = n + 1$$

There exist ample scope for more such results.

REFERENCES:

[1] "Amarnath Murthy" , 'Generalization Of Partition Function, Introducing 'Smarandache Factor Partition', SNJ, Vol. 11, No. 1-2-3,

2000.
[2] "Amarnath Murthy" , 'A General Result On The "Smarandache Star Function" , SNJ, Vol. 11, No. 1-2-3, 2000.

[3] "Amarnath Murthy" , 'More Results And Applications Of The Generalized Smarandache Star Function' SNJ, Vol. 11, No. 1-2-3, 2000.

[4] "Amarnath Murthy" , 'Expansion Of X^n In Smarandache Terms Of Permutations' SNJ, Vol. 11, No. 1-2-3, 2000.

[5] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

SMARANDACHE MAXIMUM RECIPROCAL REPRESENTATION FUNCTION

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

ABSTRACT: Smarandache Maximum Reciprocal Representation

(SMRR) Function $f_{SMRR}(n)$ is defined as follows

$f_{SMRR}(n) = t$ if

$$\sum_{r=1}^{t} 1/r \leq n \leq \sum_{r=1}^{t+1} 1/r$$

SMARANDACHE MAXIMUM RECIPROCAL REPRESENTATION SEQUENCE

SMRRS is defined as $T_n = f_{SMRR}(n)$

$f_{SMRR}(1) = 1$

$f_{SMRR}(2) = 3$, $\quad (1+ 1/2 + 1/3 < 2< 1+ 1/2 + 1/3 + 1/4)$

$f_{SMRR}(3) = 10$

$$\sum_{r=1}^{10} 1/r \leq 3 \leq \sum_{r=1}^{11} 1/r$$

SMRRS is

1, 3, 10, . . .

A note on The SMRR Function:

The harmonic series $\sum 1/n$ satisfies the following inequality

$$\mathbf{\log (n+1) < \sum 1/n < \log n + 1} \quad \text{-------(1)}$$

This inequality can be derived as follows
We have $e^x > 1 + x >$, $x > 0$

and $(1 + 1/n)^{(1 + 1/n)} > 1$, $n > 0$

which gives

$1/(r+1) < \log(1 + 1/r) < 1/r$

summing up for $r = 1$ to $n+1$ and with some algebraic jugallary we get (1). With the help of (1) we get the following result on the **SMRR function.**

If SMRR(n) = m then [log(m)] ≈ n - 1

Where [log(m)] stands for the integer value of log(m).

SOME CONJECTURES:

(1.1). Every positive integer can be expressed as the sum of the reciprocal of a finite number of distinct natural numbers. (in infinitely many ways.).

Let us define a function $R_m(n)$ as the minimum number of natural numbers required for such an expression.

(1.2). Every natural number can be expressed as the sum of the reciprocals of a set of natural numbers which are in Arithmetic Progression.

(1.3). Let

$\sum 1/r \leq n \leq \sum 1/(r+1)$

where $\sum 1/r$ stands for the sum of the reciprocals of first r natural numbers and let $S_1 = \sum 1/r$

let $S_2 = S_1 + 1/(r+k_1)$ such that $S_2 + 1/(r+k_1+1) > n \geq S_2$

let $S_3 = S_2 + 1/(r+k_2)$ such that $S_3 + 1/(r+k_2+1) > n \geq S_3$

and so on , then there exists a finite m such that

$$S_{m+1} + 1/(r+k_m) = n$$

Remarks : The veracity of conjecture (1.1) is deducible from conjecture (1.3) .

(1.4). (a) There are infinitely many disjoint sets of natural numbers sum of whose reciprocals is unity.

(b) Among the sets mentioned in (a) , there are sets which can be organised in an order such that the largest element of any set is smaller than the smallest element of the next set.

REFERENCES:

[1] 'Smarandache Notion Journal' Vol. 10 ,No. 1-2-3, Spring 1999. Number Theory Association of the UNIVERSITY OF CRAIOVA.

[2] "Amarnath Murthy" , 'Smarandache Reciprocal Partition Of Unity Sets And Sequences', SNJ, Vol. 11, No. 1-2-3, 2000.

[3] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

OPEN PROBLEMS AND CONJECTURES ON THE FACTOR /RECIPROCAL PARTITION THEORY:

(Amarnath Murthy ,S.E. (E &T), Well Logging Services,Oil And Natural Gas Corporation Ltd. ,Sabarmati, Ahmedbad, India- 380005.)

