SMARANDACHE COSETS

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Abstract

This paper aims to study the Smarandache cosets and derive some interesting results about them. We prove the classical Lagranges theorem for Smarandache semigroup is not true and that there does not exist a one-to-one correspondence between any two right cosets. We also show that the classical theorems cannot be extended to all Smarandache semigroups. This leads to the definition of Smarandache Lagrange semigroup, Smarandache p Sylow subgroup and Smarandache Cauchy elements. Further if we restrict ourselves to the subgroup of the Smarandache semigroup all results would follow trivially hence the Smarandache coset would become a trivial definition.

Keywords:

Smarandache cosets, Smarandache Lagrange semigroups, Smarandache p-Sylow subgroups, Smarandache Cauchy element, Smarandache Normal subgroups and Smarandache quotient groups.

Definition [2]: The *Smarandache semigroup* is defined to be a semigroup A such that a proper subset of A is a group (with respect to the same induced operation).

Definition 1. Let A be a Smarandache semigroup. A is said to be a *commutative Smarandache semigroup* if the proper subset of A that is a group is commutative.
If A is a commutative semigroup and if A is a Smarandache semigroup then A is obviously a commutative Smarandache semigroup.

Definition 2. Let A be a Smarandache semigroup. $H \subseteq A$ be a group under the same operations of A. For any $a \in A$ the *Smarandache right coset* is $Ha = \{ha / h \in H\}$. Ha is called the Smarandache right coset of H in A. Similarly left coset of H in A can be defined.

Example 1: Let $Z_{12} = \{0, 1, 2, ..., 11\}$ be the Smarandache semigroup under multiplication modulo 12. Clearly Z_{12} is a commutative Smarandache semigroup. Let $A = \{3,9\}$ be a subgroup of Z_{12} under multiplication. $9^2 = 9$ (mod 12) acts as identity with respect to multiplication. For $4 \in Z_{12}$ the right (left) coset of A in Z_{12} is $4A = \{0\}$. For $1 \in Z_{12}$ the right (left) coset of A in Z_{12}

is $1A = \{3, 9\}$. Hence we see the number of elements in nA is not the same for each $n \in Z_{12}$.

Example 2. $Z_9 = \{0, 1, 2, ..., 8\}$ be the commutative Smarandache semigroup under multiplication modulo 9. $A = \{1, 8\}$ and $A_1 = \{2, 4, 1, 5, 7, 8\}$ are the subgroups of Z_9. Clearly order of A does not divide 9. Also order of A_1 does not divide 9.

Example 3. Let S denote the set of all mappings from a 3-element set to itself. Clearly number of elements in S is 27. S is a semigroup under the composition of maps.

Now S contains S_3 the symmetric group of permutations of degree 3. The order of S_3 is 6. Clearly 6 does not divide order of S.

Thus we see from the above examples that the classical Lagrange theorem for groups do not hold good for Smarandache semigroups. It is important to mention here that the classical Cayley theorem for groups could be extended to the case of Smarandache semigroups. This result is proved in [3]. For more details please refer [3]. Thus:

Definition 3. Let S be a finite Smarandache semigroup. If the order of every subgroup of S divides the order of S then we say S is a *Smarandache Lagrange semigroup*.

Example 4. Let $Z_4 = \{0, 1, 2, 3\}$ be the semigroup under multiplication. $A = \{1, 3\}$ is the only subgroup of Z_4. Clearly $|A|/4$. Hence Z_4 is a Smarandache Lagranges semigroup.

But we see most of the Smarandache semigroups are not Smarandache Lagrange semigroup. So one has:

Definition 4. Let S be a finite Smarandache semigroup. If there exists at least one group, i.e. a proper subset having the same operations in S, whose order divides the order of S, then we say that S is a *weakly Smarandache Lagrange semigroup*.

Theorem 5. Every Smarandache Lagrange semigroup is a weakly Smarandache Lagrange semigroup and not conversely.

Proof: By the very definition 3 and 4 we see that every Smarandache Lagrange semigroup is a weakly Smarandache Lagrange semigroup.

To prove the converse is not true consider the Smarandache semigroup given in Example 3. 6 does not divide 27 so S is not a Smarandache Lagrange semigroup but S contains subgroup say $\left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\right\}$ of order 3. Clearly 3 divides 27. Thus S is a weakly Smarandache Lagrange semigroup.

Thus the class of Smarandache Lagrange semigroup is strictly contained in the class of weakly Smarandache Lagrange semigroup.

Theorem 6. Let $S = \{1, 2, ..., n\}$, $n \geq 3$, be the set with n natural elements, S(n) the semigroup of mappings of the set S to itself. Clearly S(n) is a semigroup under the composition of mapping. S(n) is a weakly Smarandache Lagrange semigroup.

