SMARANDACHE ANTI-GEOMETRY

by

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Abstract: This is an experimental geometry. All Hilbert's 20 axioms of the Euclidean GGeometry are denied in this vanguardist geometry of the real chaos! What is even more intriguing? F.Smarandache[5] has even found in 1969 a model of it!

Key Words: Hilbert's Axioms, Euclidean Geometry, Non-Euclidean Geometry, Smarandache Geometries, Geometrical Model

Introduction:

Here it is exposed the Smarandache Anti-Geometry:
It is possible to entirely de-formalize Hilbert's groups of axioms of the Euclidean Geometry, and to construct a model such that none of his fixed axioms holds.

Let's consider the following things:

- a set of <points>: A, B, C, ...
- a set of <lines>: h, k, l, ...
- a set of <planes>: alpha, beta, gamma, ...

and

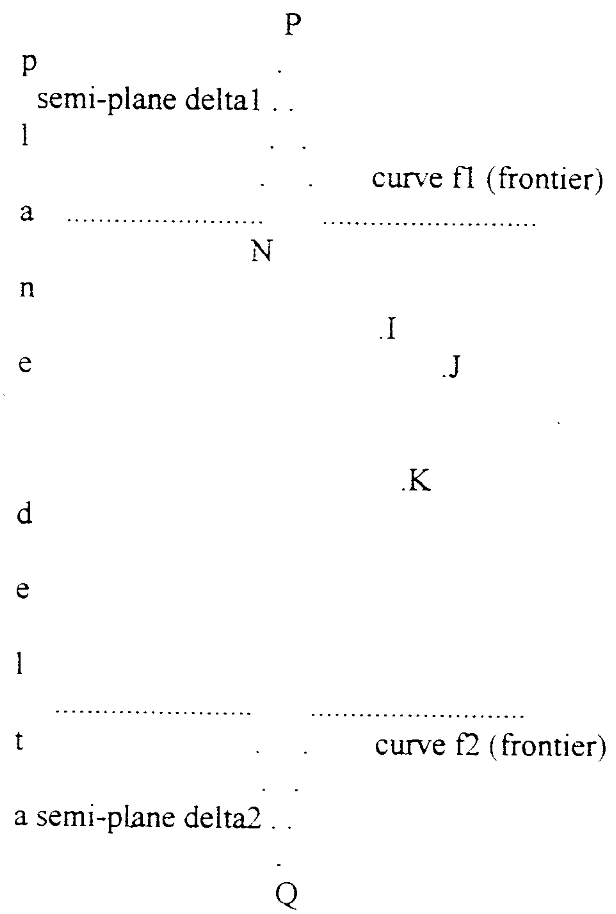
- a set of relationships among these elements: "are situated", "between", "parallel", "congruent", "continuous", etc.

Then, we can deny all Hilbert's twenty axioms [see David Hilbert, "Foundations of Geometry", translated by E. J. Townsend, 1950; and Roberto Bonola, "Non-Euclidean Geometry", 1938].
There exist cases, within a geometric model, when the same axiom is verified by certain points/lines/planes and denied by others.

GROUP I. ANTI-AXIOMS OF CONNECTION:

I.1. Two distinct points A and B do not always completely determine a line.

Let's consider the following model MD:
get an ordinary plane delta, but with an infinite hole in of the following shape:



Plane delta is a reunion of two disjoint planar semi-planes;
f1 lies in MD, but f2 does not;
P, Q are two extreme points on f that belong to MD.

One defines a LINE l as a geodesic curve: if two points A, B that belong to MD lie in l, then the shortest curve lied in MD between A and B lies in l also.
If a line passes two times through the same point, then it is called double point (KNOT).

One defines a PLANE alpha as a surface such that for any two points A, B that lie in alpha and belong to MD there is a geodesic which passes through A, B and lies in alpha also.

Now, let's have two strings of the same length:
one ties P and Q with the first string s1 such that the curve s1 is folded in two or more different planes and s1 is under the plane delta;
next, do the same with string s2, tie Q with P, but over the plane delta and such that s2 has a different form from s1;
and a third string s3, from P to Q, much longer than s1.
s1, s2, s3 belong to MD.

Let I, J, K be three isolated points -- as some islands, i.e. not joined with any other point of MD, exterior to the plane delta.

This model has a measure, because the (pseudo-)line is the shortest way (lenth) to go from a point to another (when possible).

Question 37:

Of course, this model is not perfect, and is far from the best. Readers are asked to improve it, or to make up a new one that is better.

