SOME CONVERGENCE PROBLEMS INVOLVING THE SMARANDACHE FUNCTION

by

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In this paper we consider same series attashed to the Smarandache function (Dirichlet series and other (numerical) series). Asimptotic behaviour and convergence of these series is etablished.

1. INTRODUCTION. The Smarandache function $S : N^* \to N^*$ is defined [3] such that $S(n)$ is the smallest integer n with the property that n! is divisible by n. If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_t^{\alpha_t} \qquad (1.1)$$

is the decomposition into primes of the positiv integer n, then

$$S(n) = \max_i S(p_i^{\alpha_i}) \qquad (1.2)$$

and more general if $n_1 \overset{d}{\vee} n_2$ is the smallest commun multiple of n_1 and n_2 then .

$$S(n_1 \overset{d}{\vee} n_2) = \max(S(n_1), S(n_2))$$

Let us observe that on the set N of non-negative integers, there are two latticeal structures generated respectively by $\vee = \max$, $\wedge = \min$ and $\overset{d}{\vee}$ = the last commun multiple, $\underset{d}{\wedge}$ = the greatest commun division. if we denote by $\leq$ and $\leq_d$ the induced orders in these lattices, It results

$$S(n_1 \overset{d}{\vee} n_2) = S(n_1) \vee S(n_2)$$

The calculus of $S(p^a)$ depends closely of two numerical scale, namely the standard scale

$$(p) : 1, p, p^2, \dots, p^a, \dots$$

and the generalised numerical scale [p]

$$[p] : a_1(p), a_2(p), \ldots, a_n(p), \ldots$$

where $a_k(p) = (p^k-1)/(p-1)$. The dependence is in the sens that

$$S(p^\alpha) = p(\alpha_{[p]})_{(p)} \qquad (1.3)$$

so, $S(p^\alpha)$is obtained multiplyng p by the number obtained writing α in the scale [p] and "reading" it in the scale (p).

Let us observe that if $b_n(p) = p^n$ then the calculus in the scale [p] is essentialy different from the standard scale (p), because :

$$b_{n+1}(p) = pb_n(p) \quad \text{but } a_{n+1}(p) = pa_n(p) + 1$$

(for more details see [2]).

We have also [1] that

$$S(p^\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \qquad (1.4)$$

where $\sigma_{[p]}(\alpha)$ is the sum of digits of the number α writen in the scale [p].

In [4] it is showed that if φ is Euler's totient function and we note $S_p(\alpha) = S(p^\alpha)$ then

$$S_p(p^{\alpha-1}) = \varphi(p^\alpha) + p \qquad (1.5)$$

It results that $\varphi(p_i^{\alpha_i}) = S(p_i^{p_i^{\alpha_i-1}}) - p$ so

$$\varphi(n) = \prod_{i=1}^{r}\left(S(p_i^{p_i^{\alpha_i-1}}) - p_i\right)..$$

In the same paper [4] the function S is extended to the set Q of rational numbers.

2. GENERATING FUNCTIONS. It is known that we may attashe to each numerical function f:N*-->C the Dirichlet serie :

$$D_f(z) = \sum_{n=1}^{\infty} \frac{f(n)}{n^z} \qquad (2.1)$$

which for some $z = x + iy$ may be convergent or not.

The simplest Dirichlet series is:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad (2.2)$$

called Riemann's function or zeta function where is convergent for $Re(z) > 1$.

It is said for instance that if f is Möbius function ($\mu(1) = 1$, $\mu(p_1 p_2 \cdots p_r) = (-1)^r$ and $\mu(n) = 0$ if n is divisible by the squar of a prime number) then $D_\mu(z) = 1/\zeta(z)$ for $x>1$, and if f is Euler's totient function ($\varphi(n)$ = the number of positive integers not greater than and prime to the positive integer n) then $D_\varphi(z)=\zeta(z-1)/\zeta(z)$ for $x>2$).

We have also $D_d(z) = \zeta^2(z)$, for $x > 1$, where $d(n)$ is the number of divisors of n, including 1 and n; and $D_{\sigma_k}(n) = \zeta(z)\cdot\zeta(z-k)$ (for $x > 1$, $x > k+1$), where $\sigma_k(n)$ is the sum of the k-th powers of the divisors of n. We write $\sigma(n)$ for $\sigma_1(n)$.

