Problems

Edited by

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Welcome to the inaugural version of what is to be a regular feature in **Smarandache Notions**! Our goal is to present interesting and challenging problems in all areas and at all levels of difficulty with the only limits being good taste. Readers are encouraged to submit new problems and solutions to the editor at one of the addresses given above. All solvers will be acknowledged in a future issue. Please submit a solution along with your proposals if you have one. If there is no solution and the editor deems it appropriate, that problem may appear in the companion column of unsolved problems. Feel free to submit computer related problems and use computers in your work. Programs can also be submitted as part of the solution. While the editor is fluent in many programming languages, be cautious in submitting programs as solutions. Wading through several pages of obtuse program to determine if the submitter has done it right is not the editors idea of a good time. Make sure you explain things in detail.

If no solution is currently available, the problem will be flagged with an asterisk*. The deadline for submission of solutions will generally be six months after the date appearing on that issue. Regardless of deadline, no problem is ever officially closed in the sense that new insights or approaches are always welcome. If you submit a problem or solution and wish to guarantee a reply, please include a self-addressed envelope or postcard with appropriate stamps attached. Suggestions for improvement or modification are also welcome at any time. All proposals in this initial offering are by the editor.

The Smarandache function S(n) is defined in the following way

For $n \geq 1$, $S(n) = m$ is the smallest nonnegative integer such that n evenly divides m factorial.

New Problems

1) The Euler phi function $\phi(n)$ is defined to the number of positive integers not exceeding n that are relatively prime to n.

a) Prove that there are no solutions to the equation

$$\phi(S(n)) = n$$

b) Prove that there are no solutions to the equation

$$S(\phi(n)) = n$$

c) Prove that there are an infinite number of solutions to the equation

$$n - \phi(S(n)) = 1$$

d) Prove that for every odd prime p, there is a number n such that

$$n - \phi(S(n)) = p+1$$

2) This problem was proposed in **Canadian Mathematical Bulletin** by P. Erdos and was listed as unsolved in the book **Index to Mathematical Problems 1980-1984** edited by Stanley Rabinowitz and published by MathPro Press.

Prove that for infinitely many n

$$\phi(n) < \phi(n - \phi(n)).$$

3) The following appeared as unsolved problem (21) in **Unsolved Problems Related To Smarandache Function**, edited by R. Muller and published by Number Theory Publishing Company.

Are there m,n,k non-null positive integers, m,n $\neq 1$ for which

$$S(mn) = m^k * S(n)?$$

Find a solution.

4) The following appeared as unsolved problem (22) in **Unsolved Problems Related to Smarandache Function**, edited by R. Muller and published by Number Theory Publishing Company.

Is it possible to find two distinct numbers k and n such that

$$\log_{S(k^n)} S(n^k)$$

is an integer?

Find two integers n and k that satisfy these conditions.

5) Solve the following doubly true Russian alphametic

ДВА	2
ДВА	2
ТРИ	3
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СЕМЬ	7

where 2 divides ДВА, 3 divides ТРИ and 7 divides СЕМЬ.

Can anyone come up with a similar Romanian alphametic?

6) Prove the Smarandache Divisibility Theorem

If a and m are integers and $m > 0$, then

$$(a^m - a)(m - 1)!$$

is divisible by m.

Which was problem (126) in **Some Notions and Questions in Number Theory**, published by Erhus University Press.

7) Let $D = \{0,1,2,3,4,5,6,7,8,9\}$. For any number $1 \leq n \leq 10$, we can take n unique digits from D and form a number, leading zero not allowed. Let P_n be the set of all numbers that can be formed by choosing n unique digits from D. If 1 is not considered prime, which of the sets P_n contains the largest percentage of primes?

This problem is similar to unsolved problem 3 part (a) that appeared in **Only Problems, Not Solutions**, by Florentin Smarandache.

*8) The following four problems are all motivated by unsolved problem 3 part (b) that appeared in **Only Problems, Not Solutions**, by Florentin Smarandache.

a) Find the smallest integer n such that n! contains all 10 decimal digits.

b) Find the smallest integer n such that the n-th prime contains all 10 decimal digits.

c) Find the smallest integer n such that n^n contains all 10 decimal digits.

d) Find the smallest integer n such that n! contains one digit 10 times. What is that digit?

