SOME CONNECTIONS BETWEEN THE SMARANDACHE FUNCTION AND THE FIBONACCI SEQUENCE

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I. INTRODUCTION

The Smarandache function $S: N^* \to N^*$ is defined [9] by the condition that $S(n)$ is the smallest positive integer k such that $k!$ is divisible by n.

If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_{21}} \cdot \ldots \cdot p_t^{\alpha_t} \qquad (1)$$

is the decomposition of the positive integer n into primes, then it is easy to verify that

$$S(n) = \max\left(S\left(p_i^{\alpha_i}\right)\right) \qquad (2)$$

One of the most important properties of this function is that a positive integer p is a fixed point of S if and only if p is a prime or $p = 4$.

This paper is aimed to provide generalizations of the Smarandache function. They will be constructed by means of sequences more general than the sequence of the factorials. Such sequences are monotonously convergent to zero sequences and divisibility sequences (in particular the Fibonacci sequence).

Our main result states that the Smarandache generalized function associated with every strong divisibility sequence (sequence satisfying the condition (15) from bellow) is a dual strong divisibility sequence (i.e. it satisfies the condition (26), the dual of (15)).

Note that the Smarandache function S is not monotonous. Indeed, $n_1 \le n_2$ does not imply $S(n_1) \le S(n_2)$. For instance $5 \le 12$ and $S(5) = 5$, $S(12) = 4$.

Let us denote by $\overset{d}{\vee}$ the least common multiple, by $\underset{d}{\wedge}$ the greatest common divisor and let $\wedge = \min$, $\vee = \max$. It is known that

$$N_0 = \left(N^*, \wedge, \vee\right) \text{ and } N_d = \left(N^*, \underset{d}{\wedge}, \overset{d}{\vee}\right)$$

are lattices. The order on N* corresponding to the lattice N_0 is the usual order:

$$n_1 \le n_2 \Leftrightarrow n_1 \wedge n_2 = n_1$$

and it is a total order. On the contrary, the order $\underset{d}{\le}$ corresponding to the lattice N_d, defined as

$$n_1 \underset{d}{\le} n_2 \Leftrightarrow n_1 \underset{d}{\wedge} n_2 = n_1$$

(the divisibility relation) is only a partial order.

More precisely we have

$n_1 \underset{d}{\leq} n_2 \Leftrightarrow n_1$ *divides* n_2.

For $n_1 \underset{d}{\leq} n_2$ we shall also write $n_2 \underset{d}{\geq} n_1$. We notice that N_d has *zero* as the greatest element, N_0 does not possess a greatest element and both lattices have 1 as the smallest element. Then it is convenient to consider in N_0 the convergence to infinity and in N_d, the convergence to zero.

Let

$n_1 = \prod p_i^{\alpha_i}$ and $n_2 = \prod p_i^{\beta_i}$

be the decompositions into primes of n_1 and n_2. Then we have

$$n_1 \overset{d}{\vee} n_2 = \prod p_i^{\max(\alpha_i, \beta_i)}.$$

The definition of the Smarandache function implies that

$$S\left(n_1 \overset{d}{\vee} n_2\right) = S(n_1) \vee S(n_2) \tag{3}$$

Also we have

$$n_1 \underset{d}{\leq} n_2 \Rightarrow S(n_1) \leq S(n_2). \tag{4}$$

In order to make explicit the lattice (so, the order) on the set N^*, we shall write N_0 instead of N^*, if the order on the set of the positive integers is the usual order and N_d instead of N^*, if we consider the order $\underset{d}{\leq}$ respectively.

Then (4) shows that the Smarandache function, considered as a function

$$S : N_d \to N_0, \tag{5}$$

is an order preserving map.