In the sequel let we suppose that z is a real number, so $z = x$.

For the Smarandache function we have:

$$D_s(x) = \sum_{n=1}^{\infty} \frac{S(n)}{n^x}$$

If we note :

$$F_f^{\circ}(n) = \sum_{k \leq_d n} f(k)$$

it is said that Möbius function make a connection between f and F_f° by the inversion formula:

$$f(n) = \sum_{k \leq_d n} F_f^{\circ}(k)\mu(\tfrac{n}{k}) \qquad (2.3)$$

The functions F_f° are also called generating functions.

In [4] the Smarandache functions is regarded as a generating function and is constructed the function s_o such that:

$$s_o(n) = \sum_{k \leq_d n} S(k)\mu(\tfrac{n}{k})$$

2.1. PROPOSITION. For all $x > 2$ we have :

(i) $\zeta(x) \leq D_S(x) \leq \zeta(x-1)$

(ii) $1 \leq D_{s_o}(x) \leq D_{\varphi}(x)$

(iii) $\zeta^2(x) \leq D_{F_S}(x) \leq \zeta(x) \cdot \zeta(x-1)$

Proof. (i) The asertion results from the fact that $1 \leq S(n) \leq n$.

(ii) Using the multiplication of Dirichlet series we have:

$$\frac{1}{\zeta(x)} \cdot D_s(x) = \left(\sum_{k=1}^{\infty} \frac{\mu(k)}{k^x}\right)\left(\sum_{k=1}^{\infty} \frac{S(k)}{k^x}\right) = \mu(1)S(1) + \frac{\mu(1)S(2)+\mu(2)S(1)}{2^x} +$$

$$+ \frac{\mu(1)S(3)+\mu(3)S(1)}{3^x} + \frac{\mu(1)S(4)+\mu(2)S(2)+\mu(4)S(1)}{4^x} + \ldots = \sum_{k=1}^{\infty} \frac{s_o(k)}{k^x} = D_{s_o}(x)$$

and the asertion result using (i).

(iii) We have

$$\zeta(x) \cdot D_S(x) = \left(\sum_{k=1}^{\infty} \frac{1}{k^x}\right)\left(\sum_{k=1}^{\infty} \frac{S(k)}{k^x}\right) = S(1) + \frac{S(1)+S(2)}{2^x} + \frac{S(1)+S(3)}{3^x} + \ldots = D_{F_S}(x)$$

so the inequalities holds using (i).

Let us observe that (iii) is equivalent to $D_d(x) \leq D_{F_S} < D_{\sigma}(x)$. These inequalities can be deduced also observing that from $1 \leq S(n) \leq n$ it result:

$$\sum_{k \leq_d n} 1 \leq \sum_{k \leq_d n} S(k) \leq \sum_{k \leq_d n}$$

so,

$$d(n) \leq F_s(n) \leq \sigma(n) \qquad (2.4)$$

But from the fact that $F_S < n + 4$ (proved in [5]) we deduce

$$d(n) \leq F_s(n) \leq n + 4 \qquad (2.5)$$

Until now it is not known a closed formula for the calculus of the functions $D_s(x)$, $D_{s,s}(x)$ or $D_{rs}(x)$, but we can deduce asimptotic behaviour of these functions using the following well known results:

2.2. THEOREM. (i) $\zeta(z) = \frac{1}{z-1} + O(1)$

(ii) $\ln\zeta(z) = \ln\frac{1}{z-1} + O(z-1)$

(iii) $\zeta'(z) = -\frac{1}{(z-1)^2} + O(1)$

for all complex number.

Then from the proposition 2.1 we can get inequalities as the fallowings:

(i) $\frac{1}{x-1} + O(1) \leq D_s(x) \leq \frac{1}{x-2} + O(1)$

(ii) $1 \leq D_{s_0}(x) \leq \frac{x-1}{e^{A(x-2)}}$ for some positive constant A

(iii) $-\frac{1}{(x-1)^2} + O(1) \leq D'_s(x) \leq -\frac{1}{(x-2)^2} + O(1)$.

