SOME PROBLEMS ON SMARANDACHE FUNCTION

by

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In this paper we shall investigate some aspects involving Smarandache function, $S:N^*\dashrightarrow N^*$, $S(n) = \min\{m \mid n \text{ divide } m!\}$.

1. THE MINIMUM OF S(n)/n

Which is minimumum of $S(n)/n$ if $n > 1$?

1.1. **THEOREM:**

a) $S(n)/n$ has no minimum for $n > 1$.

b) $\lim S(n)/n$ as n goes to infinity does not exist.

Proof:

a) Since $S(n) > 1$ for $n>1$ it follows that $S(n)/n > 0$. Assume that $S(n)/n$ has a minimum and let the rational fraction be represented by r/s. By the infinitude of the natural numbers, we can find a number m such $2/m < r/s$. Using the infinitude of the primes, we can find a prime number $p > m$. Therefore, we have the sequence

$$2/p < 2/m < r/s$$

We have $S(p \cdot p) = S(p^2) = 2p$. It is known that $S(p \cdot p) = 2p$. The ratio of $S(p^2)/(p \cdot p)$ is then

$$2p/(p^2) = 2/p$$

And this ratio is less than r/s, contradicting the assumption of the minimum.

b) Suppose $\lim S(n)/n$ exists and has value r. Now choose, $e > 0$ and $e < 1/p$ where p is a twenty digit prime. Since $S(p) = p$, $S(p)/p = 1$.

However, $S(p \cdot p) = 2p$, so the ratio $S(n)/n = 2p/(p \cdot p) = 2/p$. Since p is a twenty digit prime,

$$|S(p)/p - S(p \cdot p)/(p \cdot p)| > e \text{ by choice of } e.$$

so the limit does not exist.

2. THE DECIMAL NUMBER WHOSE DIGITS ARE THE VALUES OF SMARANDACHE FUNCTION IS IRRATIONAL.

Unsolved problem number (8) in [1] is as follows:

Is $r = 0{,}0234537465114\ldots$, where the sequence of digits is $S(n)$, $n \geq 1$, an irrational number?

The number r is indeed irrational and this claim will be proven below.

The following well-known results will be used.

DIRICHLET'S THEOREM:

If $d > 1$ and $a \neq 0$ are integers that are relativey prime, then the arithmetic progression a, a, + d, a + 2d, a + 3d,...

contains infinitely many primes.

Proof of claim:

Assume that r as defined above is rational. Then after some m digits, there must exist a series of digits t_1, t_2, t_3, ..., t_n, such that

$$r = 0{,}023453746114\ldots s\overline{t_1 t_2 t_3 t_4 t_5 \ldots t_n}$$

where s is the m-th digit in the decimal expansion.

Now, construct the repunit number consisting of 10n 1's.

a = 11111 ... 111

10n times

and let **d = 1000 ... 00**

10n + 1 0's

Since the only prime factors of d are 2 and 5, it is clear that a and d are relatively prime and by Dirichlet's Theorem, the sequence

a, a + d, a + 2d, ...

must contain primes. Given the number of 1's in a and the fact that S(p) = p, it follows that the sequence of repeated digits in r must consist entirely of 1's.

Now, construct the repdigit number constructed from 10n 3's

a = 3333...333

10n *times*

and using

d = 10000...00

10n + 1 0's

we again have a and d relatyvely prime. Arguments similar to those used before forces the conclusion that the sequence of repeted digits must consist entirely of 3's.

This is of course impossibile and therefore the assumption of rationality must be false.

3. ON THE DISTRIBUTION OF THE POINTS OF S(n)/n IN THE INTERVAL (0,1).

The following problem is listed as unsolved problem number (7) in [1]

Are the points $p(n) = S(n)/n$ uniformly distributed in the interval (0,1)?

The answer is no, the interval (0.5,1.0) contains only a finite number of points p(n).

3.1. **LEMMA:**

$$\frac{S(p^k)}{p^k} \geq \frac{S(p^{k+1})}{p^{k+1}} .$$

For p prime and k>0.

Proof:

It is well-known that $S(p^k) = j \cdot p$ where $j \le k$.

Therefore, forming the expressions

$$\frac{S(p^k)}{p^k} = \frac{j \cdot p}{p^k} = \frac{j}{p^{k-1}}$$

$$\frac{S(p^{k+1})}{p^{k+1}} = \frac{m \cdot p}{p^{k+1}} = \frac{m}{p^k}$$

where m must have one of the two values $\{j, j+1\}$.

