

Smarandache Continued Fractions

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Abstract: The theory of general continued fractions is developed to the extent required in order to calculate Smarandache continued fractions to a given number of decimal places. Proof is given for the fact that Smarandache general continued fractions built with positive integer Smarandache sequences having only a finite number of terms equal to 1 is convergent. A few numerical results are given.

Introduction

The definitions of Smarandache continued fractions were given by Jose Castillo in the Smarandache Notions Journal, Vol. 9, No 1-2 [1].

A Smarandache Simple Continued Fraction is a fraction of the form:

$$a(1) + \frac{1}{a(2) + \frac{1}{a(3) + \frac{1}{a(4) + \frac{1}{a(5) + \dots}}}}$$

where $a(n)$, for $n \geq 1$, is a Smarandache type Sequence, Sub-Sequence or Function.

Particular attention is given to the Smarandache General Continued Fraction defined as

$$a(1) + \frac{b(1)}{a(2) + \frac{b(2)}{a(3) + \frac{b(3)}{a(4) + \frac{b(4)}{a(5) + \dots}}}}$$

where $a(n)$ and $b(n)$, for $n \geq 1$, are Smarandache type Sequences, Sub-Sequences or Functions.

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As a particular case the following example is quoted

$$1 + \frac{1}{12 + \frac{21}{123 + \frac{321}{1234 + \frac{4321}{12345 + \dots}}}}$$

Here 1, 12, 123, 1234, 12345, ... is the Smarandache Consecutive Sequences and 1, 21, 321, 4321, 54321, ... is the Smarandache Reverse Sequence.

The interest in Castillo's article is focused on the calculation of such fractions and their possible convergents when the number of terms approaches infinity. The theory of simple continued fractions is well known and given in most standard textbooks in Number Theory. A very comprehensive theory of continued fractions, including general continued fractions is found in *Die Lehre von den Kettenbrüchen* [2]. The symbols used to express facts about continued fractions vary a great deal. The symbols which will be used in this article correspond to those used in Hardy and Wright *An Introduction to the Theory of Numbers* [3]. However, only simple continued fractions are treated in the text of Hardy and Wright. Following more or less the same lines the theory of general continued fractions will be developed in the next section as far as needed to provide the necessary tools for calculating Smarandache general continued fractions.

General Continued Fractions

We define a finite general continued fraction through

$$C_n = q_0 + \frac{r_1}{q_1 + \frac{r_2}{q_2 + \frac{r_3}{q_3 + \frac{r_4}{q_4 + \dots \frac{r_n}{q_n}}}}} = q_0 + \frac{r_1}{q_1 + \frac{r_2}{q_2 + \frac{r_3}{q_3 + \frac{r_4}{q_4 + \dots \frac{r_n}{q_n}}}}} \quad (1)$$

where $\{q_0, q_1, q_2, \dots, q_n\}$ and $\{r_1, r_2, r_3, \dots, r_n\}$ are integers which we will assume to be positive.

The above definition is an extension of the definition of a simple continued fraction where $r_1=r_2=\dots=r_n=1$. The theory developed here will apply to simple continued fractions as well by replacing r_k ($k=1, 2, \dots$) in formulas by 1 and simply ignoring the reference to r_k when not relevant.

The formula (1) will usually be expressed in the form

$$C_n = [q_0, q_1, q_2, q_3, \dots, q_n, r_1, r_2, r_3, \dots, r_n] \quad (2)$$

For a simple continued fraction we would write

$$C_n = [q_0, q_1, q_2, q_3, \dots, q_n] \quad (2')$$

If we break off the calculation for $m \leq n$ we will write

$$C_m = [q_0, q_1, q_2, q_3, \dots, q_m, r_1, r_2, r_3, \dots, r_m] \quad (3)$$

Equation (3) defines a sequence of finite general continued fractions for $m=1, m=2, m=3, \dots$. Each member of this sequence is called a **convergent** to the continued fraction

Working out the general continued fraction in stages, we shall obviously obtain expressions for its convergents as quotients of two sums, each sum comprising various products formed with $q_0, q_1, q_2, \dots, q_m$ and r_1, r_2, \dots, r_m .

