Some inequalities concerning Smarandache's function

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The objectives of this article are to study the sum $\sum_{d|n} S(d)$ and to find some upper bounds for Smarandache's function. This sum is proved to satisfy the inequality $\sum_{d|n} S(d) \le n$ at most all the composite numbers. Using this inequality, some new upper bounds for Smarandache's function are found. These bounds improve the well-known inequality $S(n) \le n$.

1. Introduction

The object that is researched is Smarandache's function. This function was introduced by Smarandache [1980] as follows:

$$S: N^* \to N \text{ defined by } S(n) = \min\{k \in N \mid k! = \underline{M}n\} \quad (\forall n \in N^*). \tag{1}$$

The following main properties are satisfied by S:

$$(\forall a,b \in N^*)\ (a,b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\}. \tag{2}$$

$$(\forall a \in N^*)\ S(a) \le a \text{ and } S(a) = a \text{ iif } a \text{ is prim}. \tag{3}$$

$$(\forall p \in N^*, p \text{ prime})(\forall k \in N^*)\ S(p^k) \le p \cdot k. \tag{4}$$

Smarandache's function has been researched for more than 20 years, and many properties have been found. Inequalities concerning the function S have a central place and many articles have been published [Smarandache, 1980], [Cojocaru, 1997], [Tabirca, 1997], [Tabirca, 1988]. Two important directions can be identified among these inequalities. First direction and the most important is represented by the inequalities concerning directly the function S such as upper and lower bounds. The second direction is given by the inequalities involving sums or products with the function S.

2. About the sum $\sum_{d|n} S(d)$

The aim of this section is to study the sum $\sum_{d|n} S(d)$.

Let $SS(n) = \sum_{d|n} S(d)$ denote the above sum. Obviously, this sum satisfies

$SS(n) = \sum_{1\neq d|n} S(d)$. Table 1 presents the values of *S(n)* and *SS(n)* for *n*<50 [Ibstedt, 1997]. From this table, it can be seen that the inequality $SS(n) \leq n+2$ holds for all *n*=1, 2, ..., 50 and *n*≠12. Moreover, if *n* is a prim number, then the inequality becomes equality $SS(n) = n$.

***Remarks* 1.**

a) If *n* is a prime number, then $SS(n) = S(1) + S(n) = n$.

b) If *n*>2 is a prim number, then

$$SS(2 \cdot n) = S(1) + S(2) + S(n) + S(2 \cdot n) = 2 + n + n = 2 \cdot n + 2,$$

c) $SS(n^2) = S(1) + S(n) + S(n^2) = n + 2 \cdot n = 3 \cdot n \leq n^2$.

N	*S*	*SS*	*n*	*S*	*SS*	*n*	*S*	*SS*	*n*	*S*	*SS*	*n*	*S*	*SS*
1	0	0	11	11	11	21	7	17	31	31	31	41	41	41
2	2	2	12	4	16	22	11	24	32	8	24	42	7	36
3	3	3	13	13	13	23	23	23	33	11	25	43	43	43
4	4	6	14	7	16	24	4	24	34	17	36	44	11	39
5	5	5	15	5	13	25	10	15	35	7	19	45	6	25
6	3	8	16	6	16	26	13	28	36	6	34	46	23	48
7	7	7	17	17	17	27	9	18	37	37	37	47	47	47
8	4	10	18	6	20	28	7	27	38	19	40	48	6	36
9	6	9	19	19	19	29	29	29	39	13	29	49	14	21
10	5	12	20	5	21	30	5	28	40	5	30	50	10	32

Table 1. The values of *n, S, SS.*

The inequality $SS(n) \le n$ is proved to be true for the following particular values $n = p^k, 2\cdot p^k, 3\cdot p^k$ and $6\cdot p^k$.

***Lemma* 1.** If $p>2$ is a prime number and $k>1$, then the inequality $SS(p^k) \le p^k$ holds.

Proof

The following inequality holds according to inequality (4) and the definition of SS.

$$SS(p^k) = \sum_{i=1}^{k} S(p^i) \le \sum_{i=1}^{k} p\cdot i = p\cdot\frac{k\cdot(k+1)}{2}.$$

The inequality

$$\sum_{i=1}^{k} p\cdot i = p\cdot\frac{k\cdot(k+1)}{2} \le p^k \tag{5}$$

is proved to be true by analysing the following cases.

- $k=2 \Rightarrow 3\cdot p \le p^2.$ (6)
- $k=3 \Rightarrow 6\cdot p \le p^3.$ (7)
- $k=4 \Rightarrow 10\cdot p \le p^4.$ (8)

Inequalities (6-8) are true because $p>2$.

