Some Properties of The Happy Numbers and the Smarandache H-Sequence

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Abstract:

The happy numbers are those where the iterated sums of the squares of the digits terminates at 1. A Smarandache Concatenate Sequence is a set of numbers formed by the repeated concatenation of the elements of another set of numbers. In this paper, we examine some of the properties of the happy numbers as well as a concatenation sequence constructed from the happy numbers.

Introduction:

Definition: Given any positive integer n, the repeated iteration of the sum of the squares of the decimal digits either terminates at 1 or enters the cycle

4 -> 16 -> 37 -> 58 -> 89 -> 145 -> 42 -> 20 -> 4.

If the iteration terminates at 1, the number is said to be Happy[1].

For example, 13 is Happy, as

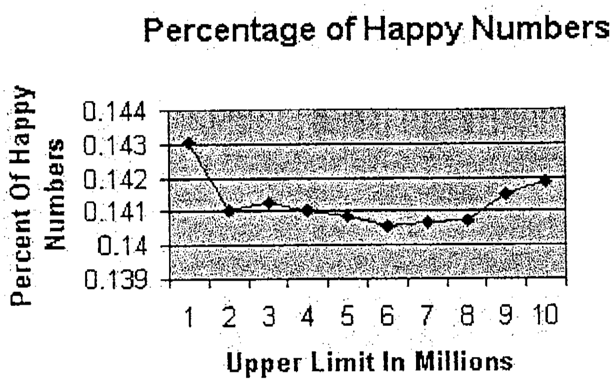
$1 + 9 = 10 = 1 + 0 = 1,$

the Happy numbers less than or equal to 100 are { 1, 7, 10, 13, 19, 23, 28, 31, 32, 44, 49, 68, 70, 79, 82, 86, 91, 94, 97, 100 }. If a number is Happy, then the number formed by appending an arbitrary number of zeros to the right is also Happy. Therefore, the set of Happy numbers is infinite.

Happy numbers turn out to be rather common, and Guy[1] notes that about 1/7 of the positive integers appear to be Happy.

To examine this in more detail, a computer program was created to determine and count the number of Happy numbers up through an upper limit. The counts and percentages for upper limits of one million through ten million were computed and are summarized in table 1.

Table 1



From this figure, it is clear that the percentage of Happy numbers is near 1/7 = 0.142, but shows a small amount of variation.

In his paper, Gupta[2] describes the Smarandache H-Sequence[2], constructed by repeatedly appending Happy numbers on the right side. For example, the first five elements of the sequence are

SH(1) = 1
SH(2) = 17
SH(3) = 1710
SH(4) = 171013
SH(5) = 17101319.

Gupta also defines the Reversed Smarandache H-Sequence, which is constructed by appending the happy numbers to the left side. For example, the first five elements of the sequence are

RSH(1) = 1
RSH(2) = 71
RSH(3) = 1071
RSH(4) = 131071
RSH(5) = 19131071.

Primes in the SH and RSH sequences.

Gupta conducts a search for primes in both the SH and RSH sequences. Three primes were found in the first 1000 terms of the SH sequence and they are SH(2) , SH(5) and SH(43). Eight primes were found in the first 1000 terms of the RSH sequence and they are RSH(2), RSH(4), RSH(5), RSH(6), RSH(10), RSH(31), RSH(255) and RSH(368). This is hardly surprising, as happy numbers can end with any of the decimal digits, six of which { 0, 2, 4, 5, 6, 8 } immediately eliminate the SH sequence element as a possible prime. However, with the trailing digit always being 1 for elements in the RSH sequence, there is no immediate elimination of the number as a possible prime. Assuming that all digits are equally likely to be the trailing digit of a happy number, then with six out of ten immediately eliminating the possibility of it being prime, the ratio of three to eight seems quite reasonable.

The trailing digits of the set of Happy numbers.

Which brings us to a related question.