If $m=1$, we obtain the first convergent

$$C_1 = [q_0, q_1, r_1] = q_0 + \frac{r_1}{q_1} = \frac{q_0 q_1 + r_1}{q_1} \quad (4)$$

For $m=2$ we have

$$C_2 = [q_0, q_1, q_2, r_1, r_2] = q_0 + \frac{q_2 r_1}{q_1 q_2 + r_2} = \frac{q_0 q_1 q_2 + q_0 r_2 + q_2 r_1}{q_1 q_2 + r_2} \quad (5)$$

In the intermediate step the value of $q_1 + \frac{r_2}{q_2}$ from the previous calculation has been quoted, putting q_1, q_2 and r_2 in place of q_0, q_1 and r_1 . We can express this by

$$C_2 = [q_0, [q_1, q_2, r_2], r_1] \quad (6)$$

Proceeding in the same way we obtain for $m=3$

$$C_3 = [q_0, q_1, q_2, q_3, r_1, r_2, r_3] = q_0 + \frac{(q_2 q_3 + r_3) r_1}{q_1 q_2 q_3 + q_1 r_3 + q_3 r_2} = \frac{q_0 q_1 q_2 q_3 + q_0 q_1 r_3 + q_0 q_3 r_2 + q_2 q_3 r_1 + r_1 r_3}{q_1 q_2 q_3 + q_1 r_3 + q_3 r_2} \quad (7)$$

or generally

$$C_m = [q_0, q_1, \dots, q_{m-2}, [q_{m-1}, q_m, r_m], r_1, r_2, \dots, r_{m-1}] \quad (8)$$

which we can extend to

$$C_n = [q_0, q_1, \dots, q_{m-2}, [q_{m-1}, q_m, \dots, q_n, r_m, \dots, r_n], r_2, r_2, \dots, r_{m-1}] \quad (9)$$

Theorem 1:

Let A_m and B_m be defined through

$$\begin{aligned} A_0 &= q_0, & A_1 &= q_0 q_1 + r_1, & A_m &= q_m A_{m-1} + r_m A_{m-2} \quad (2 \leq m \leq n) \\ B_0 &= 1, & B_1 &= q_1, & B_m &= q_m B_{m-1} + r_m B_{m-2} \quad (2 \leq m \leq n) \end{aligned} \quad (10)$$

then $C_m = [q_0, q_1, \dots, q_m, r_1, \dots, r_m] = \frac{A_m}{B_m}$, i.e. $\frac{A_m}{B_m}$ is the m th convergent to the general continued fraction.

Proof: The theorem is true for $m=0$ and $m=1$ as is seen from $[q_0] = \frac{q_0}{1} = \frac{A_0}{B_0}$ and $[q_0, q_1, r_1] = \frac{q_0 q_1 + r_1}{q_1} = \frac{A_1}{B_1}$. Let us suppose that it is true for a given $m < n$. We will induce that it is true for $m+1$

$$\begin{aligned} [q_0, q_1, \dots, q_{m+1}, r_1, \dots, r_{m+1}] &= [q_0, q_1, \dots, q_m, [q_m, q_{m+1}, r_{m+1}], r_1, \dots, r_m] \\ &= \frac{[q_m, q_{m+1}, r_{m+1}] A_{m-1} + r_m A_{m-2}}{[q_m, q_{m+1}, r_{m+1}] B_{m-1} + r_m B_{m-2}} \\ &= \frac{(q_m + \frac{r_{m+1}}{q_{m+1}}) A_{m-1} + r_m A_{m-2}}{(q_m + \frac{r_{m+1}}{q_{m+1}}) B_{m-1} + r_m B_{m-2}} \\ &= \frac{q_{m+1} (q_m A_{m-1} + r_m A_{m-2}) + r_{m+1} A_{m-1}}{q_{m+1} (q_m B_{m-1} + r_m B_{m-2}) + r_{m+1} B_{m-1}} \\ &= \frac{q_{m+1} A_m + r_{m+1} A_{m-1}}{q_{m+1} B_m + r_{m+1} B_{m-1}} = \frac{A_{m+1}}{B_{m+1}} \end{aligned}$$

□

The recurrence relations (10) provide the basis for an effective computer algorithm for successive calculation of the convergents C_m .

