Some Considerations Concerning the Sumatory Function Associated to Smarandache Function

by
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The Smarandache Function [4] ia a numerical function $S: N^* \to N^*$ defined by
$S(n) = \min\{m | m! \, divisible \ by \ n\}$.
From the definition it results that if

$$n = p_1^{r_1} p_2^{r_2} \dots p_t^{r_t} \tag{1}$$

is the decomposition of n into primes, then

$$S(n) = \max\{S(p_i^{r_i}) | i = 1,2,\dots,t\} \tag{2}$$

It is said that for every function f it can be attashed the sumatory function

$$F(n) = \sum_{d/n} f(d) \tag{3}$$

If f is the Smarandache function and $n = p^a$, then

$$F_s(p^a) = \sum_{j=0}^{a} S(p^j) = \sum_{j=0}^{a} S_p(j) \tag{4}$$

In [2] it is proved that

$$S(p^j) = (p-1)j + \sigma_{[p]}(j) \tag{5}$$

where

$$j = \sum_{i=1}^{t_j} k_i^j a_i(p) \tag{6}$$

and

$$\sigma_{[p]}(j) = \sum_{i=1}^{t_j} k_i^j \tag{7}$$

is the sum of the digits of the integer j, written in the generalised scale
$[p] \ : a_1(p), a_2(p), \dots, a_k(p), \dots$
with

$$a_n(p) = \frac{p^n - 1}{p - 1} \quad , n = 1,2,\dots$$

For example

$$p = p \cdot a_1(p);$$
$$p^p = (p-1) \cdot a_p(p) + 1 \cdot a_1(p);$$
$$p^t = (p-1) \cdot a_t(p) + 1;$$

and

$$\sigma_{[p]}(p^p) = p;$$
$$S(p^p) = p^2; \quad S\left(p^{p^p}\right) = (p-1)p^p + p.$$

In [3] it is proved that

$$F_s(p^a) = (p-1)\frac{a(a+1)}{2} + \sum_{j=1}^{a} \sigma_{[p]}(j) \tag{8}$$

In the following we give an algorithm to calculate the sumatory function, associated to the Smarandache function:

1. Calculating the generalised scale $[p]$: $a_1(p), a_2(p), \ldots, a_n(p), \ldots$
2. Calculating the expresion of a in the scale $[p]$. Let $a_{[p]} = \overline{k_s k_{s-1} \ldots k_1}$.
3. For $i = 1,2,\ldots,s$

 3.1. If $k_i \neq 0$

 then

 3.1.1. $v_i = a - a_i(p) + 1$

 3.1.2. $z_i = \left(\overline{k_s k_{s-1} \ldots k_{i+1}}\right)_{u=a_i(p)}$

 3.1.3. $h_i = v_i - z_i$

 else

 3.1.4. $b = \overline{k_s k_{s-1} \ldots k_{i+1} - 1p00\ldots0}$

 3.1.5. $v_i = b - a_i(p) + 1$

 3.1.6. $z_i = \left(\overline{k_s k_{s-1} \ldots k_i}\right)_{u=a_i(p)}$

 3.1.7. $h_i = v_i - z_i$

 3.2. $A_i = \left[\frac{h_i}{a_{i+1}(p) - a_i(p)}\right]$

 3.3. $r_i = h_i - A_i\left(a_{i+1}(p) - a_i(p)\right)$

 3.4. $B_i = \left[\frac{r_i}{a_i(p)}\right]$

 3.5. $q_i = r_i - B_i * a_i(p)$

 3.6. $S_i = A_i a_i(p)\frac{p(p-1)}{2} + A_i p + a_i(p)\frac{B_i(B_i+1)}{2} + q_i(B_i+1)$

4. Calculating $F_s(p^a) = (p-1)\frac{a(a+1)}{2} + \sum_{i=1}^{a} S_i$, $\left(\sum_{j=1}^{a} \sigma_{[p]}(j) = \sum_{i \geq 1} S_i(a)\right)$.