PROPOSED PROBLEMS

by

M. Bencze

(i) Solve the following equations:

1) $S^k(x) + S^k(y) = S^k(z)$, $k \in Z$, $x, y, z \in Z$

where S' is the Smarandache function and $S(-n) = -S(n)$

2) $\frac{4}{n} = \frac{1}{S(x)} + \frac{1}{S(y)} + \frac{1}{S(z)}$, $n > 4$

3) $\frac{5}{n} = \frac{1}{S(x)} + \frac{1}{S(y)} + \frac{1}{S(z)}$, $n > 5$

4) $S^{S(y)}(x) = S^{S(x)}(y)$

5) $S\left(\sum_{k=1}^{n} x_k^u\right) = S^u\left(\sum_{k=1}^{n} x_k\right)$, $u \in Z$

6) $S^y(x) - S^t(z) = S^{y-t}(x-z)$

7) $\sum_{k=1}^{n} S^m(x_k) = \sum_{k=n}^{2n} S^m(x_k)$

8) $2S(x^4) - S^2(y) = 1$

9) $S\left(\frac{x+y+z}{3}\right) + \frac{S(x)+S(y)+S(z)}{3} = \frac{2}{3}\left[S\left(\frac{x+y}{2}\right) + S\left(\frac{y+z}{2}\right) + S\left(\frac{z+x}{2}\right)\right]$

10) $S(x_1^{x_1}) \cdot S(x_2^{x_2}) \ldots S(x_n^{x_n}) = S(x_{n+1}^{x_{n+1}})$

11) $S(x_1^{x_2}) \cdot S(x_2^{x_3}) \ldots S(x_{n-1}^{x_n}) = S(x_n^{x_1})$

12) $S(x) = \mu(y)$, where μ is the Möbius function

13) $S^2(Q_n) = \sum_{Q_{n-1} | Q_n} \ldots \sum_{Q_2 | Q_3} \sum_{Q_1 | Q_2} \mu^2(Q_1)$

14) $S(x) = B_y$, where B_y is a Bernoulli number

15) $S(x+y)\,(\,S(x) - S(y)\,) = S(x-y)\,(\,S(x) + S(y)\,)$

16) $S(x) = F_y$, where F_y is a Fibonacci number

17) $\sum_{k=1}^{n} S(k^p) = \sum_{k=1}^{n} S^p(k)$

18) $\sum_{k=1}^{n} S(k) = S\left(\frac{n(n+1)}{2}\right)$

19) $\sum_{k=1}^{n} S(k^2) = S\left(\frac{n(n+1)(2n+1)}{6}\right)$

20) $\sum_{k=1}^{n} S(k^3) = S\left(\frac{n^2(n+1)^2}{4}\right)$

21) $\sum_{k=1}^{n} k(S(k)!) = (S(n+1))! - 1$

22) $\sum_{k=1}^{n} \frac{1}{S(k)S(k+1)} = \frac{S(n)}{S(n+1)}$

(ii) Solve the system

$$\begin{cases} S(x) + S(y) = 2S(z) \\ S(x) \cdot S(y) = S^2(z) \end{cases}$$

(iii) Find n such that n divides the sum

$$1^{S(n-1)} + 2^{S(n-1)} + \ldots (n-1)^{S(n-1)} + 1$$

(iv) May be writen every positive integer n as

$$n = S^3(x) + 2\,S^3(y)\;3\,S^3(z) \quad ?$$

(v) Prove that

$$|S(x) + S(y) + S(z)| + |S(x)| + |S(y)| + |S(z)| \geq$$

$$\geq |S(x) + S(y)| + |S(y) + S(z)| + |S(z) + S(x)|$$

for all $x, y, z \in Z$

(vi) Find all the positive integers x, y, z for which

$$(x+y+z) + S(x) + S(y) + S(z) \geq S(x+y) + S(y+z) + S(z+x)$$

(vii) There exists an infinity of prime numbers which may be writen under the form

$$P = S^3(x) + S^3(y) + S^3(z) + S^3(t) \quad ?$$

(viii) Let $M_1, M_2, \ldots, M_n$ be finite sets and $a_{ij} = \mathrm{card}(M_i \cap M_j)$, $b_{ij} = S(a_{ij})$. Prove that $\det(a_{ij}) \geq 0$ and $\det(b_{ij}) \geq 0$.

