Some Elementary Algebraic Considerations Inspired by Smarandache Type Functions

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The basic ideea of this paper is the algebraic construction of some functions representing prolongations of the Smarandache type functions to more complete sets already known and having specified properties.

A. Starting from a sequence of positive integers $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ satisfying the condition

$$\forall n \in \mathbf{N}^*, \exists m_n \in \mathbf{N}^*, \forall m \geq m_n \Longrightarrow n/\sigma(m) \tag{1}$$

(such sequences-possibly satisfying an extra condition-considered by C. Christol to generalise the p-adic numbers were called also multiplicative convergent to zero; for example: $\sigma(n) = n!$)it was built an associated Smarandache type function that is $S_\sigma : \mathbf{N}^* \to \mathbf{N}^*$ defined by

$$S_\sigma(n) = \min\{m_n : m_n \text{ is given by } (1)\} \tag{2}$$

(For $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ with $\sigma(n) = n!$ the associated function S_σ is just the Smarandache function.)

The sequence is noted σ_{0d} and the associated function S_{0d}.

For each such a sequence, the associated function has a series of properties already proved, from whom we retain:

$$S_{0d}([n_1, n_2]) = \max\{S_{0d}(n_1), S_{0d}(n_2)\} \text{ (see [1, th. 2.2]).} \tag{3}$$

We can stand out:-the universal algebra $(\mathbf{N}^*, \Omega)$. the set of operations is $\Omega = \{\vee_d, \varphi_0\}$ where $\vee_d : (\mathbf{N}^*)^2 \to \mathbf{N}^*$ with $\forall x, y \in \mathbf{N}^*, x \vee_d y = [x, y]$ and

$\varphi_0 : (\mathrm{N}^*)^{\emptyset} \to \mathrm{N}^*$ the null operation that fixes 1-unique particular element with the role of neutral element for "$\vee_d$"; $1 = e_{\vee_d}$ -the universal algebra (N^*, Ω') with $\Omega' = \{\vee, \psi_0\}$ where $\vee : (\mathrm{N}^*)^2 \to \mathrm{N}^*$, $\forall x, y \in N^*$; $x \vee y = \sup\{x, y\}$ and $\psi_0 : (\mathrm{N}^*)^{\emptyset} \to \mathrm{N}^*$ a null operation that fixes 1-unique particular element with the role of neutral element for "$\vee$": $1 = e_\vee$. We observe that the universal algebra (N^*, Ω) and (N^*, Ω') are of the same type

$$\begin{pmatrix} \vee_d & \varphi_0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} \vee & \psi_0 \\ 2 & 0 \end{pmatrix}$$

and with the similarity (bijective) $\vee_d \Longleftrightarrow \vee$ and $\varphi_0 \Longleftrightarrow \psi_0$ function $S_{0d} : \mathrm{N}^* \to \mathrm{N}^*$ is a morphism between them.

We already know that $\left((\mathrm{N}^*)^I, \overline{\Omega}\right)$ with I-a some set-is an universal algebra with $\overline{\Omega} = \{w_1, w_0\}$ defined by :

$$w_1 : (\mathrm{N}^*)^I \times (\mathrm{N}^*)^I \to (\mathrm{N}^*)^I$$

with

$$\forall a = \{a_i\}_{i \in I}, \; b = \{b_i\}_{i \in I}, \; a, b \in (N^*)^I, \; w_1(a, b) = \{a_i \vee_d b_i\}_{i \in I}$$

and w_0 a null operation: $e_{w_1} = \{e_i = 1\}_{i \in I}$ (the canonical projections $\tilde{p}_j$ being, of course, morphismes between $\left((\mathrm{N}^*)^I, \overline{\Omega}\right)$ and (N^*, Ω) (see [3, th. 1.a)])).

We also know that $\left((\mathrm{N}^*)^I, \Omega'\right)$ with $\Omega' = \{\theta_1, \theta_0\}$ defined by

$$\theta_1 : (\mathrm{N}^*)^I \times (\mathrm{N}^*)^I \to (\mathrm{N}^*)^I$$

by

$$\forall a = \{a_i\}_{i \in I}, \; b = \{b_i\}_{i \in I}, \; a, b \in (N^*)^I, \; \theta_1(a, b) = \{a_i \vee b_i\}_{i \in I}$$

and θ_0 - a null operation: $e_{\theta_1} = \{e_i = 1\}_{i \in I}$ (neutral element) is an universal algebra and is of the same type as the above one.