(1.1) To derive a formula for SFPs of given length m of $p^a q^a$ for any value of a.

(1.2) To derive a formula for SFPs of

$$N = p_1^2 \; p_2^2 \; p_3^2 \; . \; . \; . \; p_r^2$$

(1.3) To derive a formula for SFPs of given length m of

$$N = p_1^\alpha \; p_2^\alpha \; p_3^\alpha \; . \; . \; . \; p_r^\alpha$$

(1.4) To derive a reduction formula for $p^a q^a$ as a linear combination of $p^{a-r} q^{a-r}$ for r =0 to a-1.

Similar reduction formulae for (1.2) and (1.3) also.

(1.5) In general , in how many ways a number can be expressed as the product of its divisors?

(1.6). Every positive integer can be expressed as the sum of the reciprocal of a finite number of distinct natural numbers. (in infinitely many ways.).

Let us define a function $R_m(n)$ as the minimum number of natural numbers required for such an expression.

(1.7). Every natural number can be expressed as the sum of the reciprocals of a set of natural numbers which are in Arithmetic Progression.

(1.8). Let

$\sum 1/r \le n \le \sum 1/(r+1)$

where $\sum 1/r$ stands for the sum of the reciprocals of first r natural numbers and let $S_1 = \sum 1/r$

let $S_2 = S_1 + 1/(r+k_1)$ such that $S_2 + 1/(r+k_1+1) > n \ge S_2$

let $S_3 = S_2 + 1/(r+k_2)$ such that $S_3 + 1/(r+k_2+1) > n \ge S_3$

and so on , then there exists a finite m such that

$S_{m+1} + 1/(r+k_m) = n$

Remarks : The veracity of conjecture (1.6) is deducible from conjecture (1.8) .

(1.9). (a) There are infinitely many disjoint sets of natural numbers sum of whose reciprocals is unity.

(b) Among the sets mentioned in (a) , there are sets which can be organised in an order such that the largest element of any set is smaller than the smallest element of the next set.

DEFINITION: We can define **Smarandache Factor Partition Sequence** as follows : T_n = factor partion of $n = F'(n)$

$T_1 = 1$. $T_8 = 3$. $T_{12} = 4$ etc.

SFPS is given by

1,1,1,2,1,2,1,3,2,2,1,4,1,2,2,5, 1,4,1,4,2,2,1,7,2, . . .,

DEFINITION: Let S be the smallest number such that F'(S) = n. We define S a **Vedam Number** and the sequence formed by Vedam numbers as the **Smarandache Vedam Sequence**.

Smarandache Vedam Sequence is given as follows: T_n = F'(S)

1, 4, 8, 12, 16, **-?-** , 24 , . . .

Note: There exist no number whose factor partition is equal to 6. hence a question mark at the sixth slot. We define such numbers as **Dull numbers.** The readers can explore the distribution (frequency) and other properties of dull numbers.

DEFINITION: A number n is said to be a **Balu number** if it satisfies the relation d(n) = F'(n) = r , and is the smallest such number .

1, 16, 36 are all Balu numbers.

d(1) = F'(1) = 1 d(16) = F'(16) = 5, d(36) = F'(36) = 9.

Each Balu number ≥ 16 , generates a **Balu Class** C_B (n) of numbers having the same canonical form satisfying the equation d(m) = F'(m).e.g. $C_B(16) = \{ x \mid x = p^4$, p is a prime.$\}$ = { 16, 81, 256, . . .}. Similarly $C_B(36) = \{ x \mid x = p^2q^2$, p and q are primes.$\}$

Conjecture

(1.10): There are only finite number of Balu Classes.

In case Conjecture (1.10) is true , to find out the largest **Balu number.**

REFERENCES

[1] George E. Andrews, "Number Theory" , Dover Publications Inc. NewYork.

[2] 'Smarandache Notion Journal' Vol. 10 ,No. 1-2-3, Spring 1999. Number Theory Association of the UNIVERSITY OF CRAIOVA.

[3] V.Krishnamurthy, "COMBINATORICS Theory and Applications" , East West Press Private Ltd. ,1985.

[4] "Amarnath Murthy" , 'Generalization Of Partition Function, Introducing 'Smarandache Factor Partition', SNJ, Vol. 11, No. 1-2-3, 2000.

[5] "Amarnath Murthy" , 'A General Result On The "Smarandache Star Function" , SNJ, Vol. 11, No. 1-2-3, 2000.