Proof: Clearly order of $S(n) = n^n$. S_n the symmetric group of order n!. Given $n \geq 3$, n! does not divide n^n for

$$\frac{n^n}{n!} = \frac{\overbrace{n \times \cdots \times n}^{n \text{ times}}}{1.2.3.4.\cdots.n-1.n} = \frac{\overbrace{n \times \cdots \times n}^{n-1 \text{ times}}}{1.2.\cdots.n-1}$$

Now since $(n-1, n) = 1$, that is $n - 1$ and n are relatively prime. We see n! does not divide n^n. Hence the class of Smarandache semigroups S(n), $n \geq 3$, are weakly Smarandache Lagrange semigroup.

Corollary. S(n), n = 2, is a Smarandache Lagrange semigroup.

Proof. Let n = 2. Then $S(n) = \left\{\begin{pmatrix}1&2\\1&1\end{pmatrix},\begin{pmatrix}1&2\\2&2\end{pmatrix},\begin{pmatrix}1&2\\2&1\end{pmatrix},\begin{pmatrix}1&2\\1&2\end{pmatrix}\right\}$, $|S(n)| = 4$. $S_2 = \left\{\begin{pmatrix}1&2\\1&2\end{pmatrix},\begin{pmatrix}1&2\\2&1\end{pmatrix}\right\}$ is the symmetric group of degree 2 and $|S_2|$ divides 4^2. Hence the claim.

Now the natural question would be: does there exist a Smarandache semigroup, which are not a Smarandache Lagrange semigroup and weakly Smarandache Lagrange semigroup? The answer is yes. The Smarandache semigroup $Z_9 = \{0, 1, 2, ..., 8\}$ under multiplication given in example 2 does not have subgroups which divides 9, hence the claim.

Now to consider the converse of the classical Lagrange theorem we see that there is no relation between the divisor of the order of the Smarandache semigroup S and the order of the subgroup S contains. The example is quite interesting.

Example 5: Let $Z_{10} = \{0, 1, 2, \ldots, 9\}$ be the semigroup of order 10. Clearly Z_{10} is a Smarandache semigroup. The subgroups of Z_{10} are $A_1 = \{1, 9\}$, $A_2 = \{2, 4, 6, 8\}$ and $A_3 = \{1, 3, 7, 9\}$, $A_4 = \{4, 6\}$. Thus 4 does not divide 10, which contradicts Lagrange's theorem (that the order of a subgroup divides the order of the group) in the case of Smarandache semigroup. Also Z_{10} has subgroups of order 5 leading to a contradiction of the classical Sylow theorem (which states that if p^{α} divides the order of the group G then G has a subgroup of order p^{α}) again in the case of Smarandache semigroup. This forces us to define Smarandache p-Sylow subgroups of the Smarandache semigroup.

Definition 7. Let S be a finite Smarandache semigroup. Let p be a prime such that p divides the order of S. If there exists a subgroup A in S of order p or p^t ($t > 1$) we say that S has a *Smarandache p-Sylow subgroup*.

Note. It is important to see that p^t needs not to divide the order of S, that is evident from Example 5, but p should divide the order of S.

Example 6. Let $Z_{16} = \{0, 1, 2, \ldots, 15\}$ be the Smarandache semigroup of order $16 = 2^4$. The subgroups of Z_{16} are $A_1 = \{1, 15\}$, $A_2 = \{1, 3, 9, 11\}$, $A_3 = \{1, 5, 9, 13\}$, and $A_4 = \{1, 3, 5, 7, 9, 11, 13, 15\}$ of order 2, 4, and 8 respectively. Clearly the subgroup A_4 is the Smarandache 2-Sylow subgroup of Z_{16}.

The Sylow classical theorems are left as open problems in case of Smarandache p-Sylow subgroups of a Smarandache semigroup.

Problem 1. Let S be a finite Smarandache semigroup. If $p/|S|$ and S has Smarandache p-Sylow subgroup. Are these Smarandache p-Sylow subgroups conjugate to each other?

Problem 2. Let S be a finite Smarandache semigroup. If p divides order of S and S has Smarandache p-Sylow subgroups. How many Smarandache p-Sylow subgroups exist in S?

Let S be a finite Smarandache semigroup of order n. Let $a \in S$ now for some $r > 1$, if $a^r = 1$ then in general r does not divide n.

Example 7. Let $S = \{1, 2, 3, 4, 5\}$ be the set with 5 elements S (5) be the semigroup of mappings of S to itself. S(5) is a Smarandache semigroup for S(5) contains S_5 the permutation group of degree 5. Clearly $|S(5)| = 5^5$. Now $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix} \in S(5)$. Clearly $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix}^4$ = identity element of S(5), but 4 does not divide $|S(5)| = 5^5$. Thus we define Smarandache Cauchy element.

Definition 8. Let S be a finite Smarandache semigroup. An element $a \in A$, $A \subset S$, A the subgroup of S, is said to be a *Smarandache Cauchy element* of S if $a^r = 1$ ($r \geq 1$), 1 unit element of A, and r divides the order of S; otherwise a is not a Smarandache Cauchy element.

Problem 3. Can we find conditions on the Smarandache semigroup S so that every element in S is a Smarandache Cauchy element of S?

Problem 4. Let Z_n be the Smarandache semigroup under usual multiplication modulo n. Is every element in every subgroup of Z_n is a Cauchy element of Z_n? (n is not a prime.)