With all these known elements we can state:

Theorem 1 *If $S_{0d} : \mathrm{N}^* \to \mathrm{N}^*$ is a Smarandache type function defined by (2), morphism between (N^*, Ω) and (N^*, Ω') and I is a some set, then there is an unique $s_{0d} : (\mathrm{N}^*)^I \to (\mathrm{N}^*)^I$, morphism between the universal algebras $\left((\mathrm{N}^*)^I, \overline{\Omega}\right)$ and $\left((\mathrm{N}^*)^I, \overline{\Omega'}\right)$ so that $p_i \circ s_{0d} = S_{0d} \circ \tilde{p}_i$, $i \in I$, where $p_j : (\mathrm{N}^*)^I \to (\mathrm{N}^*)$ with $\forall a = \{a_i\}_{i \in I} \in (\mathrm{N}^*)^I$, $p_j(a) = a_j$, $\forall j \in I$ are the canonical projections, morphismes between $\left((\mathrm{N}^*)^I, \overline{\Omega'}\right)$ and (N^*, Ω'), $\tilde{p}_i : (\mathrm{N}^*)^I \to \mathrm{N}^*$, analoguous between $\left((\mathrm{N}^*)^I, \overline{\Omega}\right)$ and (N^*, Ω).*

Proof. We use the property of universality of the universal algebra $\left((\mathrm{N}^*)^I, \overline{\Omega'}\right)$: for every $\left(A, \overline{\theta}\right)$ with $\overline{\theta} = \left\{\mathrm{T}, \overline{\theta}_0\right\}$ is an universal algebra of the same type with $\left((\mathrm{N}^*)^I, \overline{\Omega'}\right)$ and $u_i : A \to N^*$, $\forall i \in I$, morphismes between $\left(A, \overline{\theta}\right)$ and (N^*, Ω'), there is an unique $u : A \to (\mathrm{N}^*)^I$ morphism between the universal algebras $\left(A, \overline{\theta}\right)$ and $\left((\mathrm{N}^*)^I, \overline{\Omega'}\right)$, so that $p_j \circ u = u_j, \forall j \in I$, with p_j - the canonical projections. A some universal algebra can be $\left((\mathrm{N}^*)^I, \overline{\Omega}\right)$ because is of the same type and the morphismes from the assumption can be $u_i : (\mathrm{N}^*)^I \to \mathrm{N}^*$ defined by:

$$\forall a = \{a_i\}_{i \in I} \in (N^*)^I, \; u_j(a) = S_{0d}(a_j) \Longleftrightarrow u_j = S_{0d} \circ \widetilde{p}_j, \; \forall j \in I,$$

where S_{0d} is a Smarandache type function, morphisme, as we know from (3) and $\widetilde{p}_j$ - the canonical projections, morphismes between $\left((\mathrm{N}^*)^I, \overline{\Omega}\right)$ and $(\mathrm{N}^*, \Omega)(u_i$ are morphismes as a composition of two morphismes). The assumptions of the property of universality being ensured, it results that there is an unique $s_{0d} : (\mathrm{N}^*)^I \to (\mathrm{N}^*)^I$, morphismes between $\left((\mathrm{N}^*)^I, \overline{\Omega}\right)$ and $\left((\mathrm{N}^*)^I, \overline{\Omega'}\right)$ so that $p_j \circ s_{0d} = u_j, \forall j \in I$, i.e. $p_j \circ s_{0d} = S_{0d} \circ \widetilde{p}_j$, $\forall j \in I$. ■

B. A sequence of positive integers $\sigma : \mathrm{N}^* \to \mathrm{N}^*$ is called " of divisibility (d.s.)" if:

$$m/n \Longrightarrow \sigma(m)/\sigma(n) \tag{4}$$

and "of strong divisibility (s.d.s.)" if:

$$\sigma((m,n)) = (\sigma(m), \sigma(n)), \; \forall m, n \in \mathrm{N}^*, \tag{5}$$

with (m, n) the greatest common factor.

The sequence s.d.s. were studied by N. Jansen; the Fibonacci sequence defined by

$$F_{n+1} = F_n - F_{n-1} \text{ with } F_1 = F_2 = 1$$

is a s.d.s.

