SOLUTION OF TWO QUESTIONS CONCERNING THE DIVISOR FUNCTION AND THE PSEUDO – SMARANDACHE FUNCTION

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Abstract In this paper we completely solve two questions concerning the divisor function and the pseudo – Smarandache function.

Key words divisor function, pseudo – Smarandache function, functional equation

1 Introduction

Let $\mathbb{N}$ be the set of all positive integers. For any $n \in \mathbb{N}$, let

(1) $$d(n) = \sum_{d \mid n} 1,$$

(2) $$Z(n) = \min\{a \mid a \in \mathbb{N}, n \mid \sum_{j=1}^{a} j\}$$

Then $d(n)$ and $Z(n)$ are called the divisor function and the pseudo – Smarandache function of n, respectively, In[1], Ashbacher posed the following unsolved questions.

Question 1 How many solutions n are there to the functional equation.

(3) $$Z(n) = d(n), n \in \mathbb{N}?$$

Question 2 How many solutions n are there to the functional equation.

(4) $$Z(n)+d(n)=n, n\in\mathbb{N}?$$

In this paper we completely solve the above questions as follows.

Theorem 1 The equation (3) has only the solutions $n=1,3$ and 10.

Theorem 2 The equation (4) has only the solution $n=56$.

2 Proof of Theorem 1

A computer search showed that (3) has only the solutions $n=1,3$ and 10 with $n\leqslant 10000$ (see [1])

We now let n be a solution of (3) with $n\neq 1,3$ or 10. Then we have $n>10000$. Let

(5) $$n=p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$$

be the factorization of n. By [2, Theorem 273], we get from (1) and (5) that

(6) $$d(n)=(r_1+1)(r_2+1)\cdots(r_k+1).$$

On the other hand, since $\sum_{j=1}^{a} j=a(a+1)/2$ for any $a\in\mathbb{N}$, we see from (2) that $n\mid Z(n)(Z(n)+1)/2$. It implies that $Z(n)(Z(n)+1)/2\geqslant n$. So we have

(7) $$Z(n)\geqslant\sqrt{2n+\frac{1}{4}}-\frac{1}{2}$$

Hence, by (3), (5), (6) and (7), we get

(8) $$1>\sqrt{2}\prod_{i=1}^{k}\frac{p_i^{r_i/2}}{r_i+1}-\frac{1}{2}\prod_{i=1}^{k}\frac{1}{r_i+1}$$

If $p_1>3$, then from (8) we get $p_1\geqslant 5$ and

$$1\geqslant\sqrt{2}\left(\frac{\sqrt{5}}{2}\right)^k-\frac{1}{2^{k+1}}>1,$$

a contradiction. Therefore, if (8) holds, then either $p_1=2$ or $p_1=3$. By the same method, then n must satisfy one of the following conditions.

(i) $p_1=2$ and $r_1 \leqslant 4$.

(ii) $p_1=3$ and $r_1=1$.

However, by (8), we can calculate that $n<10000$, a contradiction. Thus, the theorem is proved.

3 Proof of Theorem 2

A computer search showed that (4) has only the solution $n=56$ with $n \leqslant 10000$ (see [1]). We now let n be a solution of (4) with $n \neq 56$. Then we have $n>10000$. We see from (4) that

$$(9) \qquad Z(n) \equiv -d(n) \pmod{n}$$

It implies that.

$$(10) \qquad Z(n)+1 \equiv 1-d(n) \pmod{n}$$

By the proof of Theorem 1, we have $n \mid Z(n)(Z(n)+1)/2$, by (2). It can be written as

$$(11) \qquad Z(n)(Z(n)+1) \equiv 0 \pmod{n}.$$

Substituting (9) and (10) into (11), we get

$$(12) \qquad d(n)(d(n)-1) \equiv 0 \pmod{n}.$$

Notice that $d(n)>1$ if $n>1$. We see from (12) that

$$(13) \qquad (d(n))^2>n$$

Let (5) be the factorization of n. By (5), (6) and (13), we obtain

$$(14) \qquad 1>\prod_{i=1}^{k} \frac{p_i^{r_i}}{(r_i+1)^2}$$

On the other hand, it is a well known fact that $Z(p^r) \geqslant p^r - 1 > (r+1)^2$ for any prime power p^r with $p^r > 32$. We find from (14) that $k \geqslant 2$.

If $p_1 > 3$, then $p_i^{r_i}/(r_i+1)^2 \geqslant 5/4 > 1$ for $i = 1, 2, \cdots k$, It implies that if (14) holds, then either $p_1 = 2$ or $p_1 = 3$. By the same method, then n must satisfy one of the following conditions:

(i) $p_1 = 2, p_2 = 3$ and $(r_1, r_2) = (1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (1,2), (2,2), (3,2), (4,2)$ or $(5,2)$.

(ii) $p_1 = 2, p_2 > 3$ and $r_1 \leqslant 5$.

(iii) $p_1 = 3$ and $r_1 = 1$.

However, by (14), we can calculate that $n < 10000$, a contradiction. Thus, the theorem is proved.

References

[1]C. Ashbacher, The pseudo – Smarandache function and the classical functions of number theory, Smarandache Notions J., 9(1998), 78 – 81.

[2]G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, Oxford Univ. Press, 1937.

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