With the restrictions on the values of m and p, it is clear that

$$\frac{j}{m} \ge \frac{1}{p}$$

which implies that

$$\frac{S(p^k)}{p^k} \ge \frac{S(p^{k+1})}{p^{k+1}}$$

which is the desired result. Equality occurs only when p=2, j=1 and m=2.

3.2. **LEMMA:**

The interval (0.5,1.0) contains only a finite number of points p(n), where

$$p(n) = \frac{S(n)}{n}$$ and n is a power of a prime.

Proof:

If n=p $\frac{S(p)}{p} = 1$, outside the interval.

Start with the smallest prime p=2 and move up the powers of 2

$$\frac{S(2 \cdot 2)}{(2 \cdot 2)} = \frac{4}{4} = 1$$

$$\frac{S(2 \cdot 2 \cdot 2)}{(2 \cdot 2 \cdot 2)} = \frac{4}{8}$$

$$\frac{S(2\cdot2\cdot2\cdot2)}{(2\cdot2\cdot2\cdot2)} = \frac{6}{16} < 0.5 \ .$$

And applying the previous lemma, all additional powers of 2 will yield a value less than 0.5.

Taking the next smallest prime p=3 and moving up the powers of 3

$$\frac{S(3\cdot3)}{(3\cdot3)} = \frac{6}{9}$$

$$\frac{S(3\cdot3\cdot3)}{(3\cdot3\cdot3)} = \frac{9}{27} < 0.5$$

and by the previous lemma, all additional powers of 3 also yield a value less than 0.5.

Now, if p>3 and p is prime

$$\frac{S(p\cdot p)}{(p\cdot p)} = \frac{2}{p} < 0.5$$

so all other powers of primes yield values less than 0.5 and we are done.

3.3. **THEOREM:**

The interval (0.5,1.0) contains only a finite number of points p(n) where

$$p(n) = \frac{S(n)}{n} \ .$$

Proof:

It is well-known that if

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot \ldots \cdot p_n^{\alpha_n} \ , \text{ then}$$

$$S(n) = \max\{S(P_i^{\alpha_i})\} \ .$$

Applying the well-known result with the formula for p(n)

$$p(n) = \frac{S(p_i^{\alpha_i})}{p_i^{\alpha_i}} \cdot \frac{1}{\prod_{j \neq i} p_j^{\alpha_j}}$$

which is clearly less than

$$\frac{S(p_i^{\alpha_i})}{p_i^{\alpha_i}} .$$

Theorefore, applying Lemma 2, we get the desired results.

3.4. **COROLLARY:**

The points p(n)=S(n)/n are not evenly distributed in the interval (0,1).

4. THE SMARANDACHE FUNCTION DOES NOT SATYSFY A LIPSCHITZ CONDITION

Unsolved problem number 31 in [1] is as follows.

Does the Smarandache function veryfy a Lipschitz condition? In other words, is there a real number L such that

$$| S(m) - S(n) | \leq L | m - n | \text{ for all } m,n \text{ in } \{0,1,2,3,\ldots\}.$$

4.1. **THEOREM**

The Smarandache function does not verify a Lipschitz condition.

Proof:

Suppose that Smarandache function does indeed satisfy a Lipschitz condition and let L be the Lipschitz constant.

Since the numbers of primes is infinite, is possible to fiind a prime p such that

$$p - (p + 1)/2 > L$$

Now, examine the numbers (p-1) and (p+1). Clearly, at least one must not be a power of two, so we choose that one call it m.

Factoring m into the product of all primes equal to 2 and everything else, we have

$$m = 2^k \cdot n$$

Then $S(m) = \max\{S(2^k), S(n)\}$ and because $S(2^k) \leq 2^k$. we have

$$S(m) \leq \frac{m}{2}$$

And so,

$$|S(p) - S(m)| > |p - \frac{m}{2}| > L$$

Since $|p - m| = 1$ by choice of m, we have a violation of the Lipschitz condition, rendering our original assumption false.

Therefore, the Smarandache function does not satisfy a Lipschitz condition.

5. ON THE SOLVABILITY OF THE EXPRESSION S(m) =n!

One of the unsolved problems in [1] involves a relationship between the Smarandache and factorial functions.

Solve the Diophantine Equation

$$S(m) = n!$$

where m and n are positive integers.

This equation is always solvable and the number of solutions is a function of the number of primes less than or equal to n.

5.1. **LEMMA**: Let be a prime. Then the range of the sequence

$S(p), S(p \cdot p), S(p \cdot p \cdot p), \ldots$

will contain all positive integral multiples of p.