[6] "Amarnath Murthy" , 'More Results And Applications Of The Generalized Smarandache Star Function' SNJ, Vol. 11, No. 1-2-3, 2000.

[7] "Amarnath Murthy" , ' Length / Extent of Smarandache Factor Partitions. , SNJ, Vol. 11, No. 1-2-3, 2000.

[8] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

SMARANDACHE RECIPROCAL FUNCTION AND AN ELEMENTARY INEQUALITY

(Amarnath Murthy , S.E.(E&T), Well Logging Services, ONGC , Sabarmati , Ahmedabad , INDIA)

The Smarandache Function is defined as $S(n) = k$. Where k is the smallest integer such that n divides $k!$

Let us define $S_c(n)$ **Smarandache Reciprocal Function** as follows:

$S_c(n) = x$ where $x + 1$ does not divide $n!$ and for every $y \leq x$, $y \mid n!$

THEOREM-I.

If $S_c(n) = x$,and $n \neq 3$, then $x + 1$ is the smallest prime greater than n.

PROOF: It is obvious that n! is divisible by 1 , 2, . . . up to n. We have to prove that n! is also divisible by $n + 1$, $n + 2$, . . . $n + m$ (= x) , where $n + m + 1$ is the smallest prime greater than n.. Let r be any of these composite numbers. Then r must be factorable into two factors each of which is ≥ 2. Let $r = p.q$, where $p , q \geq 2$. If one of the factors (say q) is $\geq n$ then $r = p.q \geq 2n$. But according to the Bertrand's postulate there must be a prime between n and 2n, there is a contradiction here since all the numbers from $n + 1$ to $n + m$ ($n + 1 \leq r < n + m$) are assumed to be composite. Hence neither of the two factors p , q can be $\geq n$. So each must be $< n$. Now there are two possibilities:

Case-I $p \neq q$.

In this case as each is $< n$ so $p.q = r$ divides $n!$

Case-II $p = q$ = prime

In this case $r = p^2$ where p is a prime. There are again three possibilities:

(a) $p \geq 5$. Then $r = p^2 > 4p \Rightarrow 4p < r < 2n \Rightarrow 2p < n$. Therefore both p and 2p are less than n so p^2 divides $n!$

(b) $p = 3$, Then $r = p^2 = 9$ Therefore n must be 7 or 8 . and 9 divides 7! and 8!.

(c) $p = 2$, then $r = p^2 = 4$. Therefore n must be 3 . But 4 does not divide 3! , Hence the theorem holds for all integral values of n except $n = 3$. This completes the proof.

Remarks: Readers can note that n! is divisible by all the composite numbers between n and 2n.

Note: We have the well known inequality $S(n) \leq n$. ---------(2)

From the above theorem one can deduce the following inequality.

If p_r is the r^{th} prime and $p_r \leq n < p_{r+1}$ then $S(n) \leq p_r$. (A slight improvement on (2)).

i.e. $S(p_r) = p_r$, $S(p_r + 1) < p_r$, $S(p_r + 2) < p_r$, . . . $S(p_{r+1} - 1) < p_r$, $S(p_{r+1})$
$= p_{r+1}$

Summing up for all the numbers $p_r \le n < p_{r+1}$ one gets

$$\sum_{t=0}^{p_{r+1} - p_r - 1} S (p_r + t) \le (p_{r+1} - p_r) p_r$$

Summing up for all the numbers up to the s^{th} prime, defining $p_0 = 1$, we get

$$\sum_{k=1}^{p_s} S (k) \le \sum_{r=0}^{s} (p_{r+1} - p_r) p_r \quad \text{--------(3)}$$

More generally from Ref. [1] following inequality on the nth partial sum of the Smarandache (Inferior) Prime Part Sequence directly follows.