(Let A, B be two distinct points in delta1-f1. P and Q are two points on s1, but they do not completely determine a line, referring to the first axiom of Hilbert, because A-P-s1-Q are different from B-P-s1-Q.)

I.2. There is at least a line l and at least two distinct points A and B of l, such that A and B do not completely determine the line l.

(Line A-P-s1-Q are not completely determined by P and Q in the previous construction, because B-P-s1-Q is another line passing through P and Q too.)

I.3. Three points A, B, C not situated in the same line do not always completely determine a plane alpha

(Let A, B be two distinct points in delta1-f1, such that A, B, P are not co-linear. There are many planes containing these three points: delta1 extended with any surface s containing s1, but not cutting s2 in between P and Q, for example.)

I.4. There is at least a plane, alpha, and at least three points A, B, C in it not lying in the same line, such that A, B, C do not completely determine the plane alpha.

(See the previous example.)

I.5. If two points A, B of a line l lie in a plane alpha, doesn't mean that every point of l lies in alpha.

(Let A be a point in delta1-f1, and B another point on s1 in between P and Q. Let alpha be the following plane: delta1 extended with a surface s containing s1, but not cutting s2 in between P and Q, and tangent to delta2 on a line QC, where C is a point in delta2-f2.
Let D be point in delta2-f2, not lying on the line QC.
Now, A, B, D are lying on the same line A-P-s1-Q-D, A, B are in the plane alpha, but D do not.)

I.6. If two planes alpha, beta have a point A in common, doesn't mean they have at least a second point in common.

(Construct the following plane alpha: a closed surface containing s1 and s2, and intersecting delta1 in one point only, P. Then alpha and delta1 have a single point in common.)

I.7. There exist lines where lies only one point,
or planes where lie only two points,
or space where lie only three points.

(Hilbert's I.7 axiom may be contradicted if the model has discontinuities.
Let's consider the isolated points area.
The point I may be regarded as a line, because it's not possible to add any new point to I to form a line.
One constructs a surface that intersects the model only in the points I and J.)

GROUP II. ANTI-AXIOMS OF ORDER:

II.1. If A, B, C are points of a line and B lies between A and C, doesn't mean that always B lies also between C and A.

[Let T lie in s1, and V lie in s2, both of them closer to Q, but different from it. Then:
P, T, V are points on the line P-s1-Q-s2-P
(i.e. the closed curve that starts from the point P and lies in s1 and passes through the point Q and lies back to s2 and ends in P),
and T lies between P and V
-- because PT and TV are both geodesics --,
but T doesn't lie between V and P
-- because from V the line goes to P and then to T, therefore P lies between V and T.]

[By definition: a segment AB is a system of points lying upon a line between A and B (the extremes are included).

Warning: AB may be different from BA;
for example:
the segment PQ formed by the system of points starting with P, ending with Q, and lying in s1, is different from the segment QP formed by the system of points starting with Q, ending with P, but belonging to s2.
Worse, AB may be sometimes different from AB;
for example:
the segment PQ formed by the system of points starting with P, ending with Q, and lying in s1, is different from the segment PQ formed by the system of points starting with P, ending with Q,

but belonging to s2.]

II.2. If A and C are two points of a line, then:
there does not always exist a point B lying between A and C,
or there does not always exist a point D such that C lies between A and D.

[For example:
let F be a point on f1, F different from P,
and G a point in delta1, G doesn't belong to f1;
draw the line l which passes through G and F;
then:
there exists a point B lying between G and F
-- because GF is an obvious segment --,
but there is no point D such that F lies between G and D -- because GF is right bounded in F
(GF may not be extended to the other side of F, because otherwise the line will not remain a geodesic anymore).]

II.3. There exist at least three points situated on a line such that:
one point lies between the other two,
and another point lies also between the other two.

[For example:
let R, T be two distinct points, different from P and Q, situated on the line P-s1-Q-s2-P,
such that the lenghts PR, RT, TP are all equal;
then:
R lies between P and T,
and T lies between R and P;
also P lies between T and R.]

II.4. Four points A, B, C, D of a line can not always be arranged:
such that B lies between A and C and also between A and D,
and such that C lies between A and D and also between B and D.

[For examples:
- let R, T be two distinct points, different from P and Q, situated on the line P-s1-Q-s2-P such that the lenghts PR, RQ, QT, TP are all equal, therefore R belongs to s1, and T belongs to s2;
then P, R, Q, T are situated on the same line:
such that R lies between P and Q, but not between P and T
-- because the geodesic PT does not pass through R --,
and such that Q does not lie between P and T
-- because the geodesic PT does not pass through Q --,
but lies between R and T;
- let A, B be two points in delta2-f2 such that A, Q, B are colinear, and C, D two points on s1, s2 respectively, all of the four points being different from P and Q;
then A, B, C, D are points situated on the same line A-Q-s1-P-s2-Q-B, which is the same with line A-Q-s2-P-s1-Q-B, therefore we may have two different orders of these four points in the same time:
A, C, D, B and A, D, C, B.]