Proof: It has already been proven [2] that for all integers $k > 0$, there exists another integer $m > 0$, such that

$$S(p^k)k = mp \quad \text{where} \quad m \leq k$$

and in particular

$$S(p) = p$$

So the only remaining element of the proof is to show that m takes on all possible integral values greater than 0.

Let p be an arbitrary prime number and define the set $M = \{$ all positive integers n such that there is no positive integer k such that $S(p^k) = mp$ $\}$
and assume that M is not empty.

Since M is non-empty subset of the natural numbers, it must have a least element. Call that least element m. It is clear that $m > 1$.

Now, let j be the largest integer such that

$$S(p^j) = (m - 1) \cdot p$$

and consider the exponent $j + 1$.

By the choise of j, it follows that either

1) $S(p^{j+1}) = m \cdot p$

or

2) $S(p^{j+1}) = n \cdot p$ where $n > m$

in the first case, we have a contradiction of our choise of m, so we proceed to case (2).

However, it is a direct consequence of the definition of prime numbers that if $((m - 1)\cdot p)!$ contains j instances of the prime p, then $m\cdot p$ is the smallest number such that $(m\cdot p)!$ contains more than j instances of p. Then, using the definition of Smarandache function where we choose the smallest number having the required number of instances we have a contradiction of case (2).

Therefore, it follows that there can be no least element of the set M, so M must be empty.

5.2. **THEOREM:** Let n be any integer and p a prime less than or equal to n. Then, there is some integer k such that

$$S(p^k) = n!$$

Therefore, each equation of the form $S(m) = n!$ has at least $^{\#}p$ solutions, where $^{\#}p$ is the number of primes less than or equal to n.

Proof:

Since n! is an integral multiple of p for p any prime less than or equal to n, this is a direct consequence of the lemma.

Now that the question is known to have multiple solutions, the next logical question is to determine how many solutions there are.

5.3. **DEFINITION:** Let NSF(n) be the number of integers m, such that $S(m) = n!$.

From the fact hat $S(n) = \max\{S(p_i^{a_i})\}$, we have the following obvious result.

Corollary:

Let n be a positive integer, q a prime less than or equal to n and k another positive integer such that $S(q^k) = n!$. Then, all numbers having the prime factorization form $m = q^k p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r}$ where $S(q^k) > S(p_i^{\alpha_i})$ will also be solutions the equation S(m) = n!

To proceed further, we need the following two obvious lemmas.

5.4. **LEMMA:** If p is a prime and m and n nonnegative integers m > n, then $S(p^n) \leq S(p^m)$.

5.5. **LEMA:** If p and q are primes such that p < q and k > 0, then $S(p^k) < S(q^k)$.

The following theorem gives an initial indication regarding how fast NSF(n) grows as n does.

5.6. **THEOREM:** Let q be a prime number and k an exponent such that $S(q^k) = n!$ Let $p_1, p_2, \ldots, p_r$ be the list of primes less than q. Then the number of solutions to the equation S(m) = n! where m contains exactly k instances of the prime q is at least $(k+1)^r$.

Proof: Applyng the two lemmas, the numbers $m = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r} q^k$ where all of exponents on the primes p_i are at most solutions to the equation. Since each prime pi can have (k + 1), {0,1,2,...,k} different values for the exponent, simple counting gives the result.

Since this procedure can be repeated for each prime less than or equal to n, we have an initial number of solutions given by the formula

$$\sum_{i=2}^{s} (k_i+1)^{i-1} + 1$$

where s is the number of primes less then or equal to n, k is the integer such that

$$S(p_j^{K_i}) = n!$$

And even this is a very poor lower bound on the number of solutions for n having any size.

5.7. **COROLARY:** Let q be a prime such that for some k $S(q^k) = n!$. Then if p is any prime such that there is some integer j such that $S(p^j) < S(q^k)$, then the product of any solution and p any power less than or equal to j will also be a solution.

Proof: Clear.

If q is the largest prime less than or equal to n, it is easy to show for "large" n that there are primes p > n > q that satisfy the above conditions. If p is any prime, then by Bertrand's Postulate, another prime r can be found in the interval p > r > 2p. Since q < n < 2n < n! for n > 2 and S(p) = p, we have one such prime. Expanding this reasoning, it folows that the number of such primes is at least j, where j is the largest exponent of 2 such that $q2^j \leq n!$, or put another way, the largest power of 2 that is less than or equal to n!/q.