Smarandache (Inferior) Prime Part Sequence

For any positive real number n one defines $p_p(n)$ as the largest prime number less than or equal to n. In [1] Prof. Krassimir T. Atanassov proves that the value of the n^{th} partial sum of this

sequence $X_n = \sum_{k=1}^{n} p_p(k)$ is given by

$$X_n = \sum_{k=2}^{\pi(n)} (p_k - p_{k-1}).\ p_{k-1} + (n - p_{\pi(n)} + 1).\ p_{\pi(n)} \quad \text{-----(4)}$$

From (3) and (4) we get

$$\sum_{k=1}^{n} S(k) \leq X_n$$

REFERENCES:

[1] "Krassimir T. Atanassov" , ' ON SOME OF THE SMARANDACHE'S PROBLEMS' AMERICAN RESEARCH PRESS Lupton, AZ USA. 1999. (22-23)

[2] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

[3] 'Smarandache Notion Journal' Vol. 10 ,No. 1-2-3, Spring 1999. Number Theory Association of the UNIVERSITY OF CRAIOVA

SOME FUNNY EXAMPLES OF SMARANDACHE LUCKY METHODS IN ALGEBRA , TRIGONOMETRY, AND CALCULUS

(Amarnath Murthy , S.E.(E&T), Well Logging Services,
ONGC , Sabarmati , Ahmedabad , INDIA)

ABSTRACT : A number is said to be a Smarandache Lucky Number if an incorrect calculation leads to a correct result. For example, the fraction 42/21 = (4-2)/(2-1) = 2/1 = 2 is incorrectly calculated, but the result 2 is still correct. More generally a Smarandache Lucky Method is said to be any incorrect method which leads to a correct result. In Ref. [1] The following question is asked:

(1) Are there infinitely many Smarandache Lucky Numbers ?

We claim that the answer is YES.

Also in the present note we give some fascinating Smarandache Lucky Methods in algebra , trigonometry, and calculus.

A SMARANDACHE LUCKY METHOD IN TRIGONOMETRY :

Some students at the early stage of just having introduced to the concept of function , misunderstand the meaning of f(x) as the product of f and x.
e.g. for them sin(x) = product of sin and x. This gives rise to a funny lucky method applicable to the following identity.

To prove

$$\sin^2(x) - \sin^2(y) = \sin(x + y) . \sin(x - y)$$

LHS = $\sin^2(x) - \sin^2(y)$

$$= \{\sin(x) + \sin(y)\} . \{\sin(x) - \sin(y)\} \quad \text{----------(A)}$$

Taking sin common from both the factors

$$= \{ \sin(x+y) \}.\{\sin(x-y)\}$$

$$= \text{RHS}$$

The correct method from (A) onwards should have been

$$\{2\sin((x+y)/2).\cos((x-y)/2)\}.\{2\cos((x+y)/2).\sin((x-y)/2)\}.$$

$$= \{2\sin((x+y)/2).\cos((x+y)/2)\}.\{2\cos((x-y)/2\sin((x-y)/2)\}.$$

$$= \{\sin(x+y)\}.\{\sin(x-y)\}$$

$= \text{RHS}$

Remarks : The funny thing is the wrong lucky method is a shortcut more so to get carried away.

A SMARANDACHE LUCKY METHOD IN ALGEBRA:

In vector algebra the dot product of two vectors $(a_1 i + a_2 j + a_3 k)$ and $(b_1 i + b_2 j + b_3 k)$ is given by

$$(a_1 i + a_2 j + a_3 k) \bullet (b_1 i + b_2 j + b_3 k) = a_1b_1 + a_2b_2 + a_3b_3$$

The same idea if extended to ordinary algebra would mean

$(a+b)(c+d) = ac + bd.$ ----------(B)

This wrong lucky method is applicable in proving the following algebraic identity.

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$RHS = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$= (a+b+c)\{(a^2 - bc) + (b^2 - ac) + (c^2 - ab)\}$$

applying the wrong lucky method (B) , one gets

$$= a.(a^2 - bc) + b.(b^2 - ac) + c.(c^2 - ab)$$

$$= a^3 - abc + b^3 - abc + c^3 - abc$$

$$= a^3 + b^3 + c^3 - 3abc = LHS$$

A SMARANDACHE LUCKY METHOD IN CALCULUS:

The fun involved in the following lucky method in calculus is two fold . It goes like this . A student is asked to differentiate the product of two functions. Instead of applying the formula for differentiation of product of two functions he applies the method of integration of the product of two functions (Integration by parts)and gets the correct answer. The height of coincidence is if another student given the same product of two functions and asked to integrate does the reverse of it i.e. he ends up in applying the formula for differentiation of the product of two functions and yet gets the correct answer. I would take the liberty to call such a lucky method to be **Smarandache superlucky method .** The suspense ends.

Consider the product of two functions x and sin (x) .

