

Smarandache Magic Squares

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The objective of this article is to investigate the existence of magic squares made with Smarandache's numbers [Tabirca, 1998]. Magic squares have been studied intensively and many aspects concerning them have been found. Many interesting things about magic squares can be found at the following WEB page <http://www.pse.che.tohoko.ac.jp/~msuzuki/MagicSquares.html>.

Definition 1. A Smarandache magic square is a square matrix $a \in M_n(N)$ with the following properties:

$$a) \quad \{a_{i,j} \mid i, j = \overline{1, n}\} = \{S(i) \mid i = \overline{1, n^2}\} \quad (1)$$

$$b) \quad (\forall j = \overline{1, n}) \sum_{i=1}^n a_{i,j} = k \quad (2)$$

$$c) \quad (\forall i = \overline{1, n}) \sum_{j=1}^n a_{i,j} = k \quad (3)$$

Therefore, a Smarandache magic square is a square matrix by order n that contains only the elements $S(1), S(2), \dots, S(n^2)$ [Smarandache, 1980] and satisfies the sum properties (2-3). According to these properties, the sum of elements on each row or column should be equal to the same number k . Obviously, this number satisfies the following equation

$$k = \frac{\sum_{i=1}^{n^2} S(i)}{n} .$$

Theorem 1. If the equation $n \mid \sum_{i=1}^{n^2} S(i)$ does not hold, then there is not a Smarandache magic square by order n .

Proof

This proof is obvious by using the simple remark $k = \frac{\sum_{i=1}^{n^2} S(i)}{n} \in N$. If $a \in M_n(N)$ is a

Smarandache magic square, then the equation $n \mid \sum_{i=1}^{n^2} S(i)$ should hold. Therefore, if this

equation does not hold, there is no a Smarandache magic square.

♣

Theorem 1 provides simple criteria to find the non-existence of Smarandache magic

square. All the numbers $1 < n < 101$ that do not satisfy the equation $n \mid \sum_{i=1}^{n^2} S(i)$ can be

found by using a simple computation [Ibstedt, 1997]. They are $\{2, 3, \dots, 100\} \setminus \{6, 7, 9, 58, 69\}$. Clearly, a Smarandache magic square does not exist for this numbers. If n is one

of the numbers 6, 7, 9, 58, 69 then the equation $n \mid \sum_{i=1}^{n^2} S(i)$ holds [see Table 1]. This does

not mean necessarily that there is a Smarandache magic square. In this case, a Smarandache magic square is found using other techniques such us detailed analysis or exhaustive computation.

n	$S(n)$	$\sum_{i=1}^{n^2} S(i)$
6	3	330
7	7	602
9	6	1413
58	29	1310162
69	23	2506080

Table 1. The values of n that satisfy $n \mid \sum_{i=1}^{n^2} S(i)$.

An algorithm to find a Smarandache magic square is proposed in the following. This algorithm uses *Backtracking* strategy to complete the matrix a that satisfies (1-3). The going through matrix is done line by line from the left-hand side to the right-hand side.

The algorithm computes:

- Go through the matrix
 - ◆ Find an unused element of the set $S(1), S(2), \dots, S(n^2)$ for the current cell.
 - ◆ If there is no unused element, then compute a step back.
 - ◆ If there is an used element, then
 - Put this element in the current cell.
 - Check the backtracking conditions.
 - If they are verified and the matrix is full, then a Smarandache magic square has been found.
 - If they are verified and the matrix is not full, then compute a step forward.

