SMARANDACHE FUNCTIONS OF THE SECOND KIND

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The Smarandache functions of the second kind are defined in [1] thus:

$$S^k : \mathbf{N}^* \to \mathbf{N}^*, \quad S^k(n) = S_n(k) \quad \text{for } n \in \mathbf{N}^*,$$

where S_n are the Smarandache functions of the first kind (see [3]).

We remark that the function S^1 has been defined in [4] by F. Smarandache because $S^1 = S$.

Let, for example, the following table with the values of S^2:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$S^2(n)$	1	4	6	6	10	6	14	12	12	10	22	8	26	14

Obviously, these functions S^k aren't monotony, aren't periodical and they have fixed points.

1. Theorem. *For $k, n \in \mathbf{N}^*$ is true $S^k(n) \le n \cdot k$.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$ and $S(n) = \max\limits_{1 \le i \le t} \{S_{p_i}(\alpha_i)\} = S(p_j^{\alpha_j})$.

Because $S^k(n) = S(n^k) = \max\limits_{1 \le i \le t} \{S_{p_i}(\alpha_i k)\} = S(p_r^{\alpha_r k}) \le kS(p_r^{\alpha_r}) \le kS(p_j^{\alpha_j}) = kS(n)$ and $S(n) \le n$, [see [3]], it results:

$$(1) \qquad S^k(n) \le n \cdot k \quad \text{for every } n, k \in \mathbf{N}^*.$$

2. Theorem. *All prime numbers $p \ge 5$ are maximal points for S^k, and*

$$S^k(p) = p[k - i_p(k)], \quad \text{where} \quad 0 \le i_p(k) \le \left[\frac{k-1}{p}\right]$$

Proof. Let $p \ge 5$ be a prime number. Because $S_{p-1}(k) < S_p(k)$, $S_{p+1}(k) < S_p(k)$ [see [2]] it results that $S^k(p-1) < S^k(p)$ and $S^k(p+1) < S^k(p)$, so that $S^k(p)$ is a relative maximum value.

Obviously,

(2) $S^k(p) = S_p(k) = p[k - i_p(k)]$ with $0 \le i_p(k) \le \left[\frac{k-1}{p}\right]$.

(3) $S^k(p) = pk$ for $p \ge k$.

3. Theorem. *The numbers kp, for p prime and p>k are the fixed points of* S^k.

Proof. Let p be a prime number, $m = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ be the prime factorization of m and $p > \max\{m, k\}$. Then $p_i \alpha_i \le p_i^{\alpha_i} < p$ for $i \in \overline{1,t}$, therefore we have:

$$S^k(m \cdot p) = S[(mp)^k] = \max_{1 \le i \le t}\left\{S_{p_i^{\alpha_i}}, S_p(k)\right\} = S_p(k) = kp.$$

For $m=k$ we obtain:

$S^k(kp) = kp$ so that kp is a fixed point.

4. Theorem. *The functions* S^k *have the following properties:*

$$S^k = 0\ (n^{1+\varepsilon}), \text{ for } \varepsilon > 0$$

$$\limsup_{n \to \infty} \frac{S^k(n)}{n} = k.$$

Proof. Obviously,

$$0 \le \lim_{n \to \infty} \frac{S^k(n)}{n^{1+\varepsilon}} = \lim_{n \to \infty} \frac{S(n^k)}{n^{1+\varepsilon}} \le \lim_{n \to \infty} \frac{kS(n)}{n^{1+\varepsilon}} = k \lim_{n \to \infty} \frac{S(n)}{n^{1+\varepsilon}} = 0 \quad \text{for}$$

$$S = 0\ (n^{1+\varepsilon}), \quad \text{[see[4]]}.$$

Therefore we have $S^k = 0\ (n^{1+\varepsilon})$, and:

$$\limsup_{n \to \infty} \frac{S^k(n)}{n} = \limsup_{n \to \infty} \frac{S(n^k)}{n} = \lim_{\substack{p \to \infty \\ p \text{ prime}}} \frac{S(p^k)}{p} = k$$

