SOME PROBLEMS CONCERNING THE SMARANDACHE SQUARE COMPLEMENTARY FUNCTION (IV)

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Abstract: In this paper we determine all solutions of an exponential diophantine equations concerning the Smarandache square complementary function.

Key words: Smarandache square complementary function; exponential diophantine equations

For any positive integer n, let $SSC(n)$ denote the Smarandache square complementary function of n (see [1]). In [3], Russo asked that solve the equation

$$SSC(n)^r + SSC(n)^{r-1} + \cdots + SSC(n) = n, \qquad r > 1. \tag{1}$$

In this paper we completely solve this problem as follows.

Theorem. All positive integer solutions (n, r) of (1) are given by the following two cases.

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(i) $(n, r)=(363, 5)$.

(ii) $(n, r)=(ab^2, 2)$, where a and b are coprime positive integers satisfying $a>1$, $b>1$, $a=b^2-1$ and a is square free.

The proof of our theorem needs the following lemma.

Lemma ([2]). The equation

$$\frac{x^r-1}{x-1}=y^2, \quad x>1, \quad y>1, \quad r>2 \tag{2}$$

has only the positive integer solution $(x, y, r)=(3,11,5)$.

Proof of Theorem. Let (n, r) be a positive integer solution of (1). Let $x=SSC(n)$. Then from (1) we get

$$x\left(x^{r-1}+\cdots+x+1\right)=n, \quad r>1. \tag{3}$$

Since $r>1$ we see from (3) that $n>1$.

It is a well known fact that n can be expressed as

$$n=p_1^{\alpha_1}\cdots p_s^{\alpha_s}q_1^{\beta_1}\cdots q_t^{\beta_t}, \tag{4}$$

where $p_1,\cdots,p_s$ and $q_1,\cdots,q_t$ are distinct primes, $\alpha_1,\cdots,\alpha_s$ are odd positive integers and $\beta_1,\cdots,\beta_t$ are even positive integers. We see from (4) that

$$x=SSC(n)=p_1\cdots p_s. \tag{5}$$

Since gcd $(x, x^{r-1}+\cdots+x+1)=1$, we get from (3), (4) and (5) that $\alpha_1=\cdots=\alpha_s=1$ and

$$\frac{x^r-1}{x-1}=x^{r-1}+\cdots+x+1=q_1^{\beta_1}\cdots q_t^{\beta_t}. \tag{6}$$

Since $\beta_1,\cdots,\beta_t$ are even, let $b^2=q_1^{\beta_1}\cdots q_t^{\beta_t}$. Then b is a positive integer satisfying

$$\frac{x^r - 1}{x - 1} = b^2. \quad (7)$$

By Lemma, if $r>2$, then from (7) we get $(x, b, r)=(3,11,5)$. It implies that $(n, r)=(363, 5)$ by (4) and (15).

If $r=2$, then we have

$$x+1=b^2. \quad (8)$$

Let $a=x$. By (4), (5) and (7),we obtain the case (ii) immediately. Thus, the theorem is proved.

References

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