(ix) Find the sum

$$\sum_{Q_{n-1}|Q_n} \cdots \sum_{Q_2|Q_3} \sum_{Q_1|Q_2} \frac{1}{S^2(Q_1)}$$

(x) Prove that

$$\sum_{k=1}^{\infty} \frac{1}{S^2(k) - S(k) + 1} \quad \text{is irational}$$

(xi) Find all the positive integers x for which

$$S\left(\left[\frac{x^{n+1} - 1}{(n+1)(x-1)}\right]\right) \geq S\left(\left[x^{\frac{n}{2}}\right]\right)$$

where [x] is the integer part of x.

(xii) There exists at lest a prime between S(n)!, and S(n+1)! ?

(xiii) If $\sigma \in S_n$ is a permutation, prove that

$$\sum_{k=1}^{n} \frac{\sigma(k)}{S^{m+1}(k)} \geq \sum_{k=1}^{n} \frac{1}{k^m}$$

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PROPOSED PROBLEM*

by

I. M. Radu

Show that (except for a finite set of numbers) between S(n) and S(n+1) there exist at least a prime number. (One notes by S(n) the Smarandache Function: the smallest integer such that S(n)! is divisible by n.)

If $N_s(n)$ denotes the number of primes between S(n) and S(n+1), calculate an asymptotic formula for $N_s(n)$.

Some comments:

If n or n+1 is prime, then S(n) or S(n+1) respectively is prime. And the above conjuncture is solved.

But I was not able to find a general proof. It might be a useful thing to apply the Brensch Theorem (if $n \geq 48$, then there exist at least a prime between n and $\frac{9}{8}n$), in stead of Bertrand Postulate / Tchebychev Theorem (between *n* and *2n* there exist at least a prime)

The last question may be writing as

$$N_s(n) = |\Pi(S(n+1)) - \Pi(S(n))| \ ,$$

where $\Pi(x)$ is the number of primes $\leq x$,
but how can we compose the function Π and S ?

References:

[1] Dumitrescu Constantin, "The Smarandache Function", in "Mathematical Spectrum", Sheffield, Vol. 29, No. 2, 1993, pp. 39-40.

[2] Ibstedt Henry, "The F. Smarandache Function S(n): programs, tables, graphs, comments", in "Smarandache Function Journal", Vol. 2-3, No. 1, 1993, pp. 38-71.

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*Charles Ashbacher (U.S.A.), using a computer program that computes the values of S(n) conducted a search up through n<1,033,197 and found where there is no prime p, where $S(n) \leq p \leq S(n+1)$. They are as follows:

n = 224	and S(n) = 8, S(n+1) = 10
n = 2057	and S(n) = 22, S(n+1) = 21
n = 265225	and S(n) = 206, S(n+1) = 202
n = 843637	and S(n) = 302, S(n+1) = 298

PROPOSED PROBLEMS

by

M. R. Mudge

Problem 1:

The Smarandache no prime digits sequence is defined as follows:

1,4,6,8,9,10,11,1,1,14,1,16,1,18,19,0,1,4,6,8,9,0,1,4,6,8,9,40,41, 42,4,44,4,46,4,48,49,0,...

(Take out all prime digits of n.)

Is it any number that occurs infinetely many times in this sequence? (for example 1, or 4, or 6, or 11, etc.).

Solution by Dr. Igor Shparlinski,

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It seems that: if, say n has already occured, then for example n3, n33, n333, etc. gives an infinitely many repetitions of this number.

Problem 2:

The Smarandache no square digits sequence is defined as follows:

2,3,5,6,7,8,2,3,5,6,7,8,2,2,22,23,2,25,26,27,28,2,3,3,32,33,3,35, 36,37,38,3,2,3,5,6,7,8,5,5,52,52,5,55,56,57,58,5,6,6,62,...

(Take out all square digits of n.)

Is it any number that occurs infinetely many times in this sequence? (for example 2, or 3, or 6, or 22, etc. ?)

Solution by Carl Pomerance (E-mail: carl@alpha.math.uga.edu):

If any number appears in the sequence, then clearly it occurs infinitely often, since if the number that appears is k, and it comes from n by deleting square digits, then k also comes from 10n.

