Smarandache Zero Divisors

W.B.Vasantha Kandasamy
Department of Mathematics
Indian Institute of Technology, Madras
Chennai- 600036

ABSTRACT

In this paper, we study the notion of Smarandache zero divisor in semigroups and rings. We illustrate them with examples and prove some interesting results about them.

Throughout this paper, S denotes a semigroup and R a ring. They need not in general be Smarandache semigroups or Smarandache rings respectively. Smarandache zero divisors are defined for any general ring and semigroup.

Definition 1 Let S be any semigroup with zero under multiplication (or any ring R). We say that a non-zero element $a \in S$ (or R) is a Smarandache zero divisor if there exists a non-zero element b in S (or in R) such that $a.b = 0$ and there exist $x, y \in S \setminus \{a, b, 0\}$ (or $x, y \in R \setminus \{a, b, 0\}$), $x \neq y$, with

1. $ax = 0$ or $xa = 0$
2. $by = 0$ or $yb = 0$ and
3. $xy \neq 0$ or $yx \neq 0$

Remark If S is a commutative semigroup then we will have $ax = 0$ and $xa = 0$, $yb = 0$ and $by = 0$; so what we need is at least one of xa or ax is zero 'or' not in the mutually exclusive sense.

Example 1 Let $Z_{12} = \{0,1,2,...,11\}$ be the semigroup under multiplication. Clearly, Z_{12} is a commutative semigroup with zero. We have $6 \in Z_{12}$ is a zero divisor as $6.8 \equiv 0 (\mod 12)$. Now 6 is a Smarandache zero divisor as $6.2 \equiv 0 (\mod 12)$, $8.3 \equiv 0 (\mod 12)$ and $2.3 \not\equiv 0 (\mod 12)$. Thus 6 is a Smarandache zero divisor. It is interesting to note that for $3 \in Z_{12}$, $3.4 \equiv 0 (\mod 12)$ is a zero divisor, but 3,4 is not a Smarandache zero divisor for there does not exist a $x, y \in Z_{12} \setminus \{0\}$ $x \neq y$ such that $3.x \equiv 0 (\mod 12)$ and $4y \equiv 0 (\mod 12)$ with $xy \not\equiv 0 (\mod 12)$.

This example leads us to the following theorem.

Theorem 2 Let S be a semigroup under multiplication with zero. Every Smarandache zero divisor is a zero divisor, but not reciprocally in general.

Proof: Given S is a multiplicative semigroup with zero. By the very definition of a Smarandache zero divisor in S we see it is a zero divisor in S. But if x is a zero divisor in S, it need not in general be a Smarandache zero divisor of S. We prove this by an example. Consider the semigroup Z_{12} given in example 1. Clearly 3 is a zero divisor in Z_{12} as $3.4 \equiv 0(12)$ but 3 is not a Smarandache zero divisor of 12.

Example 2 Let $S_{2\times2} = \left\{ \begin{pmatrix} a & b \\ c & b \end{pmatrix} \middle/ a,b,c,d \in Z_2 = \{0,1\} \right\}$ be the set of all 2×2 matrices with entries from the ring of integers modulo 2. $S_{2\times2}$ is a semigroup under matrix multiplication modulo two. Now $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in $S_{2\times2}$ is a zero divisor as $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in S_{2\times2}$ is such that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Now take $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ in $S_{2\times2}$. We have $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Finally, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a Smarandache zero divisor of the semigroup $S_{2\times2}$.

Example 3 Let $R_{3\times3} = \{(a_{ij}) \text{ such that } a_{ij} \in Z_4 = \{0,1,2,3\}\}$ be the collection of all 3×3 matrices with entries from Z_4. Now $R_{3\times3}$ is a ring under matrix addition and multiplication modulo four. We have $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in R_{3\times3}$ is a Smarandache zero divisor in $R_{3\times3}$.

For

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix} \in R_{3\times3} \text{ such that}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is Smarandache zero-divisor in $R_{3\times3}$.

Example 4: Let $Z_{20} = \{0, 1, 2, ..., 19\}$ be the ring of integers modulo 20. Clearly 10 is a Smarandache zero divisor. For $10 . 16 \equiv 0(\text{mod } 20)$ and there exists $5, 6 \in Z_{20} \setminus \{0\}$ with

$5 \times 16 \equiv 0 \ (\text{mod } 20)$
$6 \times 10 \equiv 0 \ (\text{mod } 20)$
$6 \times 5 \equiv 10(\text{mod } 20).$

Theorem 3 Let R be a ring; a Smarandache zero divisor is a zero divisor, but not reciprocally in general.

327

Proof: By the very definition, we have every Smarandache zero divisor is a zero divisor. We have the following example to show that every zero divisor is not a Smarandache zero divisor. Let $Z_{10} = \{0,1,2,...,9\}$ be the ring of integers modulo 10.

Clearly 2 in Z_{12} is a zero divisor as $2.5 \equiv 0(\text{mod } 10)$ which can never be a Smarandache zero divisors in Z_{10}. Hence the claim.

Theorem 4 Let R be a non-commutative ring. Suppose $x \in R \setminus \{0\}$ be a Smarandache zero divisor; with $xy = yx = 0$ and $a,b \in R \setminus \{0,x,y\}$ satisfying the following conditions:

1. $ax = 0$ and $xa \neq 0$,
2. $yb = 0$ and $by \neq 0$ and
3. $ab = 0$ and $ba \neq 0$.

Then we have $(xa + by)^2 = 0$.

Proof: Given $x \in R \setminus \{0\}$ is a Smarandache zero divisor such that $xy = 0 = yx$. We have $a,b \in R \setminus \{0,x,y\}$ such that $ax = 0$ and $xa \neq 0$, $yb = 0$ and $by \neq 0$ with $ab = 0$ and $ba \neq 0$. Consider $(xa + by)^2 = xaby + byxa + xaxa + byby$ using $ab = 0$, $yx = 0$, $ax = 0$ and $yb = 0$ we get $(xa + by)^2 = 0$.

Theorem 5 Let R be a ring having Smarandache zero divisor satisfying conditions of Theorem 5, then R has a nilpotent element of order 2.

Proof: By Theorem 5 the result is true.

We propose the following problems.

Problem 1: *Characterize rings R in which every zero divisor is a Smarandache zero divisor.*

Problem 2: *Find conditions or properties about rings so that it has Smarandache zero divisors.*

Problem 3: *Does there exists rings in which no zero divisor is a Smarandache zero divisor ?*

Problem 4: *Find group rings RG which has Smarandache zero divisors ?*

Problem 5: *Let G be a group having elements of finite order and F any field. Does the elements of finite order in G give way to Smarandache zero divisors ?*

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