

# Smarandache's function applied to perfect numbers

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Smarandache's function may be defined as follows:

$S(n)$  = the smallest positive integer such that  $S(n)!$  is divisible by  $n$ . [1]

In this article we are going to see that the value this function takes when  $n$  is a perfect number of the form  $n = 2^{k-1} \cdot (2^k - 1)$ ,  $p = 2^k - 1$  being a prime number.

**Lemma 1** *Let  $n = 2^i \cdot p$  when  $p$  is an odd prime number and  $i$  an integer such that:*

$$0 \leq i \leq E\left(\frac{p}{2}\right) + E\left(\frac{p}{2^2}\right) + E\left(\frac{p}{2^3}\right) + \dots + E\left(\frac{p}{2^{E(\log_2 p)}}\right) = e_2(p!)$$

Where  $e_2(p!)$  is the exponent of 2 in the prime number decomposition of  $p!$ .

$E(x)$  is the greatest integer less than or equal to  $x$ .

One has that  $S(n) = p$ .

**Demonstration:**

Given that  $\gcd(2^i, p) = 1$  ( $\gcd$  = greatest common divisor) one has that  $S(n) = \max\{s(2^i), S(p)\} \geq S(p) = p$ . Therefore  $S(n) \geq p$ .

If we prove that  $p!$  is divisible by  $n$  then one would have the equality.

$$p! = p_1^{e_{p_1}(p!)} \cdot p_2^{e_{p_2}(p!)} \dots p_s^{e_{p_s}(p!)}$$

where  $p_i$  is the  $i$ -th prime of the prime number decomposition of  $p!$ . It is clear that  $p_1 = 2$ ,  $p_s = p$ ,  $e_{p_s}(p!) = 1$  for which:

$$p! = 2^{e_2(p!)} \cdot p_2^{e_{p_2}(p!)} \dots p_{s-1}^{e_{p_{s-1}}(p!)} \cdot p$$

From where one can deduce that:

$$\frac{p!}{n} = 2^{e_2(p!)-i} \cdot p_2^{e_{p_2}(p!)} \dots p_{s-1}^{e_{p_{s-1}}(p!)}$$

is a positive integer since  $e_2(p!) - i \geq 0$ .

Therefore one has that  $S(n) = p$

**Proposition 1** *If  $n$  a perfect number of the form  $n = 2^{k-1} \cdot (2^k - 1)$  with  $k$  a positive integer,  $2^k - 1 = p$  prime, one has that  $S(n) = p$ .*

**Demonstration:**

For the Lemma it is sufficient to prove that  $k - 1 \leq e_2(p!)$ .

If we can prove that

$$k - 1 \leq 2^{k-1} - \frac{1}{2} \tag{1}$$

we will have proof of the proposition since:

$$k - 1 \leq 2^{k-1} - \frac{1}{2} = \frac{2^k - 1}{2} = \frac{p}{2}$$

As  $k-1$  is an integer one has that  $k - 1 \leq E\left(\frac{p}{2}\right) \leq e_2(p!)$

Proving (1) is the same as proving  $k \leq 2^{k-1} + \frac{1}{2}$  at the same time, since  $k$  is integer, is equivalent to proving  $k \leq 2^{k-1}$  (2).

In order to prove (2) we may consider the function:  $f(x) = 2^{x-1} - x \quad x \in \mathbb{R}$ .

This function may be derived and its derivate is  $f'(x) = 2^{x-1} \ln 2 - 1$ .

$f$  will be increasing when  $2^{x-1} \ln 2 - 1 > 0$  resolving  $x$ :

$$x > 1 - \frac{\ln(\ln 2)}{\ln 2} \simeq 1.5287$$

In particular  $f$  will be increasing  $\forall x \geq 2$ .

Therefore  $\forall x \geq 2 \quad f(x) \geq f(2) = 0$  that is to say  $2^{x-1} - x \geq 0 \quad \forall x \geq 2$  therefore

$$2^{k-1} \geq k \quad \forall k \geq 2 \text{ integer}$$

and thus is proved the proposition.

**EXAMPLES:**

$6 = 2 \cdot 3$	$S(6) = 3$
$28 = 2^2 \cdot 7$	$S(28) = 7$
$496 = 2^4 \cdot 31$	$S(496) = 31$
$8128 = 2^6 \cdot 127$	$S(8128) = 127$

**References:**

[1] C. Dumitrescu and R. Müller: *To Enjoy is a Permanent Component of Mathematics*. SMARANDACHE NOTIONS JOURNAL Vol. 9, No. 1-2,(1998) pp 21-26.

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