Problem 3:

How many regions are formed by joining, with straight chords, n point situated regularly on the circumference of a circle ?

The degeneracy from the maximum possible number of regions for n points on the circumference of a circle seems almost intractable in general.

Perhaps the use of regularly distributed point, i.e. separated by $\frac{2\pi}{n}$ radians, produces the Smarandache Portions of Pi (e) !!

Unsolved Problems

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Welcome to the first installment of what is to be a regular feature in **Smarandache Notions**! In this column, we will present problems where the solution is either unknown or incomplete. This is meant to be an interactive endeavor, so input from readers is strongly encouraged. Always feel free to contact the editor at any of the addresses given above. It is hoped that we can together advance the flow of mathematics in some small way. There will be no deadlines here, and even if a problem is completely solved, new insights or more elegant proofs are always welcome. All correspondents who are the first to resolve any issue appearing here will have their efforts acknowledged in a future column.

While there will almost certainly be an emphasis on problems related to Smarandache notions, it will not be exclusive. Our goal here is to be interesting, challenging and maybe at times even profound. In modern times computers are an integral part of mathematics and this column is no exception. Feel free to include computer programs with your submissions, but please make sure that adequate documentation is included. If you are someone with significant computer resources and would like to be part of a collective effort to resolve outstanding problems, please contact the editor. If such a group can be formed, then sections of a problem can be parceled out and all those who participated will be given credit for the solution.

And now, it is time to stop chatting and get to work!

Definition of the Smarandache function $S(n)$:

$S(n) = m$, smallest positive integer such that $m!$ is evenly divisible by n.

In [1], T. Yau posed the following question:

For what triplets n, $n+1$, and $n+2$ does the Smarandache function satisfy the Fibonacci relationship

$$S(n) + S(n+1) = S(n+2)?$$

And two solutions

$S(9) + S(10) = S(11)$; $S(119) + S(120) = S(121)$

were given.

In [2], C. Ashbacher listed the additional solutions

$S(4900) + S(4901) = S(4902)$; $S(26243) + S(26244) = S(26245)$;
$S(32110) + S(32111) = S(32112)$; $S(64008) + S(64009) = S(64010)$;
$S(368138) + S(368139) = S(368140)$; $S(415662) + S(415663) = S(415664)$

discovered in a computer search up through n = 1,000,000. He then presented arguments to support the conjecture that the number of solutions is in fact infinite.

Recently, Henry Ibstedt from Sweden sent a letter in response to this same problem appearing in the October issue of **Personal Computer World**. He has conducted a more extensive computer search, finding many other solutions. His conclusion was, "This study strongly indicates that the set of solutions is infinite." The complete report has been submitted to **PCW** for publication.

Another problem dealing with the Smarandache function has been given the name Radu's problem, having been first proposed by I.M. Radu[3].

Show that, except for a finite set of numbers, there exist at least one prime number between S(n) and S(n+1).

Ashbacher also dealt with this problem in [2] and conducted another computer search up through n = 1,000,000. Four instances where there are no primes between S(n) and S(n+1) were found.

n = 224 = 2*2*2*2*2*7 S(n) = 8 n+1 = 225 = 3*3*5*5 S(225) = 10
n = 2057 = 11*11*17 S(n) = 22 n+1 = 2058 = 2*3*7*7*7 S(2058) = 21
n = 265225 = 5*5*103*103 S(n) = 206 n+1 = 265226 = 2*13*101*101
S(265226) = 202
n = 843637 = 37*151*151 S(n) = 302 n+1 = 843638 = 2*19*149*149
S(843638) = 298

The fact that the last two solutions involve the pairs of twin primes (101,103) and (149,151) was one point used to justify the conjecture that there is an infinite set of numbers such that there is no prime between S(n) and S(n+1).

Ibstedt also extended the computer search for solutions and found many other cases where there is no prime between S(n) and S(n+1). His conclusion is quoted below.

"A very large set of solutions was obtained. There is no indication that the set would be finite."

This conclusion is also due to appear in a future issue of **Personal Computer World**.

The following statement appears in [4].

$$1141^6 = 74^6 + 234^6 + 402^6 + 474^6 + 702^6 + 894^6 + 1077^6$$

This is the smallest known solution for 6th power as the sum of 7 other 6th powers.