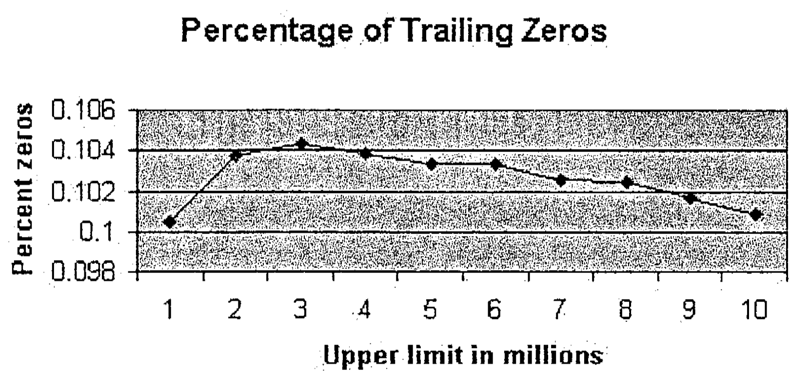
Are the trailing digits of the set of Happy numbers equally dispersed among the ten decimal digits?

At first glance, the answer to this question would appear to be false. Since zeros can be appended to any

Happy number to generate another Happy number, it would appear that the percentage of trailing zeros in Happy numbers would be greater than the average of 0.1.

A program was written in the language Java to test this question. The long data type in Java occupies eight bytes of memory and can store positive integers up to 9223372036854775807. Therefore, it is used in the computation of the Happy numbers. As the Happy numbers are generated, the trailing digit is extracted and the count of the number of times each digit appears is stored. These numbers are then displayed when the program terminates. The program was run several times, computing all Happy numbers less than n, where n was incremented in steps of one million. For each run, the percentage of the Happy numbers less than the upper bound that have a trailing zero was computed. The results for runs with upper limits from 1 million through 10 million are summarized in figure 2.

Figure 2



Note that the percentage of Happy numbers that end in zero is greater than 0.10, but the graph exhibits a decreasing rate as the upper limit increases.

This leads to the unsolved question.

Is the percentage of Happy numbers that end with a zero greater than 0.10?

The evidence here suggests that it is in fact near 0.10.

Similar questions can be asked concerning the percentages of Happy numbers that terminate with each of the remaining nine digits. Figure 3 is a chart of the percentage of Happy numbers that end in a one as the upper limit steps from one through ten million.

Figure 3

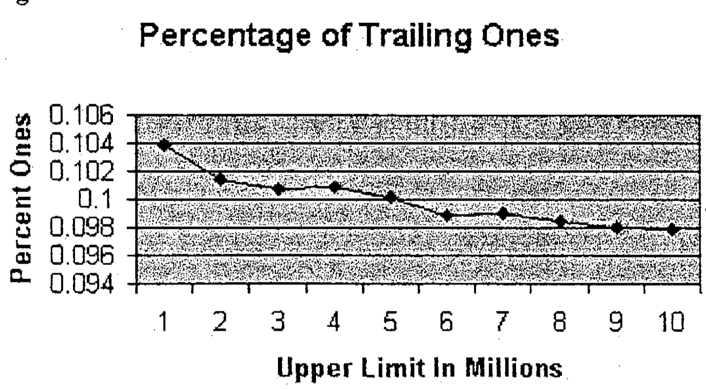


Figure 4 is a chart of the percentage of Happy numbers that end in a two as the upper limit step from one million through ten million.

Figure 4

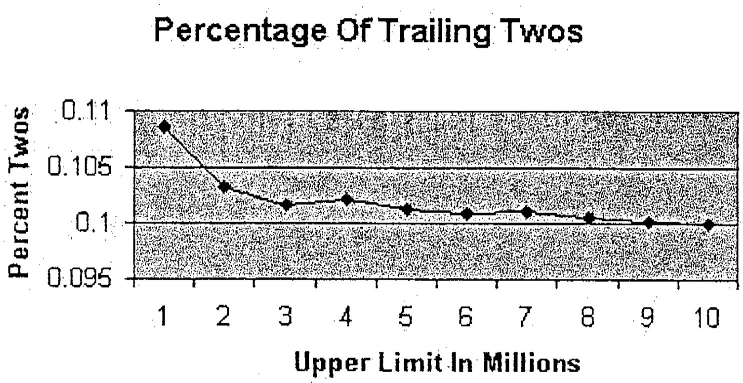


Figure 5 is a chart of the percentage of Happy numbers that in a three as the upper limits step from one million through ten million.

