SMARANDACHE NEAR-RINGS AND THEIR GENERALIZATIONS

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Abstract: *In this paper we study the Smarandache semi-near-ring and near-ring, homomorphism, also the Anti-Smarandache semi-near-ring. We obtain some interesting results about them, give many examples, and pose some problems. We also define Smarandache semi-near-ring homomorphism.*

Keywords: *Near-ring, Semi-near-ring, Smarandache semi-near-ring, Smarandache near-ring, Anti-Smarandache semi-near-ring, Smarandache semi-near-ring homomorphism,*

Definition [1 Pilz]: An algebraic system (N, +, •) is called a *near-ring* (or a *right near-ring*) if it satisfies the following three conditions:

(i) (N, +) is a group (not necessarily abelian).

(ii) (N, •) is a semigroup.

(iii) $(n_1 + n_2) \bullet n_3 = n_1 \bullet n_3 + n_2 \bullet n_3$ (right distributive law) for all $n_1, n_2, n_3 \in N$.

Definition [1 Pilz]: An algebraic system (S, +, •) is called a *semi-near-ring* (or *right semi-near-ring)* if it satisfies the following three conditions:

(i) (S, +) is a semigroup (not necessarily abelian).

(ii) (S, •) is a semigroup.

(iii) $(n_1 + n_2) \bullet n_3 = n_1 \bullet n_3 + n_2 \bullet n_3$ for all $n_1, n_2, n_3 \in S$ (right distributive law).

Clearly, every near-ring is a semi-near-ring and not conversely. For more about semi-near-rings please refer [1], [4], [5], [6], [7], [8] and [9].

Definition 1: A non-empty set N is said to be a *Smarandache semi-near-ring* if (N, +, •) is a semi-near-ring having a proper subset A ($A \subset N$) such that A under the same binary operations of N is a near-ring, that is (A, +, •) is a near-ring.

Example 1: Let $Z_{18} = \{0, 1, 2, 3, ..., 17\}$ integers modulo 18 under multiplication. Define two binary operations $\times$ and • on Z_{18} as follows:
$\times$ is the usual multiplication so that $(Z_{18}, \times)$ is a semigroup;

$a \bullet b = a$ for all $a, b \in Z_{18}$.
Clearly $(Z_{18}, \bullet)$ is a semigroup under $\bullet$. $(Z_{18}, \times, \bullet)$ is a semi-near-ring. $(Z_{18}, \times, \bullet)$ is a Smarandache semi-near-ring, for take $A = \{1, 3, 5, 7, 11, 13, 17\}$. $(A, \times, \bullet)$ is a near ring. Hence the claim.

Theorem 2: Not all semi-near-rings are in general Smarandache semi-near-rings.

Proof: By an example.
Let $Z^+ = \{$set of positive integers$\}$. Z^+ under + is a semigroup. Define $\bullet$ a binary operation on Z^+ as $a \bullet b = a$ for all $a, b \in Z^+$. Clearly Z^+ under $\bullet$ is a semigroup. Now $(Z^+, +, \bullet)$ is a semi-near-ring which is not a Smarandache semi-near-ring.

Now we give an example of.

Example 2 (of an infinite Smarandache semi-near-ring)*:*
Let $M_{n\times n} = \{(a_{ij}) / a_{ij} \in Z\}$. Define matrix multiplication as an operation on $M_{n\times n}$. $(M_{n\times n}, \times)$ is a semigroup. Define '$\bullet$' on $M_{n\times n}$ as $A \bullet B = A$ for all $A, B \in M_{n\times n}$. Clearly $(M_{n\times n}, \times, \bullet)$ is a Smarandache semi-near-ring, for take the set of all n×n matrices A such that $|A| \neq 0$. Denote the collection by $A_{n\times n}$. $A_{n\times n} \subset M_{n\times n}$ Clearly $(A_{n\times n}, \times, \bullet)$ is a near-ring.

Example 3:
Let $Z_{24} = \{0, 1, 2, \ldots, 23\}$ be the set of integers modulo 24. Define usual multiplication $\times$ on Z_{24}. $(Z_{24}, \times)$ is a semigroup. Define '$\bullet$' on Z_{24} as $a \bullet b = a$ for all $a, b \in Z_{24}$. Clearly Z_{24} is a semi-near-ring. Now Z_{24} is also a Smarandache semi-near-ring. For take $A = \{1, 5, 7, 11, 13, 17, 19, 23\}$. $(A, \times, \bullet)$ is a near-ring. So, Z_{24} is a Smarandache semi-near-ring.

Motivated by the examples 3 and 4 we propose the following open problem.

Problem 1: Let $Z_n = \{0, 1, 2, \ldots, n-1\}$ set of integers. $n = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$, where $p_1, p_2, \ldots, p_t$ are distinct primes, $t > 1$. Define two binary operations '×' and '$\bullet$' on Z_n. × is the usual multiplication. Define '$\bullet$' on Z_n as $a \bullet b = a$ for all $a, b \in Z$. Let $A = \{1, q_1, \ldots, q_r\}$ where $q_1, \ldots, q_r$ are all odd primes different from $p_1, \ldots, p_t$ and $q_1, \ldots, q_r \in Z_n$.