Theorem 2:

$$A_m B_{m-1} - B_m A_{m-1} = (-1)^{m-1} \prod_{k=1}^m r_k \quad (11)$$

Proof: For $m=1$ we have $A_1 B_0 - B_1 A_0 = q_0 q_1 + r_1 - q_0 q_1 = r_1$.

$$\begin{aligned} A_m B_{m-1} - B_m A_{m-1} &= (q_m A_{m-1} + r_m A_{m-2}) B_{m-1} - (q_m B_{m-1} + r_m B_{m-2}) A_{m-1} \\ &= -r_m (A_{m-1} B_{m-2} - B_{m-1} A_{m-2}) \end{aligned}$$

By repeating this calculation with $m-1, m-2, \dots, 2$ in place of m , we arrive at

$$A_m B_{m-1} - B_m A_{m-1} = \dots = (A_1 B_0 - B_1 A_0) (-1)^{m-1} \prod_{k=2}^m r_k = (-1)^{m-1} \prod_{k=1}^m r_k$$

□

Theorem 3:

$$A_m B_{m-2} - B_m A_{m-2} = (-1)^m q_m \prod_{k=1}^{m-1} r_k \quad (12)$$

Proof: This theorem follows from theorem 3 by inserting expressions for A_m and B_m

$$\begin{aligned} A_m B_{m-2} - B_m A_{m-2} &= (q_m A_{m-1} + r_m A_{m-2}) B_{m-2} - (q_m B_{m-1} + r_m B_{m-2}) A_{m-2} \\ &= q_m (A_{m-1} B_{m-2} - B_{m-1} A_{m-2}) = (-1)^m q_m \prod_{k=1}^{m-1} r_k \end{aligned}$$

□

Using the symbol $C_m = \frac{A_m}{B_m}$ we can now express important properties of the number sequence $C_m, m=1, 2, \dots, n$. In particular we will be interested in what happens to C_n as n approaches infinity.

From (11) we have

$$C_n - C_{n-1} = \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = \frac{(-1)^{n-1} \prod_{k=1}^n r_k}{B_{n-1} B_n} \quad (13)$$

while (12) gives

$$C_n - C_{n-2} = \frac{A_n}{B_n} - \frac{A_{n-2}}{B_{n-2}} = \frac{(-1)^{n-1} q_n \prod_{k=1}^{n-1} r_k}{B_{n-2} B_n} \quad (14)$$

We will now consider infinite positive integer sequences $\{q_0, q_1, q_2, \dots\}$ and $\{r_1, r_2, \dots\}$ where only a finite number of terms are equal to 1. This is generally the case for Smarandache sequences. We will therefore prove the following important theorem.

Theorem 4:

A general continued fraction for which the sequences q_0, q_1, q_2, \dots and r_1, r_2, \dots are positive integer sequences with at most a finite number of terms equal to 1 is convergent.

Proof: We will first show that the product $B_{n-1} B_n$, which is a sum of terms formed by various products of elements from $\{q_1, q_2, \dots, q_n, r_1, r_2, \dots, r_{n-1}\}$, has one term which is a multiple of $\prod_{k=2}^n r_k$. Looking at the process by which we calculated C_1, C_2 , and C_3 , equations

4, 5 and 7, we see how terms with the largest number of factors r_k evolve in numerators and denominators of C_k . This is made explicit in figure 1.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
Num. A_m	r_1	$q_0 r_2$	$r_1 r_3$	$q_0 r_2 r_4$	$r_1 r_3 r_5$	$q_0 r_2 r_4 r_6$	$r_1 r_3 r_5 r_7$	$q_0 r_2 r_4 r_6 r_8$
Den. B_m	-	r_2	$q_1 r_3$	$r_2 r_4$	$q_1 r_3 r_5$	$r_2 r_4 r_6$	$q_1 r_3 r_5 r_7$	$r_2 r_4 r_6 r_8$

Figure 1. The terms with the largest number of r-factors in numerators and denominators.

As is seen from figure 1 two consecutive denominators $B_n B_{n-1}$ will have a term with $r_2 r_3 \dots r_n$ as factor. This means that the numerator of (13) will not cause $C_n - C_{n-1}$ to diverge. On the other hand $B_{n-1} B_n$ contains the term $(q_1 q_2 \dots q_{n-1})^2 q_n$ which approaches ∞ as $n \rightarrow \infty$. It follows that $\lim_{n \rightarrow \infty} (C_n - C_{n-1}) = 0$.

From (14) we see that

1. If n is odd, say $n=2k+1$, then $C_{2k+1} < C_{2k-1}$ forming a monotonously decreasing number sequence which is bounded below (positive terms). It therefore has limit.

$$\lim_{k \rightarrow \infty} C_{2k+1} = C_1.$$

2. If n is even, $n=2k$, then $C_{2k} > C_{2k-2}$ forming a monotonously increasing number sequence. This sequence has an upper bound because $C_{2k} < C_{2k+1} \rightarrow C_1$ as $k \rightarrow \infty$. It therefore has limit.

$$\lim_{k \rightarrow \infty} C_{2k} = C_2.$$

3. Since $\lim_{n \rightarrow \infty} (C_n - C_{n-1}) = 0$ we conclude that $C_1 = C_2$. Consequently $\lim_{n \rightarrow \infty} C_n = C$ exists.