From (2) it follows that the determination of $S(n)$ reduces to the computation of $S(p^\alpha)$. In addition, it is proved [1] that if the sequence

$$(p): \; 1, \; p, \; p^2, \; \ldots, \; p^i, \; \ldots \tag{6}$$

is the standard $p-scale$ and the sequence

$$[p]: \; a_1(p), \; a_2(p), \; \ldots, \; a_i(p), \; \ldots$$

is the generalized numerical scale determined by the sequence

$$a_i(p) = \frac{p^i - 1}{p - 1}$$

then

$$S(p^\alpha) = p(\alpha_{[p]})_{(p)} \; . \tag{7}$$

In other words, $S(p^\alpha)$ can be obtained by multiplying by p the number obtained writing the exponent α in the generalized scale $[p]$ and "reading" it in the scale (p).

For instance, in order to calculate $S(3^{99})$ let us consider the scale

[3] 1, 4, 13, 40, 121, ...

Then, for $\alpha = 99$, we have

$$\alpha_{[3]} = 2a_4(3) + a_3(3) + a_2(3) + 2a_1(3) = 2112_{[3]}$$

and "reading" this number in the usual scale

(3) $\quad 1,\ 3,\ 3^2,\ 3^3,\ \ldots$

we get $S(3^{99}) = 3(2\cdot 3^3 + 3^2 + 3 + 2)$=204. So, 204 is the smallest positive integer whose factorial is divisible by 3^{99}.

We quote also the following formula used to compute $S(p^\alpha)$:

$$S(p^\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha), \tag{8}$$

where $\sigma_{[p]}(\alpha)$ stands for the sum of the digits of the integer α written in the scale $[p]$.

2. GENERALIZED SMARANDACHE FUNCTIONS

A scquence of positive integers is a mapping $\sigma : N^* \to N^*$ and it is usualy denoted by $(\sigma_n)_{n\in N^*}$ (i.e. the set of its values). Since in the sequel an essential point is to make cvident the structure (the lattice) on the domain and on the range of this function respectively, we adopt the notation from (5).

Then

$$\sigma : N_0 \to N_d \tag{9}$$

shows that σ is a sequence of positive integers defined on the set N^*. This set was structured as a lattice by $\wedge$ and $\vee$ and its range has also a structure of lattice, induced by $\underset{d}{\wedge}$ and $\overset{d}{\vee}$.

Definition 2.1. [3] The sequence (9) is a *multiplicatively convergent to zero* sequence (mcz) if

$$(\forall) n \in N^* \quad (\exists) \quad m_n \in N^* \quad (\forall) m \geq m_n \Rightarrow n \underset{d}{\leq} \sigma(m). \tag{10}$$

In other words, a (mcz) sequence is a sequence defined as in (9), which is *convergent to zero*.

These sequences, satisfying in addition the condition

$$\sigma(n) \underset{d}{\leq} \sigma(n+1) \tag{11}$$

(that is $\sigma(n)$ divides $\sigma(n+1)$) were considered by G. Christol [3] in order to obtain a generalization of $p-adic$ numbers.

As an example of a (mcz) sequence we may consider the sequence defined by $\sigma(n) = n!$. This sequence also satisfies the condition (11).

Remark 2.1. We find that the value $S(n)$ of tbe Smarandache function at the point n is the smallest integer m_n provided by (10), whenever $\sigma(n) = n!$. This enables us to define a Smarandache type function for each (mcz) sequence. Indeed, for an arbitrary (mcz) sequence σ, we may define $S_\sigma(n)$ as the smallest integer m_n given by (10).

The (mcz) sequences satisfying the extra-condition (11) generalize thc factorial. Indeed, if

$$\sigma(n+1) = k_{n+1}\sigma(n) \tag{12}$$

then

$$\sigma(n) = k_1 \cdot k_2 \cdot \ldots \cdot k_n, \text{ with } k_1 = 1 \text{ and } k_i \in N^* \text{ for } i > 1.$$

Starting with the lattices N_0 and N_d, we can construct sequences

$$\sigma : N_d \to N_d \ . \tag{13}$$

Definition 2.2. A sequence (13) is called a *divisibility sequence (ds)* if

$$n \underset{d}{\leq} m \Rightarrow \sigma(n) \underset{d}{\leq} \sigma(m) \tag{14}$$

(that is if the mapping σ from (13) ia monotonous). The sequence (13) is called a *strong divisibility sequence (sds)* if

$$\sigma\left(n \underset{d}{\wedge} m\right) = \sigma(n) \underset{d}{\wedge} \sigma(m) \text{ for every } n, m \in N^* . \tag{15}$$

Strong divisibility sequences are considered, for instance, by N. Jensen in [5].