The Smarandache functions S may be extended to all the nonnegative integers defining S(-n) = S(n).

In [3] it is proved that the serie

$$\sum_{k=1}^{\infty} \frac{S(k)}{k!}$$

is convergent and has the sum $q \in (e-1,2)$.

We can consider the function

$$f(z) = \sum_{k=1}^{\infty} \frac{S(k)}{(k+1)!} z^k$$

convergent for all $z \in C$ because

$$\frac{a_{n+1}}{a_n} = \frac{S(n+1)}{(n+2)S(n)} \leq \frac{n+1}{(n+2)S(n)} \leq \frac{1}{S(n)}$$

and so $\frac{a_{n+1}}{a_n} \to 0$

2.3. PROPOSITION. The function f statisfies $|f(z)| \leq qz$ an the unit disc $U(0,1) = \{z \mid |z| < 1\}$.

Proof. A lema does to Schwartz asert that if the function f is olomorphe on the unit disc $U(0,1) = \{z \mid |z| < 1\}$ and satisfies $f(0) = 0$, $|f(z)| \leq 1$ for $z \in U(0,1)$ then $|f(z)| \leq |z|$ on $U(0,1)$ and $|f'(0)| \leq 1$.

For $|z| < 1$ we fave $|f(z)| < q$ so $(1/q) f(z)$ satisfies the conditions of Schwartz lema.

3. SERIES INVOLVING THE SMARANDACHE FUNCTION. In this section we shall studie the convergence of some series concerning the function S.

Let b: N*-->N* be the function defined by: b(n) is the complement of n until the smallest factorial. From this definition it results that $b(n) = (S(n)!)/n$ for all $n \in N^*$.

3.1. PROPOSITION. The sequences $(b(n))_{n\geq 1}$ and also $(b(n)/n^k)_{n\geq 1}$ for $k\in R$, are divergent.

Proof. (i) The asertion results from the fact that $b(n!) = 1$ and if $(p_n)_{n\geq 1}$ is the sequence of prime members then

$$b(p_n) = \frac{S(p_n)!}{p_n} = \frac{p_n!}{p_n} = (p_n - 1)!$$

(ii) Let we note $x_n = b(n)/n^k$. Then

$$x_n = \frac{S(n)!}{n^{k+1}}$$

and for k > 0 it results

$$x_{n!} = \frac{S(n!)!}{(n!)^{k+1}} = \frac{n!}{(n!)^{k+1}} \to 0$$

$$x_{p_n} = \frac{p_n!}{(p_n)^{k+1}} = \frac{(p-1)!}{(p_n)^{k+1}} > \frac{p_1 \cdot p_2 \cdots p_{n-1}}{p_n^{k+1}} > p_n$$

because it in said [6] that $p_1\, p_2 \cdots p_{n-1} > p_n^{k+2}$ for n sufficiently large.

3.2. PROPOSITION. The sequence $T(n) = 1 + \sum_{i=2}^{n} \frac{1}{b(n)} - \ln b(n)$ is divergent.

Proof. If we suppose that $\lim_{n\to\infty} T(n) = l < \infty$, then because $\sum_{i=2}^{\infty} \frac{1}{b(n)} = \infty$ (see [3]) it results the contradiction $\lim_{n\to\infty} \ln b(n) = \infty$.

If we suppose $\lim_{n\to\infty} T(n) = -\infty$, from the equality $\ln b(n) = 1 + \sum_{i=2}^{\infty} \frac{1}{b(n)} - T(n)$ it results $\lim_{n\to\infty} \ln b(n) = \infty$.

We can't have $\lim_{n\to\infty} T(n) = +\infty$ because $T(n) < 0$. Indeed, from $i \leq S(i)!$ for $i \geq 2$ it results

$$i / S(i)! \leq 1 \text{ for all } i \geq 2$$

so

$$T(p_n) = 1 + \frac{2}{S(2)!} + \ldots + \frac{p_n}{S(p_n)!} - \ln((p_n - 1)!) < 1 + (p_n - 1) - \ln((p_n - 1)!) =$$

$$= p_n - \ln((p_n - 1)!).$$

But for k sufficiently large we have $e^k<(k-1)!$ that is there exists $m\in N$ so that $p_n < \ln((p_n - 1)!)$ for $n \geq m$. It results $p_n - \ln((p_n - 1)!) < 0$ for $n \geq m$, and so $T(n) < 0$.