Since there are so many solutions to the equation S(m) = n!, it is logical to consider the order of growth of the number of solutions rather than the actual number.

It is well known that the number of primes less than or equal to n is asymptotic to the ratio n/ln(n). Now, let p be the largest prime less than n. As n gets larger, it is clear that the factor m such that mp = n! grows on the order of a factorial. Since $m \le k$, where k is the exponent on the power p, it follows that the number grows on the order of the product of factorials. Since the number of items in the product depends on the number of primes q such that $q < mp = n!$, it follows that this number also grows on the order of a factorial.

Putting it all together, we have the following behavior of NSF(n).

NSF(n) grows on the order of product of items all on the order of the factorial of n, where the number of elements in the product also grows on the order of a factorial of n.

Cleary, this function grows at an astronomical rate.

6. THE NUMBER OF PRIMES BETWEEN S(n) and S(n+1)

I read the letter by I.M.Radu that appeared in [3] stating that there is always a prime between S(n) and S(n+1) for all numbers 0<n<4801, where S(n) is the Smarandache function.

Since 1 have a computer program that computes the values of S(n), I decided to investigate the problem further. The serch was conducted up through n<1,033,197 and for instances where there is no prime p, where $S(n) \le p \le S(n+1)$. They are as follows:

$n=224=2\cdot2\cdot2\cdot2\cdot2\cdot7$ $S(n)=8$ $n=225=3\cdot3\cdot5\cdot5$ $S(n)=10$

$n=2057=11\cdot11\cdot17$ $S(n)=22$ $n=2058=2\cdot3\cdot7\cdot7\cdot7$ $S(n)=21$

$n=265225=5\cdot 5\cdot 103\cdot 103$ $S(n)=206$ $n=265226=2\cdot 13\cdot 101\cdot 101$ $S(n)=202$

$n=843637=37\cdot 151\cdot 151$ $S(n)=302$

$n=843638=2\cdot 19\cdot 149\cdot 149$ $S(n)=298$

As can be seen, the first two values contradict the assertion made by I.M.Radu in his letter. Notice that the last two cases involve pairs of twin primes. This may provide a clue in the search for additional solutions.

7. ADDITIONAL VALUES WHERE THE SMARANDACHE FUNCTION SATISFIES THE FIBONACCI RELATIONSHIP S(n)+S(n+1)=S(n+2)

In [4] T.Yau poses the following problem:

For what triplets n, n+1 and n+2 does the Smarandache function satisfy the Fibonacci relationship

$$S(n)+S(n+1) = S(n+2) \ ?$$

Two solutions

$$S(9)+S(10) = S(11) \quad 6+5 = 11$$

$$S(119)+S(120) = S(121) \quad 17+5 = 22$$

were found, but no general solution was given.

To further investigate this problem, a computer program was written that tested all values for n up to 1,000,000. Additional solutions were found and all known solutions with their prime factorizations appear in the table below.

$S(9)+S(10) = S(11)$ $9 = 3\cdot 3$ $10 = 2\cdot 5$ $11 = 11$

$S(119) + S(120) = S(121)$ $119 = 7\cdot 17$ $120 = 2\cdot 2\cdot 2\cdot 3\cdot 5$ $121 = 11\cdot 11$

S(4900) + S(4901) = S(4902); S(26243) + S(26244) = S(26245) S(32110) + S(32111) = S(32112); S(64008) + S(64009) = S(64010); S(368138) + S(368139) = S(368139); S(415662) + S(415663) = S(415664);

I am unable to discern a pattern in these numbers that would lead to a proof that there is an infinite family of solutions. Perhaps another reader will be able to do so.

8. WILL SOME PROBLEMS INVOLING THE SMARANDACHE FUNCTION ALWAYS REMAIN UNSOLVED?

The most unsolved problems of the same subject are related to the **Smarandache function** in the Analytic Number Theory:

$S:Z$---N , S(n) is defined as the smallest integer such

that S(n)! is divisible by n.

The number of these unsolved problems concerning the function is equal to... an infinity!! Therefore, they will never be all solved!

One must be very careful in using such arguments when dealing with infinity. As is the case with number theoretic functions, a result in one area can have many aplications to other problems. The most celebrated recent instance is the "prof" of "Fermat's Last Theorem". In this case a result in elliptical functions has the proof as a consequence.

Since S(n) is still largely unexplored, it is quite possible that the resolution of one problem leads to the resolution of many, perhaps infinitely many, others. If that is indeed the case, then all problems may eventually be resolved.

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