A Pascal program has been designed to the calculus of $F_s(p^a)$:

```
uses dos,crt;
          type tablou=array[1..100] of real;
          var a,k,amare,bmare,niu,z,alfaa,betaa,ro,r,s:tablou;
alfa,p,ik,amax,beta,suma,fsuma,u:real;
i,dim,max,j:longint;
 hour,min,sec,sec100:word;
{*****************************************************}
 {Calc. scale p right}
 procedure bazapd(var b:tablou;var p:real;var a:real; var dim:longint);
 var i:longint;
 begin
    for i:=1 to 100 do
    bi]:=0;
    b[1]:=1;
    i:=0;
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      repeat
        i:=i+1;
          b[i]:=b[i-1]*p+1;
      until b[i]>a;
    dim:=i;
end;

{***************************************************}
{write alfa in the scale p right}
procedure nrbazapd(var a:tablou; var p:real;
var alfa:real; var k:tablou; var max:longint);
var m,i:longint;
d,r,prod:real;
begin
for i:=1 to 100 do
   k[i]:=0;
d:=alfa;
max:=trunc(ln((p-1)*d+1)/ln(p));
repeat
 m:=trunc(ln((p-1)*d+1)/ln(p));
  k[m]:=trunc(d/a[m]);
  r:=d-a[m]*k[m];
  d:=r ;
until r<p;
if r<>0 then
 k[1]:=r;
end;
{******************************************}
{calc. z for given i }

procedure calcz(var k:tablou;var a:tablou;
var i:longint;var u:real; var z:tablou; var p:real);
var j,i1,ind:longint;
prod:real;
begin
 z[i]:=0;
    ind:=1;
 for j:=i+1 to max do
   begin
    if k[j]<>0 then
     begin
       prod:=1;
       if ind>1 then
          begin
          for i1:=1 to ind-1 do
            prod:=prod*p; {****}
          prod:=prod*u+a[ind-1];
        end
       else
         prod:=u;
       z[i]:=z[i]+k[j]*prod;
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```
    end;
    ind:=ind+1;
   end;
  end;
{*************************************************}
begin
clrscr;
write('        give p=');
readln(p);
write('        give alfa=');
readln(alfa);
gettime(hour,min,sec,sec100);
writeln('        Timp Start:',hour,':',min,':',sec,':',sec100);
bazapd(a,p,alfa,dim);
nrbazapd(a,p,alfa,k,max);
for i:=1 to max do
  begin
   if k[i]<>0 then
    begin
     niu[i]:=alfa-a[i]+1;
     u:=a[i];
     calcz(k,a,i,u,z,p);
     alfaa[i]:=niu[i]-z[i];
    end
   else
    begin
     for j:=1 to max do
     betaa[j]:=k[j];
     betaa[i]:=p;
     betaa[i+1]:=betaa[i+1]-1;
     for j:=1 to i-1 do
      betaa[j]:=0;
       {Write beta in the scale 10}
        beta:=0;
         for j:=1 to max do
     beta:=beta+betaa[j]*a[j];
     niu[i]:=beta-a[i]+1;
     u:=a[i];
     calcz(betaa,a,i,u,z,p);
     alfaa[i]:=niu[i]-z[i];
     end;
     amare[i]:=int(alfaa[i]/(a[i+1]-a[i]));
     r[i]:=alfaa[i]-amare[i]*(a[i+1]-a[i]);
     bmare[i]:=int(r[i]/a[i]);
     ro[i]:=r[i]-bmare[i]*a[i];
     s[i]:=amare[i]*a[i]*(p*(p-1)/2)+amare[i]*p;
     s[i]:=s[i]+a[i]*(bmare[i]*(bmare[i]+1)/2);
     s[i]:=s[i]+ro[i]*(bmare[i]+1);
    end;
suma:=0;
```

```
for i:=1 to max do
 suma:=suma+s[i];
fsuma:=(p-1)*((alfa*(alfa+1))/2)+suma;
writeln('        fsuma=',fsuma);
gettime(hour,min,sec,sec100);
writeln('        Timp Stop:',hour,':',min,':',sec,':',sec100);
end.
```