- $k>4 \Rightarrow p^k \ge p\cdot p^{k-1} \ge p\cdot 2^{k-1} = p\cdot\sum_{i=0}^{k-1}\binom{k-1}{i}$. The first and the last three terms of this sum are kept and it is found $p^k \ge p\cdot\left(2\cdot\binom{k-1}{0} + 2\cdot\binom{k-1}{1} + 2\cdot\binom{k-1}{2}\right) = p\cdot(k^2 - k + 2)$. The inequality $k^2 - k + 2 \ge \frac{k\cdot(k+1)}{2}$ holds because $k>4$, therefore $p^k \ge p\cdot\frac{k\cdot(k+1)}{2}$ is true.

Therefore, the inequality $S(p^k) \le p^k$ holds. ♣

***Remark* 2.** The inequality $S(p^k) \le p^k$ is still true for $p=2$ and $k>3$ because (8) holds for these values. Table 1 shows that the inequality is not true for $p=2$ and $k=2,3$.

***Lemma* 2.** If $p>2$ is a prime number and $k>1$, then the inequality $SS(2\cdot p^k) \le 2\cdot p^k$ holds.

Proof

The definition of SS gives the following equation

$$SS(p^k) = S(2) + \sum_{i=1}^{k} S(p^i) + \sum_{i=1}^{k} S(2\cdot p^i).$$

Applying the inequality $S(2\cdot p^i) \le p\cdot i$ and (4), we have

$$SS(2\cdot p^k) \le 2 + \sum_{i=1}^{k} p\cdot i + \sum_{i=1}^{k} p\cdot i = 2 + p\cdot k\cdot (k+1). \tag{9}$$

The inequality

$$2 + p\cdot k\cdot (k+1) \le 2\cdot p^k \tag{10}$$

is proved to be true as before.

- $k=2 \Rightarrow 2 + 6\cdot p \le 2\cdot p^2.$ (11)
- $k=3 \Rightarrow 2 + 12\cdot p \le 2\cdot p^3.$ (12)
- $k=4 \Rightarrow 2 + 20\cdot p \le 2\cdot p^4.$ (13)
- $k=5 \Rightarrow 2 + 30\cdot p \le 2\cdot p^5.$ (14)
- $k=6 \Rightarrow 2 + 42\cdot p \le 2\cdot p^5.$ (15)

These above inequalities (11-15) are true because $p>2$.

- $k>6 \Rightarrow \; p^k \ge p\cdot p^{k-1} \ge p\cdot 2^{k-1} = p\cdot \sum_{i=0}^{k-1} \binom{k-1}{i}$. The first and the last fourth terms

  of this sum are kept finding

$$p^k \ge p\cdot \left(2\cdot \binom{k-1}{0} + 2\cdot \binom{k-1}{1} + 2\cdot \binom{k-1}{2} + 2\cdot \binom{k-1}{3} \right) \ge$$

$$\ge p\cdot \left(2\cdot \binom{k-1}{0} + 2\cdot \binom{k-1}{1} + 2\cdot \binom{k-1}{2} + 2\cdot \binom{k-1}{2} \right) =$$

$$= p\cdot \left(2\cdot k^2 - 4\cdot k + 4\right) \ge 2 + p\cdot (k^2 + k)$$

The last inequality holds because $k>6$, therefore $2\cdot p^k \ge 2 + p\cdot k\cdot (k+1)$ is true.

The inequality $SS(2\cdot p^k) \le 2\cdot p^k$ holds because (10) has been found to be true.

♣

Remark 3. Similarly, the inequality $SS(3\cdot p^k) \le 3\cdot p^k$ can be proved for all ($p>3$ and $k\ge 1$) or ($p=2$ and $k\ge 3$).

***Lemma* 3.** If $p>3$ is a prime number and $k\geq 1$, then the inequality $SS(6\cdot p^k)\leq 6\cdot p^k$ holds.

Proof

The starting point is given by the following equation (16)

$$SS(6\cdot p^k)=S(2)+S(3)+S(6)+\sum_{i=1}^{k}S(p^i)+\sum_{i=1}^{k}S(2\cdot p^i)+\sum_{i=1}^{k}S(3\cdot p^i)+\sum_{i=1}^{k}S(6\cdot p^i). \tag{16}$$

The inequalities $S(p^i), S(2\cdot p^i), S(3\cdot p^i), S(6\cdot p^i)\leq p\cdot i$ hold for all $i>1$ because $p\geq 5$. Therefore, the inequality

$$SS(6\cdot p^k)\leq 8+\sum_{i=1}^{k}p\cdot i+\sum_{i=1}^{k}p\cdot i+\sum_{i=1}^{k}p\cdot i+\sum_{i=1}^{k}p\cdot i=8+4\cdot\sum_{i=1}^{k}p\cdot i \tag{17}$$

holds. The inequality $SS(6\cdot p^k)\leq 8+4\cdot p^k\leq 6\cdot p^k$ is found to be true by applying (5) in (17).