It is known that the Fibonacci sequence is also (sds).

For a sequence σ of positive integers, concepts as (usual) monotonicity, multiplicatively convcrgence to zero, divisibility, have been independently studied by many authors. A unifying treatement of these concepts can be achieved if we remark that they are monotonicity or convergence conditions of a given sequence $\sigma : N^* \to N^*$, for adequate lattices on N^*.

We shall consider now all the possibilities to define a sequence of positive integers, with respect to the lattices N_0 and N_d. To make briefly evident thc kind of the lattice considered on the domain and on the range of α, we shall use the following notation:

(a) *a sequence* $\sigma_{oo} : N_o \to N_o$ *is an* (oo)*− sequence*

(b) *a sequence* $\sigma_{od} : N_o \to N_d$ *is an* (od)*− sequence*

(c) *a sequence* $\sigma_{do} : N_d \to N_o$ *is an* (do)*− sequence*

(d) *a sequence* $\sigma_{dd} : N_d \to N_d$ *is a* (dd)*− sequence*

We have already seen (Remark 2.1) that, considering (mcz) sequences, the Smarandache function may be generalized.

In order to generalize tbe Smarandache function for each type of the above sequences, it is necessary to consider the monotonicity and the existence of a limit corresponding to each of the cases (*a*) - (*d*).

Of course, the limit is *infinit* for N_o-valued sequence and it is *zero* for the others. We have four kinds of monotonicity.

For a (do)*− squence* σ_{do}, the monotonicity reads:

$$(m_{do}) \quad (\forall) n_1, n_2 \in N^*, \quad n_1 \underset{d}{\leq} n_2 \Rightarrow \sigma_{do}(n_1) \leq \sigma_{do}(n_2)$$

and the condition of convergence to infinity is:

$$(c_{do}) \quad (\forall) n \in N^* \quad (\exists) m_n \in N^* \quad (\forall) m \underset{d}{\geq} m_n \Rightarrow \sigma_{do}(m) \geq n .$$

Similarly, for a (dd)*− sequence* σ_{dd}, the monotonicity reads:

$(m_{dd})\quad (\forall) n_1, n_2 \in N^*, \quad n_1 \underset{d}{\leq} n_2 \Rightarrow \sigma_{dd}(n_1) \underset{d}{\leq} \sigma_{dd}(n_2)$

and the convergence to zero is:

$(c_{dd})\quad (\forall) n \in N^* \quad (\exists) m_n \in N^* \quad (\forall) m \underset{d}{\geq} m_n \Rightarrow \sigma_{dd}(m) \underset{d}{\geq} n$.

Definition 2.3. The *generalized Smarandache function* associated to a sequence σ_{ij} satisfying tbe condition (c_{ij}), with $i, j \in \{o, d\}$, is

$$S_{ij}(n) = \min \{m_n \mid m_n \text{ given by the condition } (c_{ij})\} \tag{16}$$

Remark that (oo)-*sequences* are the classical sequences of positive integers. As examples of (od)-*sequences* we quote the (mcz) sequences. Examples of (dd)-*sequences* are (ds) and (sds)-*sequences*. Finally, the generalized Smarandache functions S_{od} associated with (od)-*sequences* satisfying the condition (c_{od}) are (do)-*sequences*.

The functions S_{ij} have the following properties:

Theorem 2.1. *Every function S_{oo} satisfies:*

$(i)\quad (\forall) n_1, n_2 \in N^*, \quad n_1 \leq n_2 \Rightarrow S_{oo}(n_1) \leq S_{oo}(n_2),$

that is S_{oo} satisfies (m_{oo}).