We applied the algoritm for $p=3$ and $a = 300$ we obtain

TIMES START: 10:34:1:56

TIMES STOP: 10:34:1:57

We applied the formulas [4] for $p=3$ and $a = 300$ we obtain

TIMES START : 10:33:31:2

TIMES STOP: 10:33:31:95

A consequence of this work is that the proposed algoritm is faster then formula [4] .
From the Legendre formula it results that [1]

$$S_p(j) = p\left(j - i_p(j)\right) \text{ with } 0 \le i_p(j) \le \left[\frac{j-1}{p}\right],$$

and

$$F_s(p^a) = \sum_{j=0}^{a} p\left(j - i_p(j)\right) = p\sum_{j=0}^{a} j - p\sum_{j=0}^{a} i_p(j) ,$$

consequently

$$F_s(p^a) = \frac{pa(a+1)}{2} - p\sum_{i=0}^{a} i_p(j) \tag{9}$$

In [1] it is showed that

$$i_p(j) = \frac{j - \sigma_{[p]}(j)}{p} .$$

In particular,

$$i_p(p) = 0 ; \quad i_p(p^t) = p^{t-1} - 1$$

and

$$\sum_{j=0}^{a} i_p(j) = \frac{1}{p}\left[\sum_{j=0}^{a} j - \sum_{i=1}^{t_j} k_i^j\right] .$$

If $p \ge a$, then

$$j = j \cdot a_1(p), \quad \sigma_{[p]}(j) = j, \quad (j = 1,2,\ldots,a), \quad S(p^a) = pa$$

and

$$\sum_{j=0}^{a} \sigma_{[p]}(j) = \frac{a(a+1)}{2} ; \quad \sum_{j=0}^{a} i_p(j) = 0$$

$$F_s(p^a) = \frac{pa(a+1)}{2} \tag{10}$$

For example

$$F_s(11^3) = s(1) + S(11) + S(11^2) + S(11^3) = 66 \text{ or}$$

$$F_s(11^3) = \frac{11 \cdot 3 \cdot 4}{2} = 66 .$$

In particular,

$$F_s(p^p)=\frac{p^2(p+1)}{2} \tag{11}$$

If $p \le a$, then $a = pQ + R$ with $0 \le R \le p$, and

$$\sum_{j=0}^{a} i_p(j)=\sum_{j=0}^{a}\left\{\left[\frac{i}{p}\right]-\left[\frac{\sigma_p(j)}{p}\right]\right\}\Rightarrow$$

$$\sum_{j=0}^{a} i_p(j)=\frac{pQ(Q-1)}{2}+Q(R+1)-\sum_{j=p}^{a}\left[\frac{\sigma_p(j)}{p}\right],$$

consequently,

$$F_s(p^a)=\frac{pa(a+1)}{2}-\frac{p^2Q(Q-1)}{2}-pQ(R+1)+p\sum_{j=0}^{a}\left[\frac{\sigma_{[p]}(p)}{p}\right] \tag{12}$$

In particular, for $p = a$ then $Q = 1$, $R = 0$ and

$$F_s(p^p)=\frac{p^2(p+1}{2} \tag{13}$$

For example,

$$F_s(3^3)=18;\quad F_s(5^5)=75;$$

$$F_s(n^n)=\frac{2(2^p+1)^2(2^{p-2}+1)}{27},\ \text{for } n=\frac{2^p+1}{3},\ \text{with } 3<p\le 31 \text{ and } p \text{ prime.}$$