Figure 5

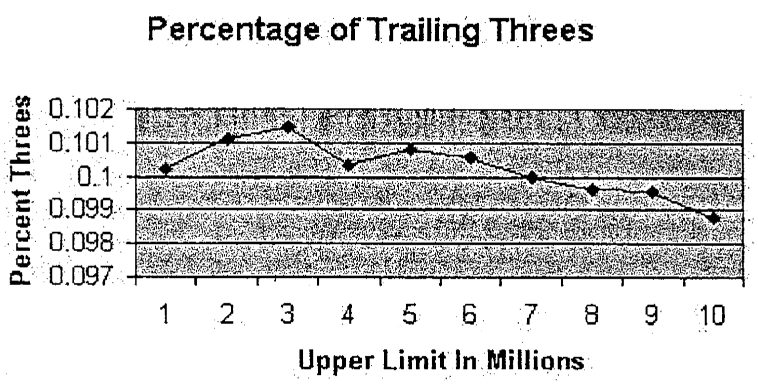


Figure 6 is a chart of the percentage of Happy numbers that end in a four as the upper limits step from one million through ten million.

Figure 6

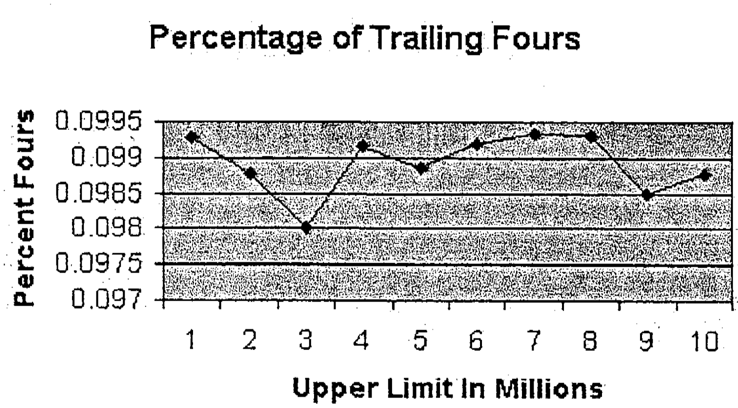


Figure 7 is a chart of the percentage of Happy numbers that end in a five as the upper limits step from one million through ten million.

Figure 7

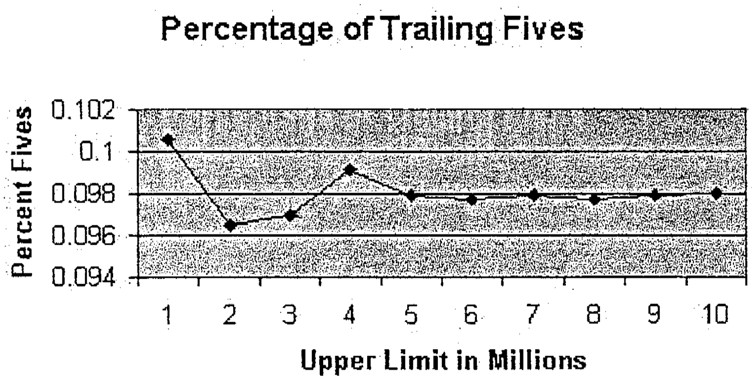


Figure 8 is a chart of the percentage of Happy numbers that end in a six as the upper limits step from one million through ten million.