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procedure Smar_magic_square(n);
begin
col:=1; row:=1;a[col, row]:=0;
while row>0 do begin
  while a[col, row]<n*n do begin
    a[col,row]:=a[col,row]+1;
    call check(col,row,n,a,cont);
    if cont=0 then exit;
  end
  if cont =0 then call back(col,row);
  if cont=1 and col=n and row=n
    then call write_square(n,a)
    else call forward(col,row);
end;
write('result negative');
end;

procedure back(col, row);
begin
col:=col-1;
if col=0 then begin
  col:=n;row:=row-1;
end;
end;

procedure forward(col, row);
begin
col:=col+1;
if col=n+1 then begin
  col:=1;row:=row+1;
end;
end;

procedure write_square(n,a);
begin
for i:=1 to n do begin
  for j:=1 to n do write (S(a[i,j]), ' ');
  writeln;
end;
stop;
end;

procedure check(col,row,n,k,a,cont);
begin
cont:=1; sum:=0;
for i:=1 to col do sum:=sum+S(a[i,j]);
if (sum>k) or (col=n and sum<>k) then begin
  cont:=0;

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return;
end;
sum:=0
for j:=1 to row do sum:=sum+S(a[i,j]);
if (sum>k) or (row=n and sum<>k) then begin
cont:=0;
return;
end;
end;

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Figure 1. Detailed algorithm for Smarandache magic squares.

The backtracking conditions are the following:

$$\left(\forall j = \overline{1, n}\right) \sum_{i=1}^{col} a_{i,j} \leq k \text{ and } \left(\forall j = \overline{1, n}\right) \sum_{i=1}^n a_{i,j} = k \quad (4a)$$

$$\left(\forall i = \overline{1, n}\right) \sum_{j=1}^{row} a_{i,j} \leq k \text{ and } \left(\forall i = \overline{1, n}\right) \sum_{j=1}^n a_{i,j} = k. \quad (4b)$$

$$\left(\forall (i, j) < (row, col)\right) a_{i,j} \neq a_{row,col} \quad (5)$$

These conditions are checked by the procedure check. A detailed algorithm is presented in a pseudo-cod description in Figure 1.

Theorem 2. If there is a Smarandache magic square by order n , then the procedure Smar_magic_square finds it.

Proof

This theorem establishes the correctness property of the procedure Smar_magic_square. The *Backtracking* conditions are computed correctly by the procedure check that verifies if the equations (4-5) hold. The correctness this algorithm is given by the correctness of the *Backtracking* strategy. Therefore, this procedure finds a Smarandache magic square. ♣

Theorem 3. The complexity of the procedure complexity Smar_magic_square is $O(n^{2 \cdot n^2 + 1})$.

Proof

The complexity of the procedure Smar_magic_square is studied in the worst case when there is not a Smarandache magic square. In this case, this procedure computes all the checking operations for the *Backtracking* strategy. Therefore, all the values

$S(1), S(2), \dots, S(n^2)$ are gone through for each cell. For each value put into a cell, at most $O(n)$ operations are computed by the procedure check. Therefore, the complexity is $O\left(n \cdot (n^2)^{n^2}\right) = O\left(n^{2 \cdot n^2 + 1}\right)$. ♣

Remark 1. The complexity $O(n^{2 \cdot n^2 + 1})$ is not polynomial. Moreover, because this is a very big complexity, the procedure Smar_magic_square can be applied only for small values of n . For example, this procedure computes at most $6^{73} > 10^{56}$ operations in the case $n=6$.

The above procedure has been translated into a C program that has been run on a Pentium MMX 233 machine. The execution of this program has taken more than 4 hours for $n=6$. Unfortunately, there is not a Smarandache magic square for this value of n . The result of computation for $n=7$ has not been provided by computer after a twelve hours execution. This reflects the huge number of operations that should be computed ($7^{99} > 10^{83}$).

According to these negative results, we believe that Smarandache magic squares do not exist. If n is a big number that satisfy the equation $n \mid \sum_{i=1}^{n^2} S(i)$, then we have many possibilities to change, to permute and to arrange the numbers $S(1), S(2), \dots, S(n^2)$ into a square matrix. In spite of that Equations (2-3) cannot be satisfied. Therefore, we may conjecture the following: *“There are not Smarandache magic squares”*. In order to confirm or infirm this conjecture, we need more powerful method than the above computation. Anyway, the computation looks for a particular solution, therefore it does not solve the problem.

References

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