SOME CONJECTURES ON PRIMES (I)

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Abstract. For any complex number x, let $\exp(x)=e^x$. For any positive integer n, let p_n be the nth prime. In this paper we prove that $\exp\left(\sqrt{(n+1)/p_{n+1}}\right)/\exp\left(\sqrt{p_n/n}\right) < \exp\left(\sqrt{3/5}\right)/\exp\left(\sqrt{3/2}\right)$.

Key words: prime, inequality.

For any complex number x, let $\exp(x)=e^x$. For any positive integer n, let p_n be the nth prime. Recently, Russo [2] proposed the following conjecture:

Conjecture For any positive integer n,

$$\frac{\exp\left(\sqrt{\frac{n+1}{p_{n+1}}}\right)}{\exp\left(\sqrt{\frac{p_n}{n}}\right)} < \frac{\exp\left(\sqrt{\frac{3}{5}}\right)}{\exp\left(\sqrt{\frac{3}{2}}\right)}. \tag{1}$$

In [2], Russo verified (1) for $p_n \leqslant 10^7$. In this paper we completely solve the above-mentioned conjecture as follows.

Theorem For any positive integer n, the inequality (1) holds.

Proof We may assume that $p_n > 10^7$. Then we have $n > 1000$. It is a well known fact that

$$p_n > n \log n, \tag{2}$$

for any positive integer n (see [1]). By (2), we get

(3) $$\exp\left(\sqrt{\frac{p_n}{n}}\right) > \exp\left(\sqrt{\log n}\right) > \exp\left(\sqrt{\log 1000}\right) > \exp(2.6).$$

On the other hand, since $p_{n+1} > n+1$, we get

(4) $$\exp\left(\sqrt{\frac{3}{2}} - \sqrt{\frac{3}{5}} + \sqrt{\frac{n+1}{p_{n+1}}}\right) < \exp\left(\sqrt{\frac{3}{2}} - \sqrt{\frac{3}{5}} + 1\right) < \exp(1.5).$$

The combination of (3) and (4) yields

(5) $$\exp\left(\sqrt{\frac{p_n}{n}}\right) > \exp\left(\sqrt{\frac{3}{2}} - \sqrt{\frac{3}{5}} + \sqrt{\frac{n+1}{p_{n+1}}}\right).$$

Thus, by (5), we get (1) immediately. The theorem is proved.

References

[1] B. Rosser, The *n*th prime is greater than *n*log*n*, Proc. London Math. Soc. (2) 45(1938), 21-44.

[2] F. Russo, Ten conjectures on prime numbers, Smarandache Notions J. 12(2001), 295-296.

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