Figure 8

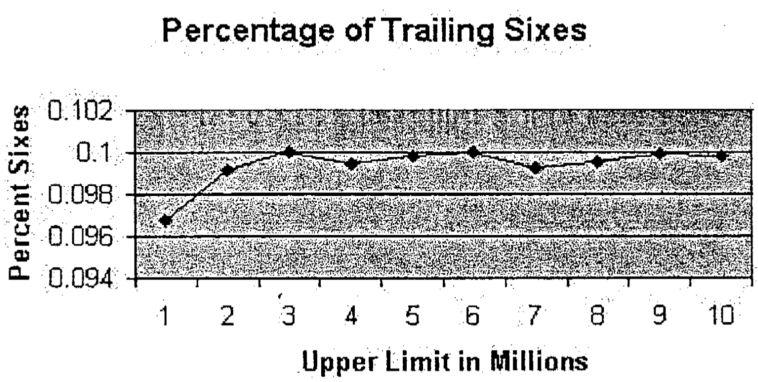


Figure 9 is a chart of the percentage of Happy numbers that end in a seven as the upper limits step from one million through ten million.

Figure 9

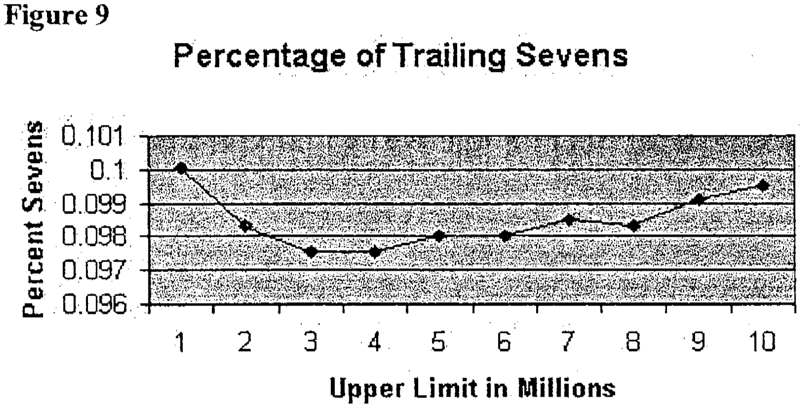


Figure 10 is a chart of the percentage of Happy numbers that end in an eight as the upper limits step from one million through ten million.

Figure 10

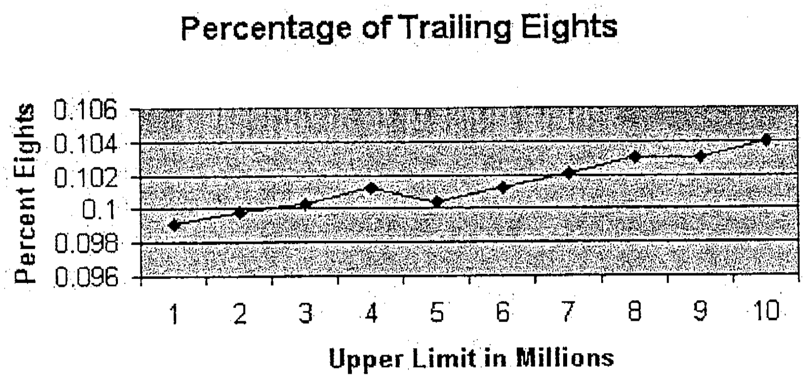
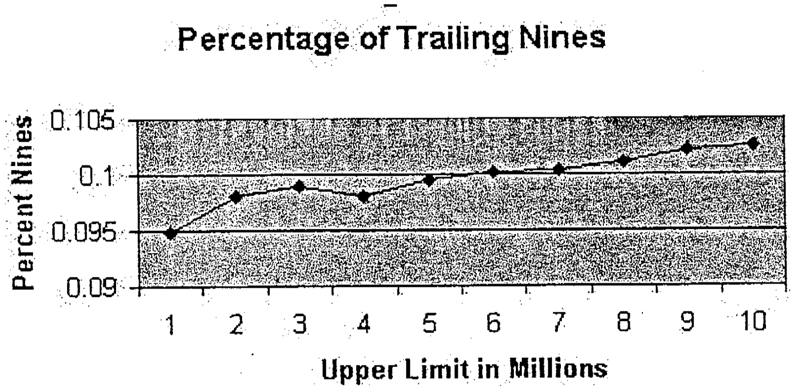


Figure 11 is a chart of the percentage of Happy numbers that end in a nine as the upper limits step from one million through ten million.