II.5. Let A, B, C be three points not lying in the same line, and l a line lying in the same plane ABC and not passing through any of the points A, B, C. Then, if the line l passes through a point of the segment AB, it doesn't mean that always the line l will pass through either a point of the segment BC or a point of the segment AC.

[For example:
let AB be a segment passing through P in the semi-plane delta1, and C a point lying in delta1 too on the left side of the line AB;
thus A, B, C do not lie on the same line;
now, consider the line Q-s2-P-s1-Q-D, where D is a point lying in the semi-plane delta2 not on f2:
therefore this line passes through the point P of the segment AB, but do not pass through any point of the segment BC, nor through any point of the segment AC.]

GROUP III. ANTI-AXIOM OF PARALLELS.

In a plane alpha there can be drawn through a point A, lying outside of a line l, either no line, or only one line, or a finite number of lines, or an infinite number of lines which do not intersect the line l. (At least two of these situations should occur.) The line(s) is (are) called the parallel(s) to l through the given point A.

[For examples:
- let l0 be the line N-P-s1-Q-R, where N is a point lying in delta1 not on f1, and R is a similar point lying in delta2 not on f2, and let A be a point lying on s2, then: no parallel to l0 can be drawn through A (because any line passing through A, hence through s2, will intersect s1, hence l0, in P and Q);
- if the line l1 lies in delta1 such that l1 does not intersect the frontier f1, then: through any point lying on the left side of l1 one and only one parallel will pass;
- let B be a point lying in f1, different from P, and another point C lying in delta1, not on f1; let A be a point lying in delta1 outside of BC; then: an infinite number of parallels to the line BC can be drawn through the point A.

Theorem. There are at least two lines l1, l2 of a plane, which do not meet a third line l3 of the same plane, but they meet each other, (i.e. if l1 is parallel to l3, and l2 is parallel to l3, and all of them are in the same plane, it's not necessary that l1 is parallel to l2).
[For example:
consider three points A, B, C lying in f1, and different from P, and D a point in delta1 not on f1; draw the lines AD, BE and CE such that E is a point in delta1 not on f1 and both BE and CE do not intersect AD;
then: BE is parallel to AD, CE is also parallel to AD, but BE is not parallel to CE because the point E belong to both of them.]

GROUP IV. ANTI-AXIOMS OF CONGRUENCE

IV.1. If A, B are two points on a line l, and A' is a point upon the same or another line l', then: upon a given side of A' on the line l', we can not always find only one point B' so that the segment AB is congruent to the segment A'B'.

[For examples:
- let AB be segment lying in delta1 and having no point in common with f1, and construct the line C-P-s1-Q-s2-P (noted by l') which is the same with C-P-s2-Q-s1-P, where C is a point lying in delta1 not on f1 nor on AB;
take a point A' on l', in between C and P, such that A'P is smaller than AB;
now, there exist two distinct points B1' on s1 and B2'on s2, such that A'B1' is congruent to AB and A'B2' is congruent to AB,
with A'B1' different from A'B2';
- but if we consider a line l' lying in delta1 and limited by the frontier f1 on the right side (the limit point being noted by M),
and take a point A' on l', close to M, such that A'M is less than A'B', then: there is no point B' on the right side of l' so that A'B' is congruent to AB.]

A segment may not be congruent to itself!

[For example:
- let A be a point on s1, closer to P,
and B a point on s2, closer to P also;
A and B are lying on the same line A-Q-B-P-A which is the same with line A-P-B-Q-A,
but AB meseared on the first representation of the line is strictly greater than AB meseared on the second representation of their line.]

IV.2. If a segment AB is congruent to the segment A'B' and also to the segment A"B", then not always the segment A'B' is congruent to the segment A"B".

[For example:
- let AB be a segment lying in delta1-f1, and consider the line C-P-s1-Q-s2-P-D, where C, D are two distinct points in delta1-f1 such that C, P, D are colinear. Suppose that the segment AB is congruent to the segment CD (i.e. C-P-s1-Q-s2-P-D). Get also an obvious segment A'B' in delta1-f1, different from the preceding ones, but congruent to AB.
Then the segment A'B' is not congruent to the segment CD (considered as C-P-D, i.e. not passing through Q.)