Remark: $Z_n = \{0, 1, 2, ..., n-1\}$ is a Smarandache semigroup under multiplication. Clearly every x in Z_n is such that $x^r = 1$ ($r > 1$), but we do not whether every element in every subgroup will satisfy this condition. This is because the subgroups may not have $1 \in Z_n$ as the identity element.

Definition 9. Let S be a finite Smarandache semigroup, if every element in every subgroup of S is a Smarandache Cauchy element; then we say S is a *Smarandache Cauchy semigroup.*

Theorem 10. Let S(n) be the Smarandache semigroup for some positive integer n. S(n) is not a Smarandache Cauchy semigroup.

Proof: Clearly S_n is a subgroup of S (n). We know $|S(n)| = n^n$ and $|S_n| = \underline{n}$. But S_n contains elements x of order (n-1), and (n-1) does not divide n^n. So S (n) is not a Smarandache Cauchy semigroup.

Thus we see the concept of the classical theorem on Cauchy group cannot be extended to finite Smarandache semigroups.

Theorem 11. There does not exist in general a one-to-one correspondence between any two Smarandache right cosets of A in a Smarandache semigroup S.

Proof: We prove this by the following example. Let $S = Z_{10} = \{0, 1, 2, ..., 9\}$. $A = \{1, 9\}$ is a subgroup of S. $A_2 = \{2, 4, 6, 8\}$ is a subgroup of S. $3A = \{3,7\}$ and $5A = \{5\}$. Also $5A_2 = \{0\}$ and $3A_2 = A_2$. So there is no one-to-one correspondence between Smarandache cosets in a Smarandache semigroup.

Theorem 12. The Smarandache right cosets of A in a Smarandache semigroup S does not in general partition S into either equivalence classes of same order or does not partition S at all.

Proof: Consider Z_{10} given in the proof of Theorem 12. Now for A = {1, 9} the subgroup of Z_{10} that is the coset division of Z_{10} by A are {0}, {5}, {1, 9}, {2,8}, {3, 7} and {4, 6}. So A partitions S as cosets the Smarandache semigroup into equivalence classes but of different length. But for A_2 = {2,4,6,8} is a subgroup of Z_{10}. 6 acts as the identity in A_2. Now the coset of division of Z_{10} by A_2 is {2,4,6,8} and {0} only. Hence this subsets do not partition Z_{10}.

Problem 5. Does there exist any Smarandache semigroup S such that there is one-to-one correspondence between cosets of A in S?

Now we proceed to define Smarandache double cosets of a Smarandache semigroup S.

Definition 13. Let S be a Smarandache semigroup. $A \subset S$ and $B \subset S$ be any two proper subgroups of S. For $x \in S$ define AxB = {axb / $a \in A$, $b \in B$}. AxB is called a *Smarandache double coset* of A and B in S.

Example 8: Let Z_{10} = {0, 1, 2, ..., 9}. A = {1, 9} and B = {2, 4, 6, 8} be subgroups of the commutative Smarandache semigroup of order 10. Take x = 5 then AxB = {0}. Take x = 3 then AxB = {2, 4, 6, 8}. For x = 7, AxB = {2, 4, 6, 8}. Thus Z_{10} is not divided into equivalence classes by Smarandache double cosets hence we have the following theorem.

Theorem 14. Smarandache double coset relation on Smarandache semigroup S is not an equivalence relation on S.

Definition 15. Let S be a Smarandache semigroup. Let A be a proper subset of S that is a group under the operations of S. We say A is a *Smarandache normal subgroup* of the Smarandache semigroup S if $xA \subseteq A$ and $Ax \subseteq A$ or xA = {0} and Ax = {0} for all $x \in S$ if 0 is an element in S.

Note. As in case of normal subgroups we cannot define $xAx^{-1} = A$ for every $x \in S$, x^{-1} may not exist. Secondly if we restrict our study only to the subgroup A it has nothing to do with Smarandache semigroup for every result is true in A as A is a group.

Example 9. Let Z_{10} = {0, 1, 2, ..., 9} be a Smarandache semigroup of order 10. A = {2,4,6,8} is a subgroup of Z_{10} which is a Smarandache normal subgroup of Z_{10}. It is interesting to note that that order of the normal subgroup of a Smarandache semigroup needs in general not to divide the order of the Smarandache semigroup. So if we try to define a Smarandache quotient group it will not be in general a group.

Definition 16. Let S be a Smarandache semigroup and A a Smarandache normal subgroup of S. The *Smarandache quotient group* of the Smarandache semigroup S is $\frac{S}{A} = \{Ax / x \in S\}$.

Note. $\frac{S}{A}$ in general is not a group, it is only a semigroup. Further, as in classical group theory, the number of elements in $\frac{S}{A}$ or in A or in S look in general not to be related. Earlier example of Z_{10}, $|Z_{10}| = 10$, $|A| = 4$ and $\left|\frac{Z_{10}}{A}\right| = 2$ proves this note.

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