Prove A is a group under ×. Solution to this problem will give the following:

Result: $Z_n = \{0, 1, 2, \ldots, n-1\}$ is a Smarandache semi-near-ring under × and $\bullet$ defined as in Examples 1 and 3. Thus we get a class of Smarandache semi-near-rings for every positive composite integer. Now when $t = 1$ different cases arise.

Example 4: $Z_4 = \{0, 1, 2, 3\}$ is a Smarandache semi-near-ring as $(Z_4, \times, \bullet)$ is a semi-near-ring and $(A = \{1, 3\}, \times, \bullet)$ is a near-ring.

Example 5: $Z_9 = \{0, 1, 2, 3, 4, ..., 8\}$. Now $(Z_9, \times, \bullet)$ is a semi-near-ring. $(A = \{1, 8\}, \times, \bullet)$ is a near-ring so Z_9 is a Smarandache near-ring. Clearly 8 is not a prime number.

Example 6: Let $Z_{25} = \{0, 1, 2, 3, ..., 24\}$. Now $(Z_{25}, \times, \bullet)$ is a semi-near-ring. $\{A = \{1, 24\}, \times, \bullet\}$ is a near-ring. Thus Z_{25} is a Smarandache semi-near-ring.

Theorem 3: Let $(Z_{p^2}, \times, \bullet)$ be a semi-near-ring. Clearly $(Z_{p^2}, \times, \bullet)$ is a Smarandache semi-near-ring.

Proof: Let $(A = \{1, p^2\text{-}1\}, \times, \bullet)$ is a near-ring. Hence $\{Z_{p^2}, \times, \bullet)$ is a Smarandache semi-near-ring.

Hence we assume t >1, for non primes one can contribute to near -ring under $(\times,\bullet)$.

Corollary: Let $(Z_{p^n}, \times, \bullet)$ be a semi-near-ring. $(Z_{p^n}, \times, \bullet)$ is a Smarandache near-ring.

Proof: Take $A = \{1, p^n\text{-}1\}$ is a near-ring. Hence $(Z_{p^n}, \times, \bullet)$ is a Smarandache semi-near-ring.

Thus we have a natural class of finite Smarandache semi-near-rings.

Definition 4 (in the classical way)**:**
N is said to be a *Smarandache near-ring* if $(N, +, \bullet)$ is a near-ring and has a proper subset A such that $(A, +, \bullet)$ is a near-field.

Now many near-rings contain subsets that are semi-near-rings, so we are forced to check:

Definition 5: N is said to be an *Anti-Smarandache semi-near-ring* if N is a near-ring and has a proper subset A of N such that A is a semi-near-ring under the same operations of N.

Example 7: Let Z be the set of integers under usual + and multiplication '$\bullet$' by $a \bullet b = a$ for all $a, b \in Z$. $(Z, +, \bullet)$ is a near-ring. Take $A = Z^+$ now $(Z^+, +, \bullet)$ is a semi-near-ring. So Z is an Anti-Smarandache semi-near-ring.

Example 8: Let $M_{n\times n} = \{(a_{ij}) / a_{ij} \in Z\}$. Define + on $M_{n\times n}$ as the usual addition of matrices and define • on $M_{n\times n}$ by A • B = A for all A, B ∈ $M_{n\times n}$. ($M_{n\times n}$, +, •) is a near-ring. Take $A_{n\times n} = \{(a_{ij}) / a_{ij} \in Z^+\}$. Now (A, +, •) is a semi-near-ring. Thus $M_{n\times n}$ is an Anti-Smarandache semi-near-ring.

We propose the following:

Problem 2: Does there exist an infinite near-ring constructed using reals or integers, which is not an Anti-Smarandache semi-near-ring?

Example 9: Z[x] is the polynomial ring over the ring of integers. Define + on Z[x] as the usual addition of polynomials. Define an operation • on Z[x] as p(x) • q(x) = p(x) for all p(x), q(x) ∈ Z[x]. Clearly (Z[x], +, •) is an Anti-Smarandache semi-near-ring, for (Z^+[x], +, •) is a semi-near-ring.

Now it is still more interesting to find a solution to the following question (or Problem 2 worded in a negative way):

Problem 3: Find a finite Anti-Smarandache semi-near-ring.

Definition 6: Let N and N_1 be two Smarandache semi-near-rings. A mapping h: N → N_1 is a *Smarandache semi-near-ring homomorphism* if h is a homomorphism.

Similarly one defines the Anti-Smarandache semi-near-ring homomorphism:

Definition 7: Let N and N_1 be two Anti-Smarandache semi-near-rings. Then h: N → N_1 is an *Anti-Smarandache semi-near-ring homomorphism* if h is a homomorphism.

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