IV.3. If AB, BC are two segments of the same line l which have no points in common aside from the point B,
and A'B', B'C' are two segments of the same line or of another line l' having no point other than B' in common, such that AB is congruent to A'B' and BC is congruent to B'C',
then not always the segment AC is congruent to A'C'.

[For example:
let l be a line lying in delta1, not on f1, and A, B, C three distinct points on l, such that AC is greater than s1;
let l' be the following line: A'-P-s1-Q-s2-P where A' lies in delta1, not on f1, and get B' on s1 such that A'B' is congruent to AB, get C' on s2 such that BC is congruent to B'C' (the points A, B, C are thus chosen);
then: the segment A'C' which is first seen as A'-P-B'-Q-C' is not congruent to AC, because A'C' is the geodesic A'-P-C' (the shortest way from A' to C' does not pass through B') which is strictly less than AC.]

Definitions. Let h, k be two lines having a point O in common. Then the system (h, O, k) is called the angle of the lines h and k in the point O.
(Because some of our lines are curves, we take the angle of the tangents to

the curves in their common point.)

The angle formed by the lines h and k situated in the same plane, noted by <(h, k), is equal to the arithmetic mean of the angles formed by h and k in all their common points.

IV.4. Let an angle (h, k) be given in the plane alpha, and let a line h' be given in the plane beta. Suppose that in the plane beta a definite side of the line h' be assigned, and a point O'. Then in the plane beta there are one, or more, or even no half-line(s) k' emanating from the point O' such that the angle (h, k) is congruent to the angle (h', k'),
and at the same time the interior points of the angle (h', k') lie upon one or both sides of h'.

[Examples:

- Let A be a point in delta1-f1, and B, C two distinct points in delta2-f2;
  let h be the line A-P-s1-Q-B, and k be the line A-P-s2-Q-C; because h and k intersect in an infinite number of points (the segment AP), where they normally coincide -- i.e. in each such point their angle is congruent to zero, the angle (h, k) is congruent to zero.
  Now, let A' be a point in delta1-f1, different from A, and B' a point in delta2-f2, different from B, and draw the line h' as A'-P-s1-Q-B'; there exist an infinite number of lines k', of the form A'-P-s2-Q-C' (where C' is any point in delta2-f2, not on the line QB'), such that the angle (h, k) is congruent to (h', K'), because (h', k') is also congruent to zero, and the line A'-P-s2-Q-C' is different from the line A'-P-s2-Q-D' if D' is not on the line QC'.
- If h, k, and h' are three lines in delta1-P, which intersect the frontier f1 in at most one point, then there exist only one line k' on a given part of h' such that the angle (h, k) is congruent to the angle (h', k').

- *Is there any case when, with these hypotheses, no 'k' exists ?
- Not every angle is congruent to itself; for example:

  <(s1, s2) is not congruent to <(s1, s2)

  [because one can construct two distinct lines: P-s1-Q-A and P-s2-Q-A, where A is a point in delta2-f2, for the first angle, which becomes equal to zero;
  and P-s1-Q-A and P-s2-Q-B, where B is another point in delta2-f2, B different from A, for the second angle, which becomes strictly greater than zero!].

IV. 5. If the angle (h, k) is congruent to the angle (h', k',) and the angle (h", k"), then the angle (h', k') is not always congruent to the angle (h", k").

(A similar construction to the previous one.)

IV. 6. Let ABC and A'B'C' be two triangles such that

AB is congruent to A'B',
AC is congruent to A'C',
<BAC is congruent to <B'A'C'.

Then not always

<ABC is congruent to <A'B'C'
and <ACB is congruent to <A'C'B'.

[For example:
Let M, N be two distinct points in delta2-f2, thus obtaining the triangle PMN;
Now take three points R, M', N' in delta1-f1, such that RM' is congruent to PM, RN' is congruent to RN, and the angle (RM', RN') is congruent to the angle (PM, PN). RM'N' is an obvious triangle.
Of course, the two triangle are not congruent, because for example PM and PN cut each other twice -- in P and Q -- while RM' and RN' only once -- in R.
(These are geodesical triangles.)]

Definitions:

Two angles are called supplementary if they have the same vertex, one side in common, and the other sides not common form a line.

A right angle is an angle congruent to its supplementary angle.

Two triangles are congruent if its angles are congruent two by two, and its sides are congruent two by two.

Propositions:

A right angle is not always congruent to another right angle.