Let now be the function

$$H_b(x) = \sum_{2\leq n\leq x} b(n).$$

3.3. PROPOSITION. The serie

$$\sum_{n\geq 2} H_b^{-1}(n) \qquad (3.1)$$

is convergent.

Proof. the sequence $(b(2)+b(3)+ \ldots + b(n))_n$ is strictley increasing to ∞ and

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}>\frac{S(2)!}{2}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}>\frac{S(5)!}{5}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}+\frac{S(6)!}{6}>\frac{S(5)!}{5}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}+\frac{S(6)!}{6}+\frac{S(7)}{7}>\frac{S(7)!}{7}$$

....

so we have

$$\sum_{n\geq 2} H_b^{-1}(n)=\frac{1}{\frac{S(2)!}{2}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}}+\dots$$

$$\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\dots+\frac{S(n)!}{n}}+\dots<$$

$$<\frac{2}{\frac{S(2)!}{2}}+\frac{1}{\frac{S(3)!}{3}}+\frac{2}{\frac{S(5)!}{5}}+\frac{4}{\frac{S(7)!}{7}}+\frac{2}{\frac{S(11)!}{11}}+\dots+\frac{p_{k+1}-p_k}{\frac{S(p_k)!}{p_k}}+\dots$$

$$<1+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k)}{S(p_k)!}=1+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k)}{p_k!}=1+\frac{1}{2}+\frac{1}{12}+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k)}{p_k!}.$$

But $(p_n-1)!>p_1p_2\dots p_n$ for $n\geq 4$ and then

$$\sum_{n\geq 2} H_b^{-1}(n)<\frac{19}{12}+\sum_{k\geq 4} a_k$$

where $a_k=\frac{p_k(p_{k+1}-p_k)}{p_k!}=\frac{p_{k+1}-p_k}{1\cdot 2\cdot 3\cdot\ldots\cdot(p_k-1)}<\frac{p_{k+1}-p_k}{p_1p_2\dots p_k}<\frac{p_{k+1}}{p_1p_2\dots p_k}$

Because $p_1p_2\dots p_k>p^3_{k+1}$ for k sufficiently large, it results

$$a_k<\frac{p_{k+1}}{p_{k+1}^3}=\frac{1}{p_{k+1}^2}\text{ for } k\geq k_o$$

and the convergence of the serie (3.1) follows from the convergence of the serie $\sum_{k\geq k_0}\frac{1}{p_{k+1}^2}$.

In the followings we give an elementary proof of the convergence of the series $\sum_{k=2}^{\infty}\frac{1}{S(k)^{\alpha}\sqrt{S(k)!}}$, $\alpha\in R$, $\alpha>1$ provides information on the convergence behavior of the series $\sum_{k=2}^{\infty}\frac{1}{S(k)!}$.

3.4. PROPOSITION. The series $\sum_{k=2}^{\infty}\frac{1}{S(k)^{\alpha}\sqrt{S(k)!}}$, converges if $\alpha\in R$ and $\alpha>1$.

Proof.

$$\sum_{k=2}^{\infty}\frac{1}{S(k)^{\alpha}\sqrt{S(k)!}}=\frac{1}{2^{\alpha}\sqrt{2!}}+\frac{1}{3^{\alpha}\sqrt{3!}}+\frac{1}{4^{\alpha}\sqrt{4!}}+\frac{1}{5^{\alpha}\sqrt{5!}}+\frac{1}{3^{\alpha}\sqrt{3!}}+\frac{1}{7^{\alpha}\sqrt{7!}}+\frac{1}{4^{\alpha}\sqrt{4!}}+\dots=\sum_{t=2}^{\infty}\frac{m_t}{t^{\alpha}\sqrt{t!}}$$

where m_t denotes the number of elements of the set

M_t { $k\in N^*$, $S(k)=t$ } = { $k\in N^*$, $k \mid t$ and $k \mid (t-1)!$ }.

