SOME CONJECTURES ON PRIMES (III)

Maohua Le

Abstract. For any positive integer n, let p_n be the nth prime. In this paper we prove that the equality $\left(\sqrt{p_n} - \log p_{n+1}\right)/\left(\sqrt{p_{n+1}} - \log p_n\right) \geq \left(\sqrt{3} - \log 5\right)/\left(\sqrt{5} - \log 3\right)$ for any n.

Key words: prime, inequality.

For any positive integer n, let p_n be the nth prime. Recently, Russo [2] proposed the following conjecture:

Conjecture For any positive integer n,

$$\frac{\sqrt{p_n} - \log p_{n+1}}{\sqrt{p_{n+1}} - \log p_n} \geq \frac{\sqrt{3} - \log 5}{\sqrt{5} - \log 3}. \tag{1}$$

In [2], Russo verified the equality (1) holds for $p_n \leqslant 10^7$. In this paper, we completely solve the above-mentioned problem as follows.

Theorem For any positive integer n, the equality (1) holds.

Proof We may assume that $p_n > 10^7$. Since

$$\frac{\sqrt{3} - \log 5}{\sqrt{5} - \log 3} < 0.11, \tag{2}$$

if (1) is false, then from (2) we get

$$\sqrt{p_n} < \log p_{n+1} + 0.11\sqrt{p_{n+1}}. \tag{3}$$

It is a well known fact that

(4) $$p_{n+1} < 2p_n$$

for any positive integer n (see [1, Theorem 245]). Substitute (4) into (3), we obtain

(5) $$0.84\sqrt{p_n} < \log(2p_n).$$

However, (5) is impossible for $p_n > 10^7$. Thus, the theorem is proved.

References

[1] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, Oxford Univ. Press, Oxford, 1938.

[2] F. Russo, Ten conjectures on prime numbers, Smarandache Notions J. 12(2001), 295-296.

Department of Mathematics
Zhanjiang Normal College
Zhanjiang, Guangdong
P. R. CHINA