Is this indeed the smallest such solution? No one seems to know. The editor would be interested in any information about this problem. Clearly, given enough computer time, it can be resolved. This simple problem is also a prime candidate for a group effort at resolution.

Another related problem that would be also be a prime candidate for a group effort at computer resolution appeared as problem 1223 in **Journal of Recreational Mathematics**.

Find the smallest integer that is the sum of two unequal fifth powers in two different ways, or prove that there is none.

The case of third powers is well known as a result of the famous story concerning the number of a taxicab

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

as related by Hardy[4].

It was once conjectured that there might be a solution for the fifth power case where the sum had about 25 decimal digits, but a computer search for a solution with sum $< 1.02 \times 10^{26}$ yielded no solutions[5].

Problem (24) in [6] involves the Smarandache Pierced Chain(SPC) sequence.

{ 101, 1010101, 10101010101, 101010101010101, . . . }

or

$$SPC(n) = 101 * 1\ \underbrace{0001}\ 0001 \ldots 0001$$

where the section in | ---- | appears n-1 times.

And the question is, how many of the numbers

SPC(n) / 101 are prime?

It is easy to verify that if n is evenly divisible by 3, then the number of 1's in SPC(n) is evenly divisible by 3. Therefore, so is SPC(n). And since 101 is not divisible by 3, it follows that

SPC(n) / 101

must be divisible by 3.

A simple induction proof verifies that SPC(2k)/101 is evenly divisible by 73 for k = 1,2,3, . . .

Basis step:

SPC(2)/101 = 73*137

Inductive step:

Assume that SPC(2k)/101 is evenly divisible by 73. From this, it is obvious that 73 divides SPC(2k). Following the rules of the sequence, SPC(2(k+1)) is formed by appending 01010101 to the end of SPC(2k). Since

01010101 / 73 = 13837

it follows that SPC(2(k+1)) must also be divisible by 73.

Therefore, SPC(2k) is divisible by 73 for all k > 0. Since 73 does not divide 101, it follows that SPC(2k) / 101 is also divisible by 73.

Similar reasoning can be used to obtain the companion result.

SPC(3 + 4k) is evenly divisible by 37 for all k > 0.

With these restrictions, the first element in the sequence that can possibly be prime when divided by 101 is

SPC(5) = 1010101010101010101.

However, this does not yield a prime as

SPC(5) = 41 * 101 * 271 * 3541 * 9091 * 27961.

Furthermore, since the elements of the sequence SPC(5k), k > 0 are made by appending the string

0101010101010101010101 = 41 * 101 * 271 * 3541 * 9091 * 27961

to the previous element, it is also clear that every number SPC(5k) is evenly divisible by 271 and therefore so is SPC(5k)/101.

Using these results to reduce the field of search, the first one that can possibly be prime is SPC(13)/101. However,

SPC(13)/101 = 53 * 79 * 521 * 859 * ......

SPC(17)/101 is the next not yet been filtered out. But it is also not prime as

SPC(17)/101 = 103 * 4013 * .....

The next one to check is SPC(29)/101, which is also not prime as

SPC(29)/101 = 59 * 349 * 3191 * 16763 * 38861 * 43037 * 62003 * ....

SPC(31)/101 is also not prime as

SPC(31)/101 = 2791 * ....

At this point we can stop and argue that the numerical evidence strongly indicates that there are no primes in this sequence. The problem is now passed on to the readership to perform additional testing or perhaps come up with a proof that there are no primes in this sequence.

References

1. T. Yau: `A Problem Concerning the Fibonacci Series', Smarandache Function Journal, V. 4-5, No. 1, (1994), page 42.

2. C. Ashbacher, **An Introduction To the Smarandache Function**, Erhus University Press, 1995.

3. I. M. Radu, Letter to the Editor, Math. Spectrum, Vol. 27, No. 2, (1994/1995), page 44.

4. D. Wells, **The Penguin Dictionary of Curious and Interesting Numbers**, Penguin Books, 1987.

5. R. Guy, **Unsolved Problems in Number Theory 2nd. Ed.**,Springer Verlag, 1994.

6. F. Smarandache, **Only Problems, Not Solutions!**, Xiquan Publishing House, 1993.