THE ALMOST PRESUMABLE MAXIMALITY OF SOME TOPOLOGICAL LEMMA

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Abstract

Some splitting lemma of topological nature provides fundamental information when dealing with dynamics (see [1], pg.79). Because the set involved, namely $X \setminus \mathcal{P}_s$, is neither open nor closed, a natural question arise: can this set be modified in order to obtain aditional data ? Unfortunately, the answer is negative.

For a metric space X which is locally connected and locally compact and for some continuous mapping $f : X \to X$, the set ω-set of each element x of X is given by the formula

$$\omega(x) = \left\{ y \in X | y = \lim_{n\to+\infty} f^{k_n}(x), \lim_{n\to+\infty} k_n = +\infty \right\}.$$

We also denote by $\omega_j(x)$, $1 \le j \le r$, the set

$$\omega_j(x) = \left\{ y \in X | y = \lim_{n\to+\infty} f^{m_n \cdot r+j}(x), \lim_{n\to+\infty} m_n = +\infty \right\}.$$

Now, $\omega(x)$ can be splitted according to the following lemma.

Lemma 1 *a)* $\omega(x) = \bigcup_{j=1}^{r} \omega_j(x)$;

b) $f(\omega_j(x)) \subset \omega_{(j+1)\, mod\, r}$.

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Its proof relies upon the properties of $\omega(x)$.

Lemma 2 *For some nonvoid subset S of X we consider C a component of $X \setminus S$, i.e. a maximal connected set (see [2], pg. 54). Then:*
a) $\overline{C}^X \subset C \cup \left(S \cap \partial^X S\right)$;
b) $\partial^X C \subset \left(C \cap \partial^X C\right) \cup \left(S \cap \partial^X S\right)$,
where $\overline{C}^X$ signifies the closure of C under the topology of X while $\partial^X C$ is the boundary of C under the same topology.

Remark 1 *For instance, if S is closed, then $\partial^X C \subset \partial^X S$ as the components of a locally connected space are open.*

Proof. a) First, let's show that $\overline{C}^X \subset C \cup S$. For $x \in X \setminus (C \cup S) = (X \setminus S) \setminus C$, as C is closed in $X \setminus S$, there will be some open $G \subset X$ such that

$$x \in G \cap (X \setminus S) \subset X \setminus (C \cup S).$$

Obviously,

$$[G \cap (X \setminus S)] \cap C = G \cap C = \emptyset$$

and so

$$x \notin \overline{C}^X.$$

Further on, let's assume that $x \in \overline{C}^X \cap S$. If $x \in X \setminus \partial^X S$, then $x \notin \overline{X \setminus S}^X$. There will be some open $W \subset X$ such that

$$x \in W \,;\, W \cap \overline{X \setminus S}^X = \emptyset.$$

In particular, $W \cap C = \emptyset$ and so $x \notin \overline{C}^X$.
b) According to a), we have:

$$\begin{aligned} \overline{C}^X \cap \overline{X \setminus C}^X &= \partial^X C \subset \left(C \cap \overline{X \setminus C}^X\right) \cup \left[\left(S \cap \partial^X S\right) \cap \overline{X \setminus C}^X\right] \\ &= \left(C \cap \overline{X \setminus C}^X\right) \cup \left(S \cap \partial^X S\right) \end{aligned}$$

because of $S \cap \partial^X S \subset S \subset X \setminus C$.
Obviously,

$$C \cap \overline{X \setminus C}^X = \left(C \cap \overline{X \setminus C}^X\right) \cap \overline{C}^X = C \cap \partial^X C.$$

■

Remark 2 *It worths noticing that the sets* $(C \cap \partial^X C)$ *and* $(S \cap \partial^X S)$ *are disjoint; in other words,* $\partial^X C$ *is piecewise-made. Lemma 2 works equally well in any topological space.*

Lemma 3 *(Melbourne, Dellnitz, Golubitsky)*
For some nonvoid subset S *of* X, *we denote by* $\mathcal{P}_s$ *the union*

$$\mathcal{P}s(f) = \bigcup_{n=0}^{\infty} (f^n)^{-1}(S)$$

Let x *be some element of* S. *Then either* $\omega(x) \subset \overline{\mathcal{P}}_s^X$ *or the following are valid:*
a) $\omega(x) \setminus \mathcal{P}_s$ *is covered by finitely many (connected) components* $C_0, \ldots, C_{r-1}$ *of* $X \setminus \mathcal{P}_s$;
b) These components can be ordered so that $f(C_i) \subset C_{(i+1) \bmod r}$;
c) $\omega(x) \subset \overline{C}_0^X \cup \ldots \cup \overline{C}_{r-1}^X$.

Remark 3 *Notice the splitting in relation with lemma 1. As we mentioned in the Abstract, it is quite natural to ask if* $X \setminus \mathcal{P}_s$ *can be replaced by the easier-to-work-with* $X \setminus \overline{\mathcal{P}}_s$. *The following lemma shows that this would imply no more the presence of finitely many components.*

Lemma 4 *Let* S *be some nonvoid subset of* X *which is not dense in* X, *i.e.* $\overline{S}^X \neq X$. *We consider* C *a component of* $X \setminus \overline{S}^X$ *and* D *a component of* $X \setminus S$ *such that* $C \subset D$. *Then any element* x *of* $D \setminus C$ *belongs either to* $\partial^X S$ *or to any other component of* $X \setminus \overline{S}^X$.

Proof. If $x \notin X \setminus \overline{S}$ then $x \in (X \setminus S) \cap \overline{S}^X \subset \partial^X S$. ■
An example would be appropriate: in $\mathbf{R}^2$, we denote by $\mathcal{D}(0, r)$ the r-disk centered in 0. Now, for $X = \overline{\mathcal{D}(0,2)}^{\mathbf{R}^2}$, $S = \mathcal{D}(0,1) \cup (1,2] \cup [-2,-1)$, we have

$$\overline{S}^{\mathbf{R}^2} = \overline{\mathcal{D}(0,1)}^{\mathbf{R}^2} \cup [1,2] \cup [-2,-1]$$

$$D = X \setminus S,\ C \in \left\{ \left(X \setminus \overline{S}^{\mathbf{R}^2}\right) \cap (y > 0), \left(X \setminus \overline{S}^{\mathbf{R}^2}\right) \cap (y < 0) \right\}.$$

Further exemples can be architectured easily even to obtain infinitely many components of $X \setminus \overline{S}^X$.
In other words, finitely many components of $X \setminus S$ may include infinitely many components of $X \setminus \overline{S}^X$.

References

[1] I. Melbourne, M. Dellnitz, M. Golubitsky, *The structure of symmetric attractors*, Arch. Rational Mech. Anal., 123, pp. 75-98, 1993

[2] J. L. Kelley, *General topology*, D. Von Nostrand Comp., 1955