Figure 11



From these figures, it is clear that the percentages of the trailing digits of Happy numbers are generally evenly distributed for the ranges examined.

The Smarandache H-Sequence One-Seventh Conjecture

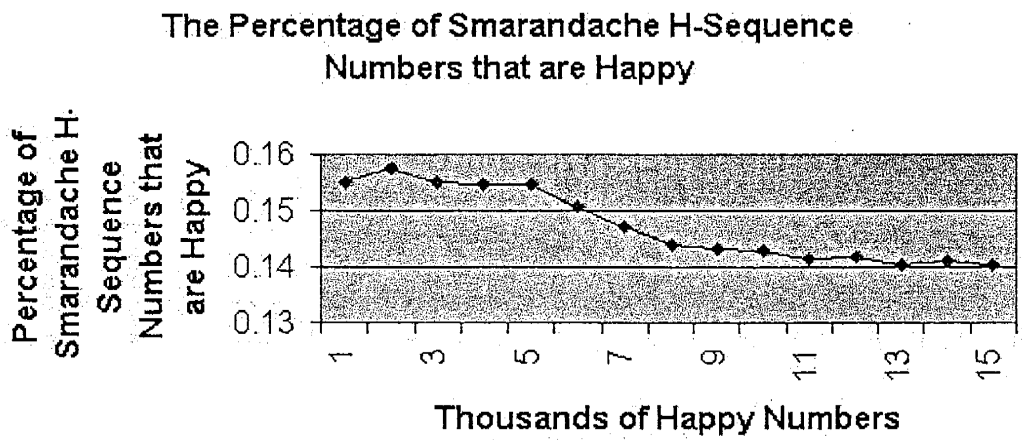
Gupta also makes the following conjecture in his paper about numbers in the Smarandache H-sequence.

Conjecture:

About one-seventh of the numbers in the Smarandache H-sequence belong to the initial H-sequence.

A computer program that uses the BigInteger class in Java was written to test this conjecture. The BigInteger class allows for the manipulation of very large integers whose only limit is the amount of machine memory. The program was run for all Happy numbers up through 15,000 and these numbers were used to construct the corresponding SH numbers. Percentages of the elements in the SH sequence that are also in the H-sequence were computed for each 1000 SH numbers and the results are summarized in figure 12 .

Figure 12



As you can see, the percentage of Smarandache H-sequence numbers that are also Happy appears to asymptotically approach 0.14. This is slightly less that the one-seventh value stated by Gupta.

Contrasting figure 12 with figure 1, it is clear that the percentage of Smarandache H-sequence numbers that are also Happy has less variation and appears to be smaller that the percentage of numbers that are Happy.

Question:

Is the percentage of Smarandache H-sequence numbers that are Happy less than the percentage of integers that are Happy?

Consecutive SH Numbers

Gupta also mentions consecutive SH numbers that are also Happy and finds the smallest such pair: SH(30) and SH(31). He moves on to find examples where three, four and five consecutive SH numbers are also Happy. He closes that section with the question:

Can you find examples of six and seven consecutive SH numbers?

The program previously mentioned that computed the percentages of H-sequence numbers that are also happy also searched for examples of six or seven consecutive Happy numbers and found no such sequences for the values of SH(10000) through SH(15000).

References:

[1] Guy, R. K., **Unsolved Problems in Number Theory**, E34, Springer-Verlag, 2nd Ed., 1994.
[2] Gupta, S. S. "Smarandache Sequence of Happy Numbers", online article at http://www.gallup.unm.edu/~smarandache/Gupta.htm.