$(ii)\quad S_{oo}(n_1 \vee n_2) = S_{oo}(n_1) \vee S_{oo}(n_2)$

$(iii)\quad S_{oo}(n_1 \wedge n_2) = S_{oo}(n_1) \wedge S_{oo}(n_2)$.

Proof: (i) The definition of $S_{oo}(n)$ implies that:

$S_{oo}(n_i) = \min \{m_{n_i} \mid (\forall) m \geq m_{n_i} \Rightarrow \sigma_{oo}(m) \geq n_i\}$, for $i = 1, 2$

Therefore

$(\forall) m \geq S_{oo}(n_2) \Rightarrow \sigma_{oo}(m) \geq n_2 \geq n_1$

and so $S_{oo}(n_1) \leq S_{oo}(n_2)$. The equalities (ii) and (iii) are consequences of (i).

Theorem 2.2. *Every function S_{od} has the following properties:*

$(iv)\quad (\forall) n_1, n_2 \in N^*, \quad n_1 \underset{d}{\leq} n_2 \Rightarrow S_{od}(n_1) \leq S_{od}(n_2)$

that is S_{od} satisfies (m_{od}).

$(v)\quad S_{od}\left(n_1 \overset{d}{\vee} n_2\right) = S_{od}(n_1) \vee S_{od}(n_2)$.

$(vi)\quad S_{od}\left(n_1 \underset{d}{\wedge} n_2\right) \leq S_{od}(n_1) \wedge S_{od}(n_2)$.

Proof: The equality (v) may be proved in the same manner as the equality (3) for the function S. Then from (v) it follows (iv).

For (vi) let us note $u = S_{od}(n_1) \wedge S_{od}(n_2)$. From

$n_1 \underset{d}{\wedge} n_2 \underset{d}{\leq} n_1, \quad n_1 \underset{d}{\wedge} n_2 \underset{d}{\leq} n_2$

and from (iv), it follows that

$$S_{od}\left(n_1 \underset{d}{\wedge} n_2\right) \leq S_{od}(n_1), \quad S_{od}\left(n_1 \underset{d}{\wedge} n_2\right) \leq S_{od}(n_2),$$

so $S_{od}\left(n_1 \underset{d}{\wedge} n_2\right) \leq S_{od}(n_1) \wedge S_{od}(n_2)$.

Theorem 2.3. *The functions S_{do} satisfy:*

(vii) $(\forall) n_1, n_2 \in N^*$, $n_1 \leq n_2 \Rightarrow S_{do}(n_1) \leq S_{do}(n_2)$.

$(viii)$ $S_{do}(n_1 \vee n_2) \leq S_{do}(n_1) \overset{d}{\vee} S_{do}(n_2)$.

(ix) $S_{do}(n_1 \vee n_2) = S_{do}(n_1) \vee S_{do}(n_2)$.

(x) $S_{do}(n_1 \wedge n_2) = S_{do}(n_1) \wedge S_{do}(n_2)$.

Proof: Let us note that (ix) and (x) are consequences of (vii). In our terms (vii) is just the fact that the Smarandache generalized function S_{do} associated with a $(do)-sequence$ is $(oo)-monotonous$. To prove this assertion, let $n_1 \leq n_2$. Then for every $m \underset{d}{\geq} m_{n_2}$, we have

$$\sigma_{do}(m) \geq n_2 \geq n_1$$

and so $S_{do}(n_1) \leq S_{do}(n_2)$.

$(viii)$ For i = 1, 2 we have:

$$S_{do}(n_i) = \min\left\{m_{n_i} \mid (\forall) m \underset{d}{\geq} m_{n_i} \Rightarrow \sigma_{do}(m) \geq n_i\right\}$$

Let us suppose that $n_1 \leq n_2$, so $n_1 \vee n_2 = n_2$ and $S_{do}(n_1 \vee n_2) = S_{do}(n_2)$. If we take $m_0 = S_{do}(n_1) \overset{d}{\vee} S_{do}(n_2)$, then for every $m \underset{d}{\geq} m_0$ it follows that $\sigma_{do}(m) \geq n_i$, for i = 1, 2, so $\sigma_{do}(m) \geq n_1 \vee n$, whence the desired inequality.

