Some Considerations Concerning the Sumatory Function Associated to Generalised Smarandache Function

E. Radescu and N. Radescu

Let us denote by $\overset{d}{\vee}$ the least common multiple, by $\underset{d}{\wedge}$ the greatest common devisor and $\wedge = \min, \vee = \max$. It is known that $N_0 = (\mathrm{N}^*, \wedge, \vee)$ and $N_d = \left(\mathrm{N}^*, \underset{d}{\wedge}, \overset{d}{\vee}\right)$ are lattices. The order on the set N^* : $n_1 \leq n_2 \Leftrightarrow n_1 \wedge n_2 = n_1$. corresponding to the first of these lattices and it is known that this is a total order. But the order $\underset{d}{\leq}$ induced on the same set by $\underset{d}{\wedge}$ and $\overset{d}{\vee}$ and defined by: $n_1 \underset{d}{\leq} n_2 \Leftrightarrow n_1 \underset{d}{\wedge} n_2 = n_1 \Leftrightarrow n_1$ divides n_2 is only a partial order.

Let $\sigma : N_0 \to N_d$ (1) a sequence of positive integers defined on the set N^*.
The sequence (1) is said to be a multiplicatively convergent to zero sequence (mcz) if:

$$\forall n \in \mathrm{N}^*, \exists m_n \in \mathrm{N}^*, \forall m > m_n \Rightarrow n \underset{d}{\leq} \sigma(m) \qquad (2).$$

The sequence

$$\sigma : N_d \to N_d \qquad (3)$$

is said to be a divisibility sequence (ds) if: $n \underset{d}{\leq} m \Rightarrow \sigma(n) \underset{d}{\leq} \sigma(m)$ and it is said to be a strong divisibility sequence (sds) if:

$$\sigma\left(n \underset{d}{\wedge} m\right) = \sigma(n) \underset{d}{\wedge} \sigma(m) \text{ for every } n, m \in \mathrm{N}^* \qquad (4).$$

Let the lattices N_0 and N_d. We'll use the following notations:

(a) a sequence $\sigma_{00} : N_0 \to N_0$ is a (oo) - sequences;
(b) a sequence $\sigma_{0d} : N_0 \to N_d$ is a (od) - sequences;
(c) a sequence $\sigma_{d0} : N_d \to N_0$ is a (do) - sequences;
(d) a sequence $\sigma_{dd} : N_d \to N_d$ is a (dd) - sequences.

Then A(do) - sequence σ_{d0} the monotonicity yields:

$$(m_{d0}) \forall n_1, n_2 \in \mathrm{N}^*, n_1 \underset{d}{\leq} n_2 \Rightarrow \sigma_{d0}(n_1) \leq \sigma_{d0}(n_2) \qquad (5)$$

and the condition of convergence to inifinity is:

$$(c_{d0}) \forall n \in \mathrm{N}^*, \exists m_n \in \mathrm{N}^*, \forall m \underset{d}{\leq} m_n \Rightarrow \sigma_{d0}(m) \geq n \qquad (6).$$

Analogously, for a (dd) - sequence σ_{dd} the monotonicity yields:

$$(m_{dd}) \forall n_1, n_2 \in \mathrm{N}^*, n_1 \underset{d}{\leq} n_2 \Rightarrow \sigma_{dd}(n_1) \underset{d}{\leq} \sigma_{dd}(n_2) \qquad (7)$$

and the convergence to zero is:

$$(c_{dd}) \forall n \in \mathrm{N}^*, \exists m_n \in \mathrm{N}^*, \forall m \underset{d}{\geq} m_n \Rightarrow \sigma_{dd}(m_n) \underset{d}{\geq} n \;. \qquad (8)$$

To each sequence σ_{ij}, with $i, j \in \{0, d\}$, satisfying the condition (c_{ij}), one may attach a sequence S_{ij} (a generalised Smarandache function) defined by:

$$S_{ij} = \min \{m_n : m_n \text{ is given by the condition } (c_{ij})\} \qquad (9).$$

For the properties the functions S_{ij}, see [2].

It is said that for every numerical function f it can be attashed the sumatory function:

$$F_f(n) = \sum_{d|n} f(d) \qquad (10)$$

The function f is expressed as:

$$f(n) = \sum_{uv=n} \mu(u) F_f(v) \qquad (11)$$

where μ is the Mobius function ($\mu(1) = 1, \mu(n) = 0$ if n is divisible by the square of a prime number, $\mu(n) = (-1)^k$ if n the product of k different prime numbers).

If f is the a generalised Smarandache function, S_{ij} then

$$F^S_{ij}(n)=\sum_{d/n} S_{ij}(d),\ i,j\in\{0,d\}. \qquad (12)$$

Now let us consider $n=p_1p_2...p_k$, with $p_1<p_2<...<p_k$ primes number and $S_{ij}(p_1)\le S_{ij}(p_2)\le...\le S_{ij}(p_k)$, for example. If i=0, j=d, then $S_{0d}\left(n_1 \overset{d}{\vee} n_2\right)=S_{0d}(n_1)\vee S_{0d}(n_2)$ and $F^S_{0d}(n)=S_{0d}(1)+\sum_{h=1}^{k} S_{0d}(p_h)+\sum_{h,t=1,h\ne t}^{k} S_{0d}(p_hp_t)+\sum_{h,t,q=1,h\ne t\ne q}^{k} S_{0d}(p_hp_tp_q)+\cdots S_{0d}(n)$. It result:

$F^S_{0d}(1)=S_{0d}(1)$;

$F^S_{0d}(p_1)=S_{0d}(1)+S_{0d}(p_1)=F^S_{0d}(1)+2^0S_{0d}(p_1)$;

$F^S_{0d}(p_1p_2)=S_{0d}(1)+S_{0d}(p_1)+S_{0d}(p_2)+S_{0d}(p_1p_2)=S_{0d}(1)+S_{0d}(p_1)+2S_{0d}(p_2)=F^S_{0d}(p_1)+2S_{0d}(p_2)$;

$F^S_{0d}(p_1p_2p_3)=F^S_{0d}(p_1p_2)+2^2S_{0d}(p_3)$;

$F^S_{0d}(p_1p_2p_3p_4)=F^S_{0d}(p_1p_2p_3)+2^3S_{0d}(p_4)$;

$F^S_{0d}(p_1p_2...p_k)=F^S_{0d}(p_1p_2...p_{k-1})+2^{k-1}S_{0d}(p_k)$.