□

Calculations

A *UBASIC* program has been developed to implement the theory of Smarandache general continued fractions. The same program can be used for classical continued fractions since these correspond to the special case of a general continued fraction where $r_1 = r_2 = \dots = r_n = 1$.

The complete program used in the calculations is given below. The program applies equally well to simple continued fractions by setting all element of the array R equals to 1.

```

10 point 10
20 dim Q(25),R(25),A(25),B(25)
30 input "Number of decimal places of accuracy: ";D
40 input "Number of input terms for R (one more for Q) ";N%
50 cls
60 for I%=0 to N%:read Q(I%):next
70 data                                     'The relevant data a_0, q_1, ...
80 for I%=1 to N%:read R(I%):next
90 data                                     'The relevant data for r_1, r_2, ...
100 print tab(10);"Smarandache Generalized Continued Fraction"
110 print tab(10);"Sequence Q:";
120 for I%=0 to 6:print Q(I%):next:print " ETC"
130 print tab(10);"Sequence R:";
140 for I%=1 to 6:print R(I%):next:print " ETC"

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150 print tab(10);"Number of decimal places of accuracy: ";D
160 A(0)=Q(0):B(0)=1 'Initiating recurrence algorithm
170 A(1)=Q(0)*Q(1)+R(1):B(1)=Q(1)
180 Delta=1:M=1 'M=loop counter
190 while abs(Delta)>10^(-D) 'Convergens check
200 inc M
210 A(M)=Q(M)*A(M-1)+R(M)*A(M-2) 'Recurrence
220 B(M)=Q(M)*B(M-1)+R(M)*B(M-2)
230 Delta=A(M)/B(M)-A(M-1)/B(M-1) 'Cm-Cm-1
240 wend
250 print tab(10);"An/Bn=";print using(2,20),A(M)/B(M) 'Cn in decimalform
260 print tab(10);"An/Bn=";print A(M);"/";B(M) 'Cn in fractional form
270 print tab(10);"Delta=";print using(2,20),Delta; 'Delta=Last difference
280 print " for n=";M 'n=number of iterations
290 print
300 end

```

To illustrate the behaviour of the convergents C_n have been calculated for $q_1=q_2=\dots=q_n=1$ and $r_1=r_2=\dots=r_n=10$. The iteration of C_n is stopped when $\Delta_n = |C_n - C_{n-1}| < 0.01$. Table 1 shows the result which is illustrated in figure 2. The factor $(-1)^{n-1}$ in (13) produces an oscillating behaviour with diminishing amplitude approaching $\lim_{n \rightarrow \infty} C_n = C$

Table 1. Value of convergents C_n for $q_n\{1,1,\dots\}$ and $r_n\{10,10,\dots\}$

n	1	2	3	4	5	6	7	8	9	10	11
C_n	11	1.91	6.24	2.6	4.84	3.07	4.26	3.35	3.99	3.51	3.85
n	12	13	14	15	16	17	18	19	20	21	22
C_n	3.6	3.78	3.65	3.74	3.67	3.72	3.69	3.71	3.69	3.71	3.7

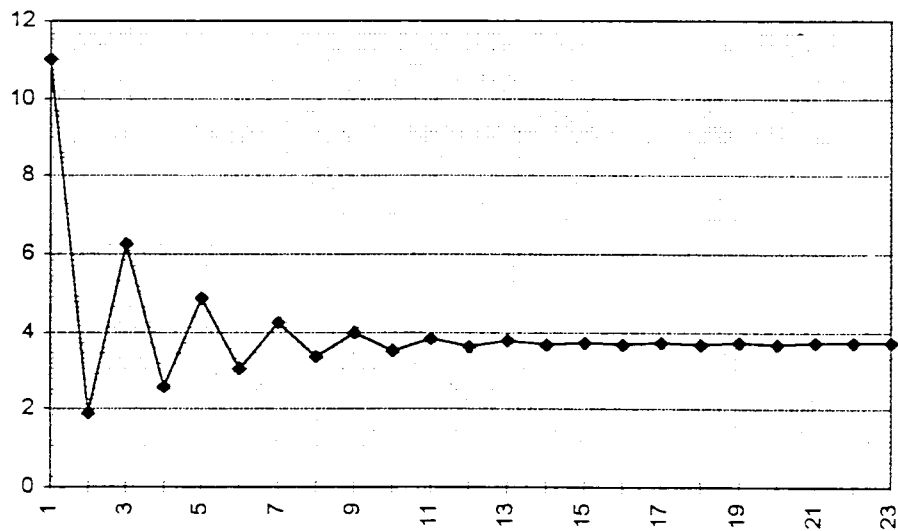


Figure 2. C_n as a function of n

A number of sequences, given below, will be substituted into the recurrence relations (10) and the convergence estimate (13).