5. Theorem, [see[1]]. *The Smarandache functions of the second kind standardise* $(\mathbf{N}^*, \cdot)$ *in* $(\mathbf{N}^*, \leq, +)$ *by:*

$$\Sigma_3: \ \max\left\{S^k(a), S^k(b)\right\} \leq S^k(ab) \leq S^k(a) + S^k(b)$$

and $(\mathbf{N}^*, \cdot)$ *in* $(\mathbf{N}^*, \leq, \cdot)$ *by:*

$$\Sigma_4: \ \max\left\{S^k(a), S^k(b)\right\} \leq S^k(ab) \leq S^k(a) \cdot S^k(b) \quad \text{for every} \quad a, b \in \mathbf{N}^*$$

6. Theorem. *The functions* S^k *are, generally speaking, increasing. It means that:*

$$\forall n \in \mathbf{N}^* \ \ \exists m_0 \in \mathbf{N}^* \ \text{so that} \ \forall m \geq m_0 \ \Rightarrow \ S^k(m) \geq S^k(n)$$

Proof. The Smarandache function is generally increasing, [see [4]], it means that :

$$(3) \qquad \forall t \in \mathbf{N}^* \ \ \exists r_0(t) \in \mathbf{N}^* \ \text{so that} \ \forall r \geq r_0 \ \Rightarrow \ S(r) \geq S(t)$$

Let $t = n^k$ and $r_0 = r_0(t)$ so that $\forall r \geq r_0 \ \Rightarrow \ S(r) \geq S(n^k)$.
Let $m_0 = \left[\sqrt[k]{r_0}\right] + 1$. Obviously $m_0 \geq \sqrt[k]{r_0} \Leftrightarrow m_0^k \geq r_0$ and $m \geq m_0 \Leftrightarrow m^k \geq m_0^k$.
Because $m^k \geq m_0^k \geq r_0$ it results $S(m^k) \geq S(n^k)$ or $S^k(m) \geq S^k(n)$.
Therefore

$$\forall n \in \mathbf{N}^* \ \ \exists m_0 = \left[\sqrt[k]{r_0}\right] + 1 \quad \text{so that}$$

$$\forall m \geq m_0 \ \Rightarrow \ S^k(m) \geq S^k(n) \quad \text{where} \quad r_0 = r_0(n^k)$$

is given from (3).

7. Theorem. *The function* S^k *has its relative minimum values for every* $n = p!$, *where* p *is a prime number and* $p \geq \max\{3, k\}$.

Proof. Let $p! = p_1^{i_1} \cdot p_2^{i_2} \cdots p_m^{i_m} \cdot p$ be the canonical decomposition of $p!$, where $2 = p_1 < 3 = p_2 < \cdots < p_m < p$. Because $p!$ is divisible by $p_j^{i_j}$ it results $S(p_j^{i_j}) \leq p = S(p)$ for every $j \in \overline{1,m}$.
Obviously,

$$S^k(p!) = S[(p!)^k] = \max_{1 \leq j \leq m}\left\{S\left(p_j^{k \cdot i_j}\right), S(p^k)\right\}$$

Because $S\left(p_j^{k \cdot i_j}\right) \leq kS(p_j^{i_j}) < kS(p) = kp = S(p^k)$ for $k \leq p$, it results that we have

$$(4) \qquad S^k(p!) = S(p^k) = kp, \ \text{for} \ k \leq p$$

Let $p!-1=q_1^{i_1}\cdot q_2^{i_2}\cdots q_t^{i_t}$ be the canonical decomposition for $p!$-1, then $q_j>p$ for $j\in\overline{1,t}$.

It follows $S(p!-1)=\max_{1\le j\le t}\left\{S(q_j^{i_j})\right\}=S(q_m^{i_m})$ with $q_m>p$.

Because $S(q_m^{i_m})>S(p)=S(p!)$ it results $S(p!-1)>S(p!)$.

Analogous it results $S(p!+1)>S(p!)$.

Obviously

$$(5)\qquad S^k(p!-1)=S\left[(p!-1)^k\right]\ge S(q_m^{k\cdot i_m})\ge S(q_m^k)>S(p^k)=kp$$

$$(6)\qquad S^k(p!+1)=S\left[(p!+1)^k\right]>k\cdot p$$

For $p\ge\max\{3,k\}$ out of (4), (5), (6) it results that $p!$ are the relative minimum points of the functions S^k.

REFERENCES

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