♣

The following propositions give the main properties of the function *SS*. Let $d(n)$ denote the number of divisors of n.

***Proposition* 1.** If a is natural numbers such that $S(a)\geq 4$, then the inequality $S(a)\geq 2\cdot d(a)$ holds.

Proof

The proof is made directly as follows:

$$S(a)=\sum_{1\neq d|a}S(d)=\sum_{1,n\neq d|a}S(d)+S(a)\geq\sum_{1,n\neq d|a}2+S(a)=2\cdot(d(a)-2)+S(a)=$$

$$=2\cdot d(a)+S(a)-4\geq 2\cdot d(a).$$ ♣

***Remark* 4.** The inequality $S(a)\geq 4$ is verified for all the numbers $a\geq 4$ and $a\neq 6$.

***Proposition* 2.** If a, b are two natural numbers such that $(a,b)=1$, then the inequality $SS(a\cdot b)\leq d(a)\cdot SS(b)+d(b)\cdot SS(a)$ holds.

Proof

This proof is made by using (2) and the simple remark that $a,b\geq 0\Rightarrow\max\{a,b\}\leq a+b$.

The set of the divisors of ab is split into three sets as follows:

$$\{1\neq d \mid a\cdot b = \underline{M}d\} = \{1\neq d \mid a=\underline{M}d\}\cup\{1\neq d \mid b=\underline{M}d\}\cup\{d_1 d_2 \mid a=\underline{M}d_1\neq 1 \wedge b=\underline{M}d_2\neq 1\wedge (d_1,d_2)=1\}. \quad (18)$$

The following transformations hold according to (18).

$$SS(a\cdot b)=\sum_{\{1\neq d \mid a\cdot b = \underline{M}d\}} S(d)=\sum_{\{1\neq d \mid a=\underline{M}d\}} S(d)+\sum_{\{1\neq d \mid b=\underline{M}d\}} S(d)+\sum_{\{1\neq d_1 \mid a=\underline{M}d_1\}}\ \sum_{\{1\neq d_2 \mid b=\underline{M}d_2\}} S(d_1\cdot d_2)=$$

$$=SS(a)+SS(b)+\sum_{\{1\neq d_1 \mid a=\underline{M}d_1\}}\ \sum_{\{1\neq d_2 \mid b=\underline{M}d_2\}} \max\{S(d_1),S(d_2)\}\leq$$

$$\leq SS(a)+SS(b)+\sum_{\{1\neq d_1 \mid a=\underline{M}d_1\}}\ \sum_{\{1\neq d_2 \mid b=\underline{M}d_2\}} [S(d_1)+S(d_2)]=$$

$$=SS(a)+SS(b)+\sum_{\{1\neq d_1 \mid a=\underline{M}d_1\}}\ \sum_{\{1\neq d_2 \mid b=\underline{M}d_2\}} S(d_1)+\sum_{\{1\neq d_1 \mid a=\underline{M}d_1\}}\ \sum_{\{1\neq d_2 \mid b=\underline{M}d_2\}} S(d_2)=$$

$$=SS(a)+SS(b)+SS(a)\cdot[d(b)-1]+SS(b)\cdot[d(a)-1]$$

Therefore, the inequality $SS(a\cdot b)\leq d(a)\cdot SS(b)+d(b)\cdot SS(a)$ holds. ♣

***Proposition* 3.** If a, b are two natural numbers such that $S(a), S(b)\geq 4$ and $(a,b)=1$, then the inequality $SS(a\cdot b)\leq SS(a)\cdot SS(b)$ holds.

Proof

Proposition 1-2 are applied to prove this proposition as follows:

$$S(a), S(b)\geq 4 \Rightarrow S(a)\geq 2\cdot d(a) \text{ and } S(b)\geq 2\cdot d(b) \quad (19)$$

$$(a,b)=1 \Rightarrow SS(a\cdot b)\leq d(a)\cdot SS(b)+d(b)\cdot SS(a). \quad (20)$$

The proof is completed if the inequality $d(a)\cdot SS(b)+d(b)\cdot SS(a)\leq SS(a)\cdot SS(b)$ is found to be true. This is given by the following equivalence

$$d(a)\cdot SS(b)+d(b)\cdot SS(a)\leq SS(a)\cdot SS(b)\Leftrightarrow$$

$$d(a)\cdot d(b)\leq [SS(a)-d(a)]\cdot[SS(b)-d(b)].$$

This last inequality holds according to (19).