$f(x) = x$ and $g(x) = \sin(x)$

The Smarandache lucky method of differentiation (integration by parts) is

$$d\{f(x).g(x)\}/dx = f(x)\int g(x)dx - \int [\{d(f(x))/dx\}.\int g(x)dx]\mathbf{dx}$$

$$d\{(x).\sin(x)\}/dx = (x)\int \sin(x)dx - \int [\{d(x)/dx\}.\int \sin(x)dx]\mathbf{dx}$$

$= -(x).(\cos(x)) + \sin(x)$

$= -x.\cos(x) + \sin(x)$

The Smarandache lucky method of Integration

$$\int \{(f(x)).\ g(x)\}\mathbf{dx}. = f(x).d\{g(x)\}/dx + g(x)\ d\{f(x)\}/dx$$

Consider the same functions again we get by this lucky method

$$\int \{(x).\sin(x)\}\mathbf{dx}. = (x).\{\cos(x)\} + \{\sin(x)\}\ (1)$$

or $\int \{(x).\sin(x)\}\mathbf{dx}. = x.\cos(x) + \sin(x)$

That, both the answers are correct, can be verified, by applying the right methods.

REFERENCE:

[1] ' M.L.PEREZ and K.ATANASSOV' Definitions, Conjectures, and Unsolved problems on Smarandache Notions. American Research Press, 1999.

[2] 'F.SMARANDACHE', Funny Problems, Special Collections, Arizona State University, Hayden Library, Tempe, AZ, USA.

Review of

Lion Hunting & Other Mathematical Pursuits: A Collection of Mathematics, Verse, and Stories by Ralph P. Boas, Jr., edited by Gerald L. Alexanderson and Dale H. Mugler, The Mathematical Association of America, Washington, D. C., 1995. 320 pp., $35.00(paper). ISBN 0-88385-323-X.

As a collection of diverse persons, mathematicians suffer from more negative stereotypes than almost any other group. This is unfortunate, discouraging and most often wrong. Widely characterized as lacking in humor, abstract and considered to be brilliant, eccentric imbeciles by much of the public, mathematicians rarely fit that description. Of course, branding a group with a stereotype is often a mask for insecurities. Ralph P. Boas Jr. is a fascinating counterexample to most of these inaccurate assumptions. Filled with humor, verse and mathematics, his optimism and love of life are captured just like the lions so prominently featured in the book.

So, how does an unarmed person capture a lion using only the weapons of mathematical thought? There are more ways than you would think. Over thirty different "proven" methods are given. My favorite is: "The lion is big game, hence certainly a game. There exists an optimal strategy. Follow it." It seems that every area of mathematics can be used to construct a way to capture a lion. Of course, some are more efficient than others.

The verse varies from limericks to some that were seeded by material from Shakespeare. All are quite good, although it is necessary to read some of them twice in order to capture the intended meaning. Most mathematicians have heard of Nicolas Bourbaki, the mathematical polyglot who is in fact a pseudonym for a collection of French mathematicians. When it came time to publish the first material on the mathematics of lion hunting, Boas and his colleagues chose the pseudonym, Hector Petard, from the Shakespearean line, "the engineer, hoist with his own petard"; Hamlet Act III, Scene IV. To complete the circle, Boas and friends also "arranged" for a wedding between Betti Bourbaki and H. Petard and duly announced the upcoming event.

Another main section of the book consists of reminiscences by Boas and those who knew him best. As a mathematical man of mischief and an educator, he had few equals. Several short papers describing some of his basic ideas for education are also included. These ideas share one common trait. Simple to understand and execute. No fancy or complex methods, just fundamental strategies to make mathematics more understandable.

The final part of the book consists of short anecdotes about his experiences in mathematics. Some are about fellow mathematicians, others about students and the rest about whatever seemed to happen during his eventful life. At times amusing, other times profound, but at all times interesting, they are simple notes describing how the mathematical world works.

Despite common misconceptions, there are some mathematicians who contain a bit of the sprite and Ralph P. Boas Jr. was such a person. That impishness is captured in this book, which is reason enough to read it.

Review of

Non-Euclidean Geometry 6th Edition, by H. S. M. Coxeter, The Mathematical Association of America, Washington, D. C., 1998. 336 pp., $30.95(paper).
ISBN 0-88385-522-4.