Starting from a sequence $\sigma_{dd} : \mathrm{N}^* \to \mathrm{N}^*$ that satisfies the condition

$$\forall n \in \mathrm{N}^*, \exists m_n \in \mathrm{N}^*, \forall m \in \mathrm{N}^*, \; m_n/n \Longrightarrow n/\sigma(m), \tag{6}$$

as associated Smarandache's function was built that is $S_{dd} : \mathrm{N}^* \to \mathrm{N}^*$ given by

$$S_{dd}(n) = \min\{m_n : m_n \text{ is given by (6)}\}, \; \forall n \in \mathrm{N}^*, \tag{7}$$

having a series of already known properties from which we retain:

if the sequence σ_{dd} is s.d.s. and satisfies (6), then

$$S_{dd}([n_1, n_2]) = [S_{dd}(n_1), S_{dd}(n_2)], \forall n_1, n_2 \in \mathrm{N}^*, \tag{8}$$

where $[a, b]$ is the smallest common multiple of a and b (see [1, th. 2.5]).

We can stand out the universal algebra (N^*, Ω) where, this time, $\Omega = \{\vee_d, \wedge_d, \varphi_0\}$ of the type $\tau = \begin{pmatrix} \vee_d & \wedge_d & \varphi_0 \\ 2 & 2 & 0 \end{pmatrix}$ with known $\vee_d$ and φ_0 (from A) and $\wedge_d : (\mathrm{N}^*)^2 \to \mathrm{N}^*$ defined by

$$x \wedge_d y = (x, y), \forall x, y \in \mathrm{N}^*.$$

It is known that then there is an universal algebra $\left((\mathrm{N}^*)^I, \overline{\Omega}\right)$ with I - a some set and here $\overline{\Omega} = \{w_1, w_2, w_0\}$ with w_1, w_0 known and $w_2 : (\mathrm{N}^*)^I \times (\mathrm{N}^*)^I \to (\mathrm{N}^*)^I$ defined by:

$$w_2(a, b) = \{a_i \wedge_d b_i\}_{i \in I}, \forall a = \{a_i\}_{i \in I}, b = \{b_i\}_{i \in I} \in (\mathrm{N}^*)^I.$$

It can be stated the same as at A:

Theorem 2 *If $S_{dd} : \mathrm{N}^* \to \mathrm{N}^*$ is a Smarandache type function defined by (7), endomorphism for the universal algebra (N^*, Ω) and I - a some set, then there is an unique $s_{dd} : (\mathrm{N}^*)^I \to (\mathrm{N}^*)^I$, an endomorphism for the above universal algebra $\left((\mathrm{N}^*)^I, \Omega\right)$ so that $p_i \circ s_{dd} = S_{dd} \circ p_i$, $\forall i \in I$.*

The analogical proof with that of th. 1. can be also done directly; the corespondence s_{dd} is defined and it is shown that is a function, endomorphism, the required conditions being obviously satisfied.

Remark 1 *If the initial sequence σ_{dd} isn't at all s.d.s. but satisfies (6) with a view to the properties of the associated function, a function can be always defined $s_{dd} : (\mathrm{N}^*)^I \to (\mathrm{N}^*)^I$ that is no more an endomorphism for the given universal algebra $\left((\mathrm{N}^*)^I, \overline{\Omega}\right)$ than in certain conditions or in particular cases (see [1, th. 2.4.]).*

C. Starting from a sequence noted σ_{d0} of positive integers $\sigma_{d0} : \mathrm{N}^* \to \mathrm{N}^*$ that satisfies the condition:

$$\forall n \in \mathrm{N}^*, \exists m_n \in \mathrm{N}^*; \forall m \in \mathrm{N}^*, m_n/m \Longrightarrow n \leq \sigma_{d0}(m) \tag{9}$$

are associated Smarandache type function was built, defined by:

$$S_{d0}(n) = \min \{m_n \backslash m_n \text{ satisfies (9) } \} \tag{10}$$

having known properties.

Standing out the universal algebra (N^*, Ω') when here $\Omega' = \{\vee, \wedge, \dagger_0\}$ with $\vee, \dagger_0$ known, and $\wedge : (\mathrm{N}^*)^2 \to \mathrm{N}^*$ by

$$x \wedge y = \inf \{x, y\}, \forall x, y \in \mathrm{N}^*$$

it can be proved the same way that there is an unique $s_{d0} : (\mathrm{N}^*)^I \to (\mathrm{N}^*)^I$ endomorphism of the universal algebra $\left((\mathrm{N}^*)^I, \overline{\Omega'}\right)$ so that

$$\tilde{p}_i \circ s_{d0} = S_{d0} \circ \tilde{p}_i, \forall i \in I.$$

Above we built the prolongations s_{ij} to more complexe sets of the Smarandache type functions noted S_{ij} (for $I = \{1\} \Longrightarrow s_{ij} = S_{ij}$). The algebric properties of the s_{ij}, for their restrictions to N^*, could bring new properties for the Smarandache type function that we considered above.

References

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