Therefore, the inequality $SS(a\cdot b)\leq SS(a)\cdot SS(b)$ is true. ♣

Theorem 1. If n is a natural number such that $n\neq 8, 12, 20$ then

a) $SS(n)=n+2$ if $(\exists\ p \text{ prime})\ n=2\cdot p$. (21)

b) $SS(n)\leq n$, otherwise. (22)

Proof

The proof of this theorem is made by using the induction on n.

Equation (21) is true according to Remark 1.a. Table 1 shows that Equation (22) holds for $n<51$ and $n \neq 8, 12, 20$. Let $n>51$ be a natural number. Let us suppose that Equation (9) is true for all the number k that satisfies $k<n$ and k does not have the form $k=2p$, p prime. The following cases are analysed:

- ***n* is prime** $\Rightarrow SS(n)=n$, therefore Equation (9) holds.
- $n=2p$, $p>2$ **prime** $\Rightarrow SS(n)=n+2$, therefore Equation (21) holds.
- ($n=2^k$ **and** $k>3$) **or** ($n=p^k$ **and** $k>1$) $\Rightarrow SS(n)\leq n$ according to Lemma 1
- $n=2\cdot p^k$, $p>2$ **prime number and** $k>1$ $\Rightarrow SS(n)\leq n$ according to Lemma 2.
- $n=3\cdot p^k$, ($p>3$ **prime number and** $k>1$) **or** ($p=2$ **and** $k>2$) $\Rightarrow SS(n)\leq n$ according to Remark 3.
- $n=6\cdot p^k$, $p>3$ **prime number and** $k\geq 1$ $\Rightarrow SS(n)\leq n$ according to Lemma 3.
- **Otherwise** $\Rightarrow$ Let $n=p_1^{k_1}\cdot p_2^{k_2}\cdot\ldots\cdot p_s^{k_s}$ be the prime number decomposition of n with $p_1<p_2<\ldots<p_s$. We prove that there is a decomposition of $n=ab$, $(a,b)=1$ such that $S(a), S(b)\geq 4$. Let us select $a=p_s^{k_s}$ and $b=p_1^{k_1}\cdot p_2^{k_2}\cdot\ldots\cdot p_{s-1}^{k_{s-1}}$. It is not difficult to see that this decomposition satisfies the above conditions. The induction's hypotheses is applied for $a,b<n$ and the inequalities $SS(a)\leq a$ and $SS(b)\leq b$ are obtained. Finally, Proposition 3 gives $SS(n)=SS(a\cdot b)\leq SS(b)\cdot SS(a)\leq b\cdot a=n$.

We can conclude that the inequality $SS(n)\leq n-2$ holds for all the natural number $n\neq 12$.

♣

Remark **5.** The above analysis is necessary to be sure that the decomposition of $n=ab$, $(a,b)=1$, $S(a)$, $S(b)\geq 4$ exists.

Theorem 1 has some interesting consequences that are presented in the following. These establish new upper bounds for Smarandache's function.

Consequence 1. If $n>1$ is a natural number, then the following inequality

$$S(n)\leq n+4-2\cdot d(n) \quad (23)$$

holds.

Proof

The proof of this inequality is made by using Theorem 1.

Obviously, (23) is true for $n=p$ or $n=2p$, p prime number.

Let $n \neq 8, 12, 20$ be a natural number.

We have the following transformations:

$$n \geq SS(n) = \sum_{1 \neq d|n} S(d) = S(n) + \sum_{1,n \neq d|n} S(d) \geq$$

$$\geq S(n) + 2 \cdot \left|\left\{d = \overline{1,n} \,|\, d \neq 1, n \wedge d \mid n\right\}\right| = S(n) + 2 \cdot (d(n) - 2) = S(n) + 2 \cdot d(n) - 4$$

Inequality (23) is also satisfied for n=8, 12, 20.

Therefore, the inequality $S(n) \leq n + 4 - 2 \cdot d(n)$ holds. ♣

Consequence 2. If $n > 1$ is a natural number, then the following inequality holds

$$S(n) \leq n + 4 - \min\{p \mid p \text{ is prime and } p|n\} \cdot d(n) \quad . \tag{24}$$

Proof

This proof is made similarly to the proof of the previous consequence by using the following strong inequality $S(d) \geq \min\{p \mid p \text{ is prime and } p|n\}$. ♣

3. Final Remark

Inequalities (23 - 24) give some generalisations of the well - known inequality $S(n) \leq n$. More important is the fact that these inequalities reflect. When n has many divisors, the value of $n + 4 - \min\{p \mid p \text{ is prime and } p|n\} \cdot d(n)$ is small, therefore the value of $S(n)$ is small as well according to Inequality (24). In spite of fact that Inequalities (23 - 24) reflect this situation, we could not say that the upper bounds are the lowest possible. Nevertheless, they offer a better upper bound than the inequality $S(n) \leq n$.

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