SuperCommuting and a second distributive law:
Subtraction and division may not commute, but they SuperCommute

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Abstract

This paper deals with teaching methods.

Elementary textbooks tell that addition and multiplication commute but subtraction and division do not. Actually they do if a simple restriction is observed. The technique is not new; but the method presented here for teaching it is believed to be new and simple enough for presentation immediately following the signed-numbers concept. The technique is dubbed *SuperCommuting* or the *shuffling property.*

SuperCommuting leads directly to a new formal algebraic distributive law, one that applies to expressions of the form 1/(a*b/c/d*e...). Also, by comparison with the first distributive law, the *duality* concept can be painlessly and unobtrusively introduced by the dedicated instructor of beginning algebra.

Introduction

The beginning algebra student is today told, almost incidentally, that a long string of numbers consisting of a mixture of additions and subtractions can be evaluated by adding all positive numbers then adding all negative numbers then subtracting the two results. This paper gives a formal treatment of that property and extends it to a similar procedure with strings of multiplications and divisions. Underlying this entire discussion is the *Order of Operations* rule in which operations of a given kind are to be performed from left to right.

Asterisk is used as the multiplication sign to prepare the student for future computer math literacy. Thus A*B is always used, never AB, A×B, or A•B. Similarly, division is always C/D never C÷D.

Subtraction and SuperCommuting

Today, the student may be told that A–B can be written –B+A; but how does one initially present this idea with total clarity? One method follows.

First write A–B as 0 +A –B where "A" has a plus sign directly attached to it, and "B" has a minus sign directly attached to it. The student would then be told it's alright to shuffle these numbers if the sign of each number is carried with it while keeping 0 at the front of the string. Thus, 0 +A –B becomes 0 –B +A or simply –B+A, the initial 0 having outlived its usefulness; and the problem in subtraction is said to be the addition of signed numbers.

That simple and obvious development suggests another one that is just as simple but perhaps less obvious.

Division and SuperCommuting

A similar process can be illustrated with division. Begin by writing C/D as 1 *C /D, where "C" has a multiplication sign directly attached to it and "D" has a division sign directly attached to it just as if *C and /D were some kind of "signed numbers"! The student would then be told it's alright to shuffle these if the sign attached to each number is carried with it, and if 1 is kept at the front of the string. Thus 1 *C /D becomes 1 /D *C, and a problem which started out as division is now seen to be multiplication by the divisor's reciprocal. In this case the initial numeral, 1, must be retained because *C and /D are not recognized as any kind of signed numbers. (But might they be?)

A second distributive law

The student would now be informed that long strings of additions and subtractions can be similarly shuffled into an arbitrary order, as can long strings of multiplications and divisions; and that the traditional commutative laws of addition and multiplication are simply narrow applications of this *shuffling property*.

The parallel, or *dual* nature of the above two developments is as obvious as the fact that lightning begets thunder. Thus is suggested a new distributive law, one that works for complex fractions like 1/(a*b/c) by considering their dual, in this case 0–(a+b–c). The suggestion is that 1/(a*b/c) becomes 1/a/b*c. A traditional proof is left to the reader.

Teaching duality early

After this, the student of elementary algebra should be perfectly comfortable with the duality concept if presented something as follows.

One's left hand is like one's right hand except that they are *mutual mirror images*, each is a mirror image or a *reflection* of the other. This is one kind of symmetry. Another kind of symmetry exists between a number and its reciprocal. A number and its reciprocal are *mutual reciprocals*. Still another kind of symmetry is the way in which the new distributive law relates to the old one.

Compare these two forms; the first illustrates the new distributive law, the second form illustrates the old familiar one:

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1 /(a *b /c) = 1 /a /b *c
0 -(a +b -c) = 0 -a -b +c
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The student can then be told that either form can be changed into the other by exchanging * and + signs, / and – signs, and constants 1 and 0; and that this kind of symmetry is called *duality*, and the two forms are said to be *mutual duals*. It might also be suggested that this kind of duality is a precise form of the usually imprecise method called *analogy*.

Finally it should be pointed out to the student that in each of the three kinds of symmetry discussed above, a double application is the same as no application. That is, if a right hand is reflected twice, the result has the shape of a right hand; if the reciprocal is taken twice, the original number results; and if the dual is taken twice, the original form results.

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