If $n = p^a q^b$ with $p < q$ and $p^a < q$, then

$$F_s(p^aq^b)=\sum_{d/p^aq^b} S(d)=\sum_{i=0}^{a}\sum_{j=0}^{a} S(p^iq^j)=(a+1)\sum_{j=0}^{b} S(q^j)=(a+1)F_s(q^b)$$

Then :

I. If $q \ge b$,

$$F_s(p^aq^b)=\frac{qb(a+1)(b+1)}{2} \tag{14}$$

II. If $q \le b$,

$$F_s(p^aq^b)=\frac{(a+1)qb(b+1)}{2}-\frac{(a+1)q^2\overline{Q}(\overline{Q}-1)}{2}-q(a+1)\overline{Q}(\overline{R}+1)+ q(a+1)\sum_{j=q}^{b}\left[\frac{\sigma_{[q]}(j)}{q}\right] \tag{15}$$

where $b = q\overline{Q} + \overline{R}$, with $0 \le \overline{R} \le q$.

If $n = p^a q$, then

$$F_s(p^aq)=\sum_{i=0}^{a} S(p^i)+\sum_{i=0}^{a} S(qp^i)\ .$$

For $p > q$, then $p^i > q$ and $S(qp^i) = S(p^i)$ with $i \ge 1$ consequently,

$$F_s(p^aq)=2F_s(p^a)+S(q)-1 \tag{16}$$

For $p < q$, there exists $x < a$ with $p^{x-1} < q < p^x$ and

$$S(qp^i)=\{S(q),\quad i=0,1,\dots,x-1$$

$$S(p^i),\quad i=x,\dots,a$$

consequently,

$$F_s(p^aq)=\sum_{i=0}^{x-1} S(p^i)+xS(q)+2\sum_{i=x}^{a} S(p^i)$$

or

$$F_s(p^a q) = F_s(p^{x-1}) + xS(q) + (a+x)(a-x+1)(p-1) + 2\sum_{j=x}^{a} \sigma_{[p]}(j) \quad (17)$$

For example, if $p \geq a$, then

$$F_S(p^a q) = F_s(p^{x-1}) + xS(q) + p(x+a)(a-x+1)$$

If $n = p^a q^k$ with $p > q$, then

$$F_s(p^a q^2) = \sum_{i=0}^{a} S(p^i) + \sum_{i=0}^{a} S(qp^i) + \sum_{i=0}^{a} S(q^2 p^i).$$

But $S(q^k p^i) = S(p^i)$ for $i \geq k$, because $\max\left(S(p^i), S(q^k)\right) = S(p^i)$ for $i \geq k$ consequently,

$$F_s(p^a q^2) = F_s(p^a q) + F_s(p^a) + S(q^2) + S(q^2 p) - p - 1 =$$
$$= 3F_s(p^a) + S(q) + S(q^2) + S(q^2 p) - p - 2$$

In short

$$F_s(p^a q) = 2F_s(p^a) + S(q) - 1$$
$$F_s(p^a q^2) = F_s(p^a q) + F_s(p^a) + S(q^2) + S(q^2 p) - p - 1$$
$$F_s(p^a q^3) = F_s(p^a q^2) + F_s(p^a) + S(q^3) + S(q^3 p) + S(q^3 p^2) - p - 2p - 1$$

....

$$F_s(p^a q^k) = F_s(p^a q^{k-1}) + F_s(p^a) + S(q^k) + S(q^k p) + S(q^k p^2) +$$
$$+ \ldots + S(q^k p^{k-1}) - p - 2p - \ldots - (k-1)p - 1$$

Hence

$$F_s(p^a q^k) = (k+1)F_s(p^a) + \sum_{i=1}^{k} s(q^i) + \sum_{i=2}^{k} S(q^i p) + \sum_{i=3}^{k} S(q^i p^2) +$$
$$+ \ldots + \sum_{i=k-1}^{k} S(q^i p^{k-2}) + S(q^k p^{k-1}) - k - \frac{pk(k^2-1)}{6} \quad (18)$$

References

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