SOME LINEAR EQUATIONS INVOLVING A FUNCTION IN THE NUMBER THEORY

We have constructed a function η which associates to eac non-null integer m the smallest positive n such that n! is a multiple of m.

(a) Solve the equation $\eta(x) = n$, where $n \in N$.

*(b) Solve the equation $\eta(mx) = x$, where $m \in Z$. Discussion.

(c) Let $\eta^{(i)}$ note $\eta \circ \eta \circ \ldots \circ \eta$ of i times. Prove tha there is a k for which

$$\eta^{(k)}(m) = \eta^{(k+1)}(m) = n_m, \text{ for all } m \in Z^*\backslash\{1\}.$$

**Find n_m and the smallest k with this property.

Solution

(a) The cases n = 0, 1 are trivial.

We note the increasing sequence of primes less or equal than n by $P_1, P_2, \ldots, P_k$, and

$$\beta_t = \sum_{h\geq 1} [n/p_t^h], \quad t = 1, 2, \ldots, k;$$

where [y] is the greatest integer less or equal than y.

Let $n = p_{i_1}^{\alpha_{i_1}} \dots p_{i_s}^{\alpha_{i_s}}$, where all p_{i_j} are distinct primes and all α_{i_j} are from N.

Of course we have $n \leq x \leq n!$

Thus $x = p_1^{\sigma_1} \dots p_k^{\sigma_k}$ where $0 \leq \sigma_t \leq \beta_t$ for all $t = 1, 2, \dots, k$ and there exists at least a $j \in \{1, 2, \dots, s\}$ for which

$$\sigma_{i_j} \in \{\beta_{i_j}, \beta_{i_j}-1, \dots, \beta_{i_j} - \alpha_{i_j} + 1\}.$$

Clearly n! is a multiple of x, and is the smallest one.

(b) See [1] too. We consider $m \in N^*$.

<u>Lemma 1</u>. $\eta(m) \leq m$, and $\eta(m) = m$ if and only if $m = 4$ or m is a prime.

Of course m! is a multiple of m.

If $m \neq 4$ and m is not a prime, the Lemma is equivalent to there are m_1, m_2 such that $m = m_1 \cdot m_2$ with $1 < m_1 \leq m_2$ and ($2m_2 < m$ or $2m_1 < m$). Whence $\eta(m) \leq 2m_2 < m$, respectively $\eta(m) \leq \max\{m_2, 2m_1\} < m$.

<u>Lemma 2</u>. Let p be a prime ≥ 5. Then $\eta(px) = x$ if and only if x is a prime $> p$, or $x = 2p$.

Proof: $\eta(p) = p$. Hence $x > p$.

Analogously: x is not a prime and $x \neq 2p \Leftrightarrow x = x_1 x_2$, $1 < x_1 \leq x_2$ and ($2x_2 < x$, $x_2 \neq p$, and $2x_1 < x$) $\Rightarrow \eta(px) \leq$

$\leq \max \{p, 2x_2\} < x$ respectively $\eta(p x) \leq \max \{p, 2x_1, x_2\} < x$.

Observations

$\eta(2x) = x \Leftrightarrow x = 4$ or x is an odd prime.

$\eta(3x) = x \Leftrightarrow x = 4, 6, 9$ or x is a prime > 3.

Lemma 3. If $(m, x) = 1$ then x is a prime $> \eta(m)$.

Of course, $\eta(mx) = \max\{\eta(m), \eta(x)\} = \eta(x) = x$. And $x \neq \eta(m)$, because if $x = \eta(m)$ then $m \cdot \eta(m)$ divides $\eta(m)!$ that is m divides $(\eta(m) - 1)!$ whence $\eta(m) \leq \eta(m) - 1$.

Lemma 4. If x is not a prime then $\eta(m) < x \leq 2\eta(m)$ and $x = 2\eta(m)$ if and only if $\eta(m)$ is a prime.

Proof: If $x > 2\eta(m)$ there are x_1, x_2 with $1 < x_1 \leq x_2$, $x = x_1 x_2$. For $x_1 < \eta(m)$ we have $(x - 1)!$ is a multiple of $m x$. Same proof for other cases.

Let $x = 2\eta(m)$; if $\eta(m)$ is not a prime, then $x = 2ab$, $1 < a \leq b$, but the product $(\eta(m) + 1)(\eta(m) + 2) \ldots (2\eta(m) - 1)$ is divided by x.

If $\eta(m)$ is a prime, $\eta(m)$ divides m, whence $m \cdot 2\eta(m)$ is divided by $\eta(m)^2$, it results in $\eta(m \cdot 2\eta(m)) \geq 2 \cdot \eta(m)$, but $(\eta(m) + 1)(\eta(m) + 2) \ldots (2\eta(m))$ is a multiple of $2\eta(m)$, that is $\eta(m \cdot 2\eta(m)) = 2\eta(m)$.

Conclusion

All x, prime number > η (m), are solutions.

If η (m) is prime, then x = 2 η (m) is a solution.

*If x is not a prime, η (m) < x < 2 η (m), and x does not divide (x - 1)!/m then x is a solution (semi-open question). If m = 3 it adds x = 9 too. (No other solution exists yet.)

(c)

Lemma 5. η (a b) $\leq$ η (a) + η (b).

Of course, η (a) = a' and η (b) = b' involves (a' + + b')! = b'! (b' + 1) ... (b' + a'). Let a' $\leq$ b'. Then η(ab) $\leq$ a' + b', because the product of a' consecutive positive integers is a multiple of a'!

Clearly, if m is a prime then k = 1 and n_m = m.

If m is not a prime then η (m) < m, whence there is a k for which $\eta^{(k)}$ (m) = $\eta^{(k+1)}$ (m).

If m $\neq$ 1 then 2 $\leq$ n_m $\leq$ m.

Lemma 6. n_m = 4 or n_m is a prime.

If n_m = n_1 n_2, 1 < n_1 $\leq$ n_2, then η (n_m) < n_m. Absurd. n_m $\neq$ 4.

(**) This question remains open.

Reference

[1] F. Smarandache, A Function in the Number Theory, An. Univ. Timisoara, seria st. mat., Vol. XVIII, fasc. 1, pp. 79-88, 1980; Mathematical Reviews: 83c: 10008.

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