Originally published in 1942, this book has lost none of its power in the last half century. It is a commentary on the recent demise of geometry in many curricula that 33 years elapsed between the publication of the fifth and sixth editions. Fortunately, like so many things in the world, trends in mathematics are cyclic, and one can hope that the geometric cycle is on the rise. We in mathematics owe so much to geometry. It is generally conceded that much of the origins of mathematics is due to the simple necessity of maintaining accurate plots in settlements. The only book from the ancient history of mathematics that all mathematicians have heard of is the **Elements** by Euclid. It is one of the most read books of all time, arguably the only book without a religious theme still in widespread use over 2000 years after the publication of the first edition. The geometry taught in high schools today is with only minor modifications found in the Euclidean classic.

There are other reasons why geometry should occupy a special place in our hearts. Most of the principles of the axiomatic method, the concept of the theorem and many of the techniques used in proofs were born and nurtured in the cradle of geometry. For many centuries, it was nearly an act of faith that all of geometry was Euclidean. That annoying fifth postulate seemed so out of place and yet it could not be made to go away. Many tried to remove it, but finally the Holmsean dictum of ,"once you have eliminated the impossible, what is left, not matter how improbable, must be true", had to be admitted. There were in fact three geometries, all of which are of equal validity. The other two, elliptic and hyperbolic, are the main topics of this wonderful book.

Coxeter is arguably the best geometer of this century but there can be no argument that he is the best explainer of geometry of this century. While fifty years is a mere spasm compared to the time since Euclid, it is certainly possible that students will be reading Coxeter far into the future with the same appreciation that we have when we read Euclid. His explanations of the non-Euclidean geometries is so clear that one cannot help but absorb the essentials. In so many ways, Euclidean geometry is but the middle way between the two other geometries. A point well made and in great detail by Coxeter.

Geometry is a jewel that was born on the banks of the Nile river and we should treasure and respect it as the seed from which so much of our basic reasoning processes sprouted. For this reason, you should buy this book and keep a copy on your shelf.

Reviewed by

Charles Ashbacher
Charles Ashbacher Technologies
e-mail 71603.522@compuserve.com

Krassimir T. Atanassov, "On Some of the Smarandache's Problems". Lupton, AZ: American Research Press, 1999, 88pp, ISBN 1-879585-72-3. (Box 199, Lupton, AZ 86508, USA.)

This attractively, but inexpensively, produced monograph introduces the reader to the flavour of the problems proposed in recent years by Smarandache and others. These are generally of two kinds: those which deal with recurring patterns within sequences and those which extend classical number theoretic results. Those who are familiar with the author's style will recognise his ingenuity in the latter and his tenacity in the former.

An example of the former is Smarandache Problem 22: "In the sequence of Smarandache Square Complements:
{1,2,3,1,5,6,7,2,1,10,11,3,14,15,1,17,2,19,5,
21,22,23,6,1,26,3,7,29,30,31,2,33,34,35,1,37,
38,39,10,41,42,43,11,5,46,47,3,1,2,51,13,53,6,55,
14,57,58,59,15,61,62,7,1,65,66,67,17,69,70,71,2,...}, for
each integer n find the smallest integer k such that nk is a perfect square."

An extension of a classical result is Smarandache Problem 117: "Let p be an odd positive number. Prove that p and p+2 are twin primes iff

$$(p-1)!\{(1/p)+(2/(p+2))\}+(1/p)+(1/(p+2)) \text{ is an integer."}$$

Atanassov develops and utilises properties of "new" functions such as the inverse factorial function defined by: x? = y iff y! =x, and his digit sum which has appeared in a number of papers over the last decade. These ensure that his solutions are always elegant rather than the result of brute force. Readers might like to try their hands at the above two problems and then buy the book to enjoy more of these problems which are easy to understand but not always easy to solve.

The book is in microfilm format too, and can be ordered from: UMI, PO Box 1346, Ann Arbor, MI 48106-1346, USA;
tel: 1-800-521-0600.

Tony Shannon PhD EdD DSc
Professor Emeritus, University of Technology, Sydney, 2007, and
Provost, KvB Institute of Technology, North Sydney, 2060, Australia

C O N T E N T S

ARTICLES:

The "Smarandache Notions" is partially online at:
http://www.gallup.unm.edu/~smarandache/
and selected papers are periodically added to our web site. The authors are pleased to send, together with their hard copy manuscripts, a floppy disk with HTML or ASCHII files to be put in the Internet as well.

Papers in electronic format are accepted. They can be e-mailed in Microsoft Word 97 (or lower) for Windows 98, WordPerfect 6.0 (or lower) for Windows 95, HTML, or text files.

$49.95