Consequence 2.1. $S_{do}(n_1) \underset{d}{\wedge} S_{do}(n_2) \leq S_{do}(n_1) \wedge S_{do}(n_2) = S_{do}(n_1 \wedge n_2) \leq$

$S_{do}(n_1) \vee S_{do}(n_2) = S_{do}(n_1 \vee n_2) \leq S_{do}(n_1) \overset{d}{\vee} S_{do}(n_2)$.

Theorem 2.4. *The functions S_{dd} satisfy:*

(xi) $S_{dd}\left(n_1 \overset{d}{\vee} n_2\right) \leq S_{dd}(n_1) \overset{d}{\vee} S_{dd}(n_2)$.

(xii) *If $n_1 \underset{d}{\leq} n_2$ or $n_2 \underset{d}{\leq} n_1$ then*

$$S_{dd}\left(n_1 \overset{d}{\vee} n_2\right) = S_{dd}(n_1) \vee S_{dd}(n_2).$$

$(xiii)$ $S_{dd}\left(n_1 \underset{d}{\wedge} n_2\right) \leq S_{dd}(n_1) \wedge S_{dd}(n_2)$.

Proof: The proof of (xi) is similar to the proof of $(viii)$ and the other assertions may be easily obtained by using the definition of S_{dd} from (17) (for $i = j = d$).

Consequence 2.2. For all $n_1, n_2 \in N^*$ we have

$$S_{dd}(n_1) \vee S_{dd}(n_2) \leq S_{dd}\left(n_1 \overset{d}{\vee} n_2\right) \leq S_{dd}(n_1) \overset{d}{\vee} S_{dd}(n_2).$$

This follows from the fact that

$$n_i \underset{d}{\leq} n_1 \overset{d}{\vee} n_2 \quad for\; i = 1, 2 \Rightarrow S_{dd}(n_i) \leq S_{dd}\left(n_1 \overset{d}{\vee} n_2\right).$$

If σ_{dd} is a divisibility sequence, the above theorem implies that the associated Smarandache function satisfies the inequality (xi). In the following we shall see that, if the sequence σ_{dd} is a divisibility sequence with additional properties, namely if it is a strong divisibility sequence, then the inequality (xi) becomes equality.

Theorem 2.5: *If σ_{dd} is a (sds) satisfying the condition (c_{dd}), then*:

$$S_{dd}\left(n_1 \overset{d}{\vee} n_2\right) = S_{dd}(n_1) \overset{d}{\vee} S_{dd}(n_2) \tag{17}$$

and

$$(\forall) n_1, n_2 \in N^*, \quad n_1 \underset{d}{\leq} n_2 \Rightarrow S_{dd}(n_1) \underset{d}{\leq} S_{dd}(n_2) \tag{18}$$

(i.e. S_{dd} satisfies the monotonicity condition (m_{dd})).

Proof: In order to prove the equality (17), it is sufficient to show that

$$S_{dd}(n_i) \underset{d}{\leq} S_{dd}\left(n_1 \overset{d}{\vee} n_2\right), \text{ for } i = 1, 2.$$

But if, for instance, the above inequality does not hold for n_1 and we denote

$$d_O = S_{dd}(n_1) \underset{d}{\wedge} S_{dd}\left(n_1 \overset{d}{\vee} n_2\right),$$

it follows that $d_O < S_{dd}(n_1)$ and taking into account that

$$\sigma_{dd}(S_{dd}(n_1)) \underset{d}{\geq} n_1 \quad and \quad n_1 \underset{d}{\leq} n_1 \overset{d}{\vee} n_2 \underset{d}{\leq} \sigma_{dd}\left(S_{dd}\left(n_1 \overset{d}{\vee} n_2\right)\right),$$

we have

$$\sigma_{dd}(d_O) = \sigma_{dd}\left(S_{dd}(n_1) \underset{d}{\wedge} S_{dd}\left(n_1 \overset{d}{\vee} n_2\right)\right) =$$

$$= \sigma_{dd}(S_{dd}(n_1)) \underset{d}{\wedge} \sigma_{dd}\left(S_{dd}\left(n_1 \overset{d}{\vee} n_2\right)\right) \underset{d}{\geq} n_1 \underset{d}{\wedge} n_1 = n_1.$$

Thus, we obtain the contradiction

$$S_{dd}(n_1) \leq d_O < S_{dd}(n_1).$$

So, if the sequence σ_{dd} is a (sds), that is if the equality (15) holds, then the corresponding Smarandache function S_{dd} satisfies the dual equality (17).