It follows that M_t { k∈N*, k | t } and there fore m_t<d(t!).
Hence $m_t < 2\sqrt{t!}$ and consequently we have

$$\sum_{t=2}^{\infty} \frac{m_t}{t^{\alpha}\sqrt{t!}} < \sum_{t=2}^{\infty} \frac{2\sqrt{t!}}{t^{\alpha}\sqrt{t!}} = 2\sum_{t=2}^{\infty} \frac{1}{t^{\alpha}}$$

So, $\sum_{t=2}^{\infty} \frac{m_t}{t^{\alpha}\sqrt{t!}}$ converges.

3.5. PROPOSITION. $t^{\alpha}\sqrt{t!} < t!$ if $\alpha \in R$, $\alpha > 1$ and $t > t_* = [e^{2\alpha+1}]$, $t \in N^*$. (where [x] means the integer part of x).

Proof. $t^{\alpha}\sqrt{t!} < t! \Leftrightarrow t^{2\alpha}t! < (t!)^2 \Leftrightarrow t^{2\alpha} < t!$ (2)

On the other hand $t^{2\alpha} < (\frac{t}{e})^t \Leftrightarrow (e\cdot\frac{t}{e})^{2\alpha} < (\frac{t}{e})^t \Leftrightarrow e^{2\alpha}\cdot(\frac{t}{e})^{2\alpha} < (\frac{t}{e})^t \Leftrightarrow e^{2\alpha} < (\frac{t}{e})^{t-2\alpha}$ (3)

If t>$e^{2\alpha+1}$ => $(\frac{t}{e})^{t-2\alpha} > (\frac{e^{2\alpha+1}}{e})^{t-2\alpha} = (e^{2\alpha})^{t-2\alpha} > (e^{2\alpha})^{e^{2\alpha+1}-2\alpha}$

Applyng the well-known result that $e^x > 1+x$ if x > 0 for x = 2α we have
$(e^{2\alpha})^{e^{2\alpha+1}-2\alpha} > (e^{2\alpha})^{2\alpha+1+1-2\alpha} = (e^{2\alpha})^2 = e^{4\alpha} > e^{2\alpha}$.
So, if t > $e^{2\alpha+1}$ we fave $e^{2\alpha} < (\frac{t}{e})^{t-2\alpha}$ (4)
It is well known that $(\frac{t}{e})^t < t!$ if t∈N*. (5)

Now, the proof of the proposition is obtained as follows:

If $t > t_* = [e^{2\alpha+1}]$, $t \in N^*$ we have $e^{2\alpha} < (\frac{t}{e})^{t-2\alpha} \Leftrightarrow t^{2\alpha} < (\frac{t}{e})^t < t!$. Hence $t^{2\alpha} < t!$ if $t > t_*$ and this proves the proposition.

CONSEQUENCE. The series $\sum_{k=2}^{\infty} \frac{1}{S(k)!}$ converges.

Proof. $\sum_{k=2}^{\infty} \frac{1}{S(k)!} = \sum_{t=2}^{\infty} \frac{m_t}{t!}$ where m_t is defined as above.

If $t > t_*$ we have $t^{\alpha}\sqrt{t!} < t! \Leftrightarrow \frac{1}{t^{\alpha}\sqrt{t!}} > \frac{1}{t!} \Leftrightarrow \frac{m_t}{t^{\alpha}\sqrt{t!}} > \frac{m_t}{t!}$.

Since $\sum_{t=2}^{\infty} \frac{m_t}{t^{\alpha}\sqrt{t!}}$ converges it results that $\sum_{t=2}^{\infty} \frac{m_t}{t!}$ also converges.

REMARQUE. From the definition of the Smarandache function it results that

card { k∈N*: S(k)=t } = card { k∈N*: k | t and k | (t-1)!} = d(t!)-d((t-1)!)

so we get

$$\sum_{t=2}^{n} car(dS^{-1}(t)) = \sum_{k=2}^{n}(d(t!) - d((t-1)!)) = d(n!) - 1$$

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