$S_1 = \{1, 1, 1, \dots\}$
 $S_2 = \{1, 2, 1, 2, 1, 2, \dots\}$
 $S_3 = \{3, 3, 3, 3, 3, 3, \dots\}$
 $S_4 = \{1, 12, 123, 1234, 12345, 123456, \dots\}$ Smarandache Consecutive Sequence.
 $S_5 = \{1, 21, 321, 4321, 54321, 654321, \dots\}$ Smarandache Reverse Sequence.
 $CS1 = \{1, 1, 2, 8, 9, 10, 512, 513, 514, 520, 521, 522, 729, 730, 731, 737, 738, \dots\}$
 $NCS1 = \{1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30, \dots\}$

The Smarandache CS1 sequence definition: CS1(n) is the smallest number, strictly greater than the previous one (for $n \geq 3$), which is the cubes sum of one or more previous distinct terms of the sequence.

The Smarandache NCS1 sequence definition: NCS1(n) is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of one or more previous distinct terms of the sequence.

These sequences have been randomly chosen from a large number of Smarandache sequences [5].

As expected the last fraction in table 2 converges much slower than the previous one. These general continued fractions are, of course, very artificial as are the sequences on which they are based. As is often the case in empirical number theory it is not the individual figures or numbers which are of interest but the general behaviour of numbers and sequences under certain operations. In the next section we will carry out some experiments with simple continued fractions.

Experiments with Simple Continued Fractions

The theory of simple continued fractions is covered in standard textbooks. Without proof we will therefore make use of some of this theory to make some more calculations. We will first make use of the fact that

There is a one to one correspondence between irrational numbers and infinite simple continued fractions.

The approximations given in table 2 expressed as simple continued fractions would therefore show how these are related to the corresponding general continued fractions.

Table 2. Calculation of general continued fractions

Q	R	n	Δ_n	C_n (dec.form)	C_n (fraction)
S ₁	S ₁	18	$-9 \cdot 10^{-8}$	1.6180339	6765
					4181
S ₂	S ₁	13	$8 \cdot 10^{-8}$	1.3660254	7953
					5822
S ₂	S ₃	22	$-9 \cdot 10^{-8}$	1.8228756	1402652240
					769472267
S ₄	S ₁	2	$-7 \cdot 10^{-6}$	1.04761	7063
					6742
		3	$5 \cdot 10^{-12}$	1.04761198457	30519245
					29132203
4	$-2 \cdot 10^{-20}$	1.0476119845794017019	1657835914708		
			1582490405905		
S ₄	S ₅	2	$-1 \cdot 10^{-3}$	1.082	540
					499
		4	$-7 \cdot 10^{-10}$	1.082166760	8245719435
					7619638429
6	$-1 \cdot 10^{-19}$	1.08216676051416702768	418939686644589150004		
			387130433063328840289		
S ₅	S ₁	2	$-7 \cdot 10^{-6}$	1.04761	7063
					6742
		3	$5 \cdot 10^{-12}$	1.04761198457	30519245
					29132203
4	$-2 \cdot 10^{-20}$	1.04761198457940170194	1657835914708		
			1582490405905		
S ₅	S ₄	2	$-8 \cdot 10^{-5}$	1.0475	2358
					2251
		3	$7 \cdot 10^{-9}$	1.04753443	2547455
					2431858
5	$1 \cdot 10^{-20}$	1.04753443663236268392	60363763803209222		
			57624610411155561		
CS1	NCS1	6	$-1 \cdot 10^{-7}$	1.540889	1376250
					893153
		7	$3 \cdot 10^{-12}$	1.54088941088	1412070090
					916399373
9	$-1 \cdot 10^{-20}$	1.54088941088788795255	377447939426190		
			244954593599743		
NCS1	CS1	16	$-5 \cdot 10^{-5}$	0.6419	562791312666017539
					876693583206100846

Table 3. Some general continued fractions converted to simple continued fractions