Example. The Fibonacci sequence $(F_n)_{n \in N^*}$ is a (sds). Therefore, the generalized Smarandache function S_F associated with this sequence satisfy:

$$S_F\left(n_1 \overset{d}{\vee} n_2\right) = S_F(n_1) \overset{d}{\vee} S_F(n_2) \tag{19}$$

By means of this equality, the computation of $S_F(n)$ reduces to the determination of $S_F(p^\alpha)$, where p is a prime number. For instance

$$S_F(52) = \min\left\{m_n \mid (\forall) m \underset{d}{\geq} m_n \Rightarrow 52 \underset{d}{\leq} F(m)\right\} =$$

$$= S_F(2^2) \overset{d}{\vee} S_F(13) = 6 \overset{d}{\vee} 7 = 42.$$

So, 42 is the smallest positive integer m such that $F(m)$ is divisible by 52.

Also, we have

$$S_F(12) = S_F(2^2 \cdot 3) = S_F(2^2) \overset{d}{\vee} S_F(3) = 6 \overset{d}{\vee} 4 = 12, \tag{20}$$

therefore $n = 12$ is a fixed point of S_F.

The values of $S_F(p^\alpha)$ may be obtained by writing all F_n in the scale (p) given by (6), which is a difficult operation. At the time being, we are not able to provide a closed formula for the computation of $S_F(p^\alpha)$. However, we shall present some partial results in this direction. In [8] it is stated that

$$3^k \underset{d}{\leq} F_n \Leftrightarrow 4 \cdot 3^{k-1} \underset{d}{\leq} n$$

$$2^k \underset{d}{\leq} F_n \Leftrightarrow 3 \cdot 2^{k-2} \underset{d}{\leq} n, \quad for\ k \geq 3.$$

It is known (see for instance [6], [7]) that if σ is a non-degenerate second-order linear recurrence sequence defined by

$$\sigma(n) = A\sigma(n-1) - B\sigma(n-2) \tag{21}$$

where A and B are fixed non-zero coprime integers and $\sigma(1) = 1$, $\sigma(2) = A$, then

$$n \in Z^*, \quad n \underset{d}{\wedge} B = 1 \Rightarrow (\exists) m \in N^* \quad n \underset{d}{\leq} \sigma(m). \tag{22}$$

The least index of these terms is called the rank of appearance of n in the sequence and is denoted by $r(n)$.

If $D = A^2 - 4B$ and (D/n) stands for the Jacobi symbol, then for $mn \underset{d}{\wedge} BD = 1$ and p a prime we have ([6])

$$\begin{gathered} n \underset{d}{\leq} \sigma(m) \Leftrightarrow r(n) \underset{d}{\leq} m; \quad r(p) \underset{d}{\leq} p - (D/p) \\ r(p) \underset{d}{\leq} \frac{p-(D/p)}{2} \Leftrightarrow (B/p) = 1; \quad r\left(m \overset{d}{\vee} n\right) = r(m) \overset{d}{\vee} r(n). \end{gathered} \tag{23}$$

Let us denote $N_B^* = \left\{n \in N^* \mid n \underset{d}{\wedge} B = 1\right\}$. Obviously, if r is considered as a function $r: N_B^* \to N^*$, then we can write:

$$r(n) = \min\left\{m \mid n \underset{d}{\leq} \sigma(m)\right\}.$$

Whence an evident parallel between the above methods described for the construction of the generalized Smarandache functions and the definition of the function r.