For example:
Let A-P-s1-Q be a line, with A lying in delta1-f1, and B-P-s1-Q another line, with B lying in delta1-f1 and B not lying in the line AP;
we consider the tangent t at s1 in P, and B chosen in a way that <(AP, t) is not congruent to <(BP, t);
let A', B' be other points lying in delta1-f1 such that <APA' is congruent to <A'P-s1-Q, and <BPB' is congruent to <B'P-s1-Q.
Then:
- the angle APA' is right, because it is congruent to its supplementary (by construction);
- the angle BPB' is also right, because it is congruent to its supplementary (by construction);
- but <APA' is not congruent to <BPB', because the first one is half of the angle A-P-s1-Q, i.e. half of <(AP, t),
while the second one is half of the B-P-s1-Q, i.e. half of <(BP, t).

The theorems of congruence for triangles [side, side, and angle in between; angle, angle, and common side; side, side, side] may not hold either in the Critical Zone (s1, s2, f1, f2) of the Model.

Property:
The sum of the angles of a triangle can be:
- 180 degrees, if all its vertexes A, B, C are lying, for example, in delta1-f1;

- strictly less than 180 degrees [any value in the interval (0, 180)],
for example:
let R, T be two points in delta2-f2 such that Q does not lie in RT, and S another point on s2;
then the triangle SRT has <(SR, ST) congruent to 0 because SR and ST have an infinite number of common points (the segment SQ), and <QTR + <TRQ congruent to 180-<TQR [by construction we may vary <TQR in the interval (0, 180)];
- even 0 degree!
let A be a point in delta1-f1, B a point in delta2-f2, and C a point on s3, very close to P;
then ABC is a non-degenerate triangle (because its vertexes are non-colinear), but <(A-P-s1-Q-B, A-P-s3-C) = <(B-Q-s1-P-A, B-Q-s1-P-s3-C) = <(C-s3-P-A, C-s3-P-s1-Q-B) = 0
(one considers the lenth C-s3-P-s1-Q-B strictly less than C-s3-B);
the area of this triangle is also 0 !
- more than 180 degrees,
for example:
let A, B be two points in delta1-f1, such that <PAB + <PBA + <(s1, s2; in Q) is strictly greater than 180 degrees;
then the triangle ABQ, formed by the intersection of the lines A-P-s2-Q, Q-s1-P-B, AB will have the sum of its angles strictly greater than 180 degrees.

Definition:
A circle of center M is a totality of all points A for which the segments MA are congruent to one another.

For example, if the center is Q, and the lenth of the segments MA is chosen greater than the lenth of s1, then the circle is formed by the arc of circle centered in Q, of radius MA, and lying in delta2, plus another arc of circle centered in P, of radius MA-lenth of s1, lying in delta1.

GROUP V. ANTI-AXIOM OF CONTINUITY (ANTI-ARCHIMEDEAN AXIOM)

Let A, B be two points. Take the points A1, A2, A3, A4, ... so that A1 lies between A and A2, A2 lies between

A1 and A3, A3 lies between A2 and A4, etc. and the segments AA1, A1A2, A2A3, A3A4, ... are congruent to one another.
Then, among this series of points, not always there exists a certain point An such that B lies between A and An.

For example:
let A be a point in delta1-f1, and B a point on f1, B different from P;
on the line AB consider the points A1, A2, A3, A4, ...
in between A and B, such that AA1, A1A2, A2A3, A3A4, etc. are congruent to one another;
then we find that there is no point behind B (considering the direction from A to B), because B is a limit point (the line AB ends in B).

The Bolzano's (intermediate value) theorem may not hold in the Critical Zone of the Model.

Can you readers find a better model for this anti-geometry?

References:

[1] Charles Ashbacher, "Smarandache Geometries", <Smarandache Notions Journal>, Vol. 8, No. 1-2-3, Fall 1997, pp. 212-215.
[2] Mike Mudge, "A Paradoxist Mathematician, His Function, Paradoxist Geometry, and Class of Paradoxes", <Smarandache Notions Journal>, Vol. 7, No. 1-2-3, August 1996, pp. 127-129.
reviewed by David E. Zitarelli, <Historia Mathematica>, Vol. 24, No. 1, p. 114, #24.1.119, 1997.
[3] Marian Popescu, "A Model for the Smarandache Paradoxist Geometry", <Abstracts of Papers Presented to the American Mathematical Society Meetings>, Vol. 17, No. 1, Issue 103, 1996, p. 265.
[4] Florentin Smarandache, "Collected Papers" (Vol. II), University of Kishinev Press, Kishinev, pp. 5-28, 1997.
[5] Florentin Smarandache, "Paradoxist Mathematics" (lecture), Bloomsburg University, Mathematics Department, PA, USA, November 1985.