Q	R	C_n (dec.form)	C_n (Simple continued fraction sequence)
S_4	S_5	1.08216676051416702768 (corresponding to 6 terms)	1,12,5,1,6,1,1,1,48,7,2,1,20,2,1,5,1,2,1,1,9,1, 1,10,1,1,7,1,3,1,7,2,1,3,31,1,2,6,38,2 (39 terms)
S_5	S_4	1.04753443663236268392 (corresponding to 5 terms)	1,21,26,1,3,26,10,4,4,19,1,2,2,1,8,8,1,2,3,1, 10,1,2,1,2,3,1,4,1,8 (29 terms)
CS1	NCS1	1.54088941088788795255 (corresponding to 9 terms)	1,1,15,1,1,1,1,2,4,17,1,1,3,13,4,2,2,2,5,1,6,2, 2,9,2,15,1.51 (28 terms)

These sequences show no special regularities. As can be seen from table 3 the number of terms required to reach 20 decimals is much larger than for the corresponding general continued fractions.

A number of Smarandache periodic sequences were explored in the author's book *Computer Analysis of Number Sequences* [6]. An interesting property of simple continued fractions is that

A periodic continued fraction is a quadratic surd, i.e. an irrational root of a quadratic equation with integral coefficients.

In terms of A_n and B_n , which for simple continued fractions are defined through

$$\begin{aligned} A_0 &= q_0, & A_1 &= q_0 q_1 + 1, & A_n &= q_n A_{n-1} + A_{n-2} \\ B_0 &= 1, & B_1 &= q_1, & B_n &= q_n B_{n-1} + B_{n-2} \end{aligned} \quad (15)$$

the quadratic surd is found from the quadratic equation

$$B_n x^2 + (B_{n-1} - A_n)x - A_{n-1} = 0 \quad (16)$$

where n is the index of the last term in the periodic sequence. The relevant quadratic surd is

$$x = \frac{A_n - B_{n-1} + \sqrt{A_n^2 + B_{n-1}^2 - 2A_n B_{n-1} - 4A_{n-1} B_n}}{2B_n} \quad (17)$$

An example has been chosen from each of the following types of Smarandache periodic sequences:

1. The Smarandache two-digit periodic sequence:

Definition: Let N_k be an integer of at most two digits. N_k' is defined through

$$N_k' = \begin{cases} \text{the reverse of } N_k \text{ if } N_k \text{ is a two digit integer} \\ N_k \cdot 10 \text{ if } N_k \text{ is a one digit integer} \end{cases}$$

N_{k+1} is then determined by

$$N_{k+1} = |N_k - N_k'|$$

The sequence is initiated by an arbitrary two digit integer N_1 with unequal digits.

One such sequence is $Q=\{9, 81, 63, 27, 45\}$. The corresponding quadratic equation is $6210109x^2-55829745x-1242703=0$

2. The Smarandache Multiplication Periodic Sequence:

Definition: Let $c>1$ be a fixed integer and N_0 and arbitrary positive integer. N_{k+1} is derived from N_k by multiplying each digit x of N_k by c retaining only the last digit of the product cx to become the corresponding digit of N_{k+1} .

For $c=3$ we have the sequence $Q=\{1, 3, 9, 7\}$ with the corresponding quadratic equation $199x^2-235x-37=0$

3. The Smarandache Mixed Composition Periodic Sequence:

Definition. Let N_0 be a two-digit integer $a_1 \cdot 10 + a_0$. If $a_1 + a_0 < 10$ then $b_1 = a_1 + a_0$ otherwise $b_1 = a_1 + a_0 + 1$. $b_0 = |a_1 - a_0|$. We define $N_1 = b_1 \cdot 10 + b_0$. N_{k+1} is derived from N_k in the same way.

One of these sequences is $Q=\{18, 97, 72, 95, 54, 91\}$ with the quadratic equation $3262583515x^2-58724288064x-645584400=0$ and the relevant quadratic surd

$$x = \frac{58724288064 + \sqrt{3456967100707577532096}}{6525167030}$$

The above experiments were carried out with *UBASIC* programs. An interesting aspect of this was to check the correctness by converting the quadratic surd back to the periodic sequence.

There are many interesting calculations to carry out in this area. However, this study will finish by this equality between a general continued fraction convergent and a simple continued fraction convergent.

$$[1,12,123,1234,12345,123456,1,21,321,4321,54321,654321]=$$

$$[1,12,5,1,6,1,1,1,48,7,2,1,20,2,1,5,1,2,1,1,9,1, 1,10,1,1,7,1,3,1,7,2,1,3,31,1,2,6,38,2]$$

References

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