For the Fibonacci sequence (F_n) we have $A = 1$, $B = -1$ and so $D = 5$.

This implies

$$p = 5k \pm 1 \Rightarrow (5/p) = 1 \tag{24}$$

$$p = 5k \pm 2 \Rightarrow (5/p) = -1 \tag{25}$$

and it follows that if (24) holds, then p divides F_{p-1}. Thus $S_F(p)$ is a divisor of p - 1. In the second case p divides F_{p+1} and $S_F(p)$ is a divisor of p + 1.

From (23) we deduce

$$S_F(p) \leq p - (5/p)$$

for any prime number p.

Lemma 2 from [6] implies that the fraction $(p-(5/p))/S_F(p)$ is unbounded. We also have

$$p^k \underset{d}{\leq} F_n \Leftrightarrow S_F(p^k) \underset{d}{\leq} n.$$

Example. For p = 11 it follows (5/p) = l, so $S_F(11) \underset{d}{\leq} 10$. In fact, we have precisely $S_F(11) = 11 - (5/11) = 10$, but there exist prime numbers such that $S_F(p) < p - (5/p)$. For instance, p =17, for which p - (5/p) = 18 and $S_F(17) = 9$.

Definition 2.4. The sequence σ is *a dual strong divisibility sequence* (*dsds*) if

$$\sigma\left(n \overset{d}{\vee} m\right) = \sigma(n) \overset{d}{\vee} \sigma(m) \quad \text{for all } n, m \in N^*. \tag{26}$$

It may be easily seen that every strong divisibility sequence is a divisibility sequence. We also have:

Proposition 2.1 *Every dual strong divisibility sequence is a divisibility sequence.*

Proof. We have to prove that (26) implies (14). But if $n \underset{d}{\leq} m$, it follows $n \overset{d}{\vee} m = m$ and then

$$\sigma(m) = \sigma\left(n \overset{d}{\vee} m\right) = \sigma(n) \overset{d}{\vee} \sigma(m) \tag{27}$$

so, $\sigma(n) \underset{d}{\leq} \sigma(m)$.

Then Theorem *2.5* asserts that the Smarandache generalized function S_σ associated with any strong divisibility sequence σ is a dual strong divisibility sequence. Of course, in this case, both sequences σ and S_σ are divisibility sequences.

It would be very interesting to prove whether the converse assertion holds. That is if S_{dd} is the generalized Smarandache function associated with a (divisibility) sequence σ_{dd} satisfying the condition (c_{dd}), then the equality (17) implies the strong divisibility.

Remarks. (1) It is known that the Smarandache function S is *onto*. But given a $(dd)-$*sequence* σ_{dd}, even if it is a (*sds*), it does not follow that the associated function S_{dd} is *onto*. Indeed, the function S_F associated with the Fibonacci sequence is not *onto*, because n = 2 is not a value of S_F.

(2) One of the most interesting diophantine equations associated with a Smarandache type function is that which provides its fixed points. We remember that the fixed points for the Smarandarche function are all the primes and the composit number $n = 4$. For the functions S_{dd} the equation providing the fixed points reads $S_{dd}(x) = x$ and for S_F we have as solutions, for instance, $n = 5$, $n = 12$.

At the end of this paper we quote the following question on the Smarandache function, also related to the Fibonacci sequence:

T. Yau [10] wondered if there exist triplets of positive integers (n, n-l, n-2) such that the corresponding values of the Smarandache function satisfy the Fibonacci recurrence relation $S(n) = S(n-1) + S(n-2)$.

He found two such triplets, namely for $n = 11$ and for $n = 121$. Indeed, we have $S(9) + S(10) = S(11)$ and $S(119) + S(120) = S(121)$.

Using a computer, Charles Ashbacher [2] found additional values. These are for $n = 4902$, $n = 26245$, $n = 32112$, $n = 64010$, $n = 368139$, $n = 415664$.

Recently H. Ibsent [4] proposed an algorithm permitting to find, by means of a computer, much more values. But the question posed by T. Yau "How many other triplets with the same property exist?"is still unsolved.

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