That is $F^S_{0d}(p_1p_2...p_k)=S_{0d}(1)+\sum_{i=1}^{k}2^{i-1}S_{0d}(p_i)$.

The equality (11) becomes:

$$S_{0d}(p_k)=S(n)=\sum_{ab=n}\mu(n)F^S_{0d}(b)=$$
$$=F^S_{0d}(n)-\sum_i F^S_{0d}\left(\tfrac{n}{p_i}\right)+\sum_{i\ne j}F^S_{0d}\left(\tfrac{n}{p_ip_j}\right)+...$$

with $F^S_{0d}(\frac{n}{p_i})=F^S_{0d}(p_1p_2...p_{i-1}p_{i+1}...p_k)=\sum_{j=1}^{i-1}2^{j-1}S_{0d}(p_j)+\sum_{j=i+1}^{k}2^{j-1}S_{0d}(p_j)=$

$$=F^S_{0d}(p_1p_2...p_{i-1})+2^{i-1}F^S_{0d}(p_{i+1}...p_k).$$

Analogously,

$$F^S_{0d}\left(\tfrac{n}{p_ip_j}\right)=F^S_{0d}(p_1...p_{i-1})+2^{i-1}F^S_{0d}(p_{i+1}...p_{j-1})+2^{j-1}F^S_{0d}(p_{j+1}...p_k)=$$
$$=\sum_{h=1}^{i-1}2^{h-1}S_{0d}(p_h)+\sum_{h=i+1}^{j-1}2^{h-2}S_{0d}(p_h)+\sum_{h=j+1}^{k}2^{h-3}S_{0d}(p_h).$$

In particulary, for $n=p^a$, p prime number, it result:

$$S_{0d}(p^a)=\sum_{u+q=a}\mu(p^u)F^S_{0d}(p^q)=F^S_{0d}(p^a)-F^S_{0d}(p^{a-1}).$$

If $n=p^aq^b$ with $\max\{S_{0d}(p),...,S_{0d}(p^a)\}\le\min\{S_{0d}(q),...,S_{0d}(q^b)\}$, then

$$F^S_{0d}(p^aq^b)=F^S_{0d}(p^a)+(a+1)F^S_{0d}(q^b).$$

If i=d, j=d and if σ_{dd} is a (sds) satisfying the condition (c_{dd}), then

$$S_{dd}\left(n_1 \overset{d}{\vee} n_2\right)=S_{dd}(n_1)\overset{d}{\vee}S_{dd}(n_2) \qquad (13)$$

and
$$F^S_{dd}(n)=S_{dd}(1)+\sum_{h=1}^{k}S_{dd}(p_h)+\sum_{h,t=1,h\ne t}^{k}\left[S_{dd}(p_h)\overset{d}{\vee}S_{dd}(p_t)\right]+$$
$$+\sum_{h,t,q=1,h\ne t\ne q}^{k}\left[S_{dd}(p_h)\overset{d}{\vee}S_{dd}(p_t)\overset{d}{\vee}S_{dd}(p_q)\right]+...+S_{dd}(n) \qquad (14)$$

$$S_{dd}(p^a)+F^S_{dd}(p^a)-F^S_{dd}(p^{a-1}). \qquad (15)$$

Example: The Fibonacci sequence $(F_n)_{n\in N^*}$ definited by $F_{n+1}=F_n+F_{n-1}$, with $F_1=F_2=1$ ia s (sds), so for the generalised Smarandache function S_F attached to this sequence we have: $S_F\left(n_1 \overset{d}{\vee} n_2\right)=S_F(n_1)\overset{d}{\vee}S_F(n_2)$, and the calculus of $S_p(n)$ is reduced to the calculus of $S_F(p^a)$, with p a prime number. For instance:

n	$S_F(n)$	n	$S_F(n)$	n	$S_F(n)$
1	1	7	8	13	21
2	3	8	6	14	24
3	4	9	24	15	20
4	6	10	15	16	12
5	5	11	20	17	
6	12	12	12	18	

$F_{dd}^{S}(4) = 10, F_{dd}^{S}(8) = 16, F_{dd}^{S}(16) = 28, F_{dd}^{S}(15) = 30$.

References:

[1] M. Andrei, C. Dumitrescu, E. Radescu, N. Radescu, Some consideration concerning the Sumatory Function associated with Smarandache Function, Smarandache Notions Journal, vol.7,No.1-3,August 1996, p.63-69;

[2] C. Dumitrescu, C. Rocsoreanu, Some connections between the Smarandache Function and the Fibonacci sequence, to appear;

[3] E. Radescu, N. Radescu, C. Dumitrescu, On the Sumatory Function associated to the Smarandache Function, Smarandache Function Journal, vol 4-5, No.1, September 1994, p.17-21.

Curent addres:
University of Craiova - Department of Mathematics, Al. I. Cuza Street # 13
1100 Craiova, Romania.