THE AVERAGE OF THE ERDOS FUNCTION

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The aim of this article is to establish the complexity order of the Erdos function average. This will be studied based on some recent results about the Smarandache function.

1. INTRODUCTION

The main results used in this paper are reviewed in the following. These deal with the main properties of the Smarandache and Erdos functions.

The Smarandache function [Smarandache, 1980] is $S: N^* \to N$ defined by

$$S(n) = \min\{k \in N | k! = \mathcal{M}n\} \ (\forall n \in N^*). \tag{1}$$

The function $P: N^* \to N$ defined by

$$P(n) = \min\{ p \in N \mid n = \mathcal{M}p \wedge p \text{ is prim}\}(\forall n \in N^* \setminus \{1\}),\ P(1) = 0 \tag{2}$$

is named classically the Erdos function. Both functions satisfy the same main properties:

$$(\forall a, b \in N^*)\ (a,b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\},\ P(a \cdot b) = \max\{P(a), P(b)\}. \tag{3}$$

$$(\forall a \in N^*)\ P(a) \le S(a) \le a \text{ and the equalities occur } iif \text{ a is prim.} \tag{4}$$

Erdos [1991] found that these two functions have the same values for all most of the natural numbers $\lim_{n\to\infty} \frac{\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right|}{n} = 0$. This important result was extended by Ford [1999] to

$$\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right| = n \cdot e^{-\left(\sqrt{2}+a_n\right)\sqrt{\ln n \cdot \ln \ln n}}, \text{ where } \lim_{n\to\infty} a_n = 0. \tag{5}$$

Obviously, both functions are neither increasing nor decreasing functions. In this situation, many researchers have tried to study properties concerning their average. Many results that have been published so far deal with complexity orders of the average.

Let us denote $E(f(n)) = \frac{1}{n} \cdot \sum_{i=1}^{n} f(n)$ the average of function $f: N^* \to R$. The average *E(S(n))* was intensively studied by Tabirca [1997, 1998] and Luca [1999]. Tabirca [1998] proved that $(\forall\ n > c_p) E(S(n)) \le a_p \cdot n + b_p$, where $\lim_{p\to\infty} a_p = \lim_{p\to\infty} b_p = 0$. This means that the order *O(n)* is not properly chosen for $E(S(n))$. Tabirca [1998] conjectured that the average $E(S(n))$ satisfies the equation $E(S(n)) \leq \frac{n}{\ln n}$. Finally and the most important, Luca [1999] proposed the equation

$$\frac{1}{2} \cdot \left[\pi(n) - \pi(\sqrt{n})\right] < E(S(n)) < \pi(n) + \frac{5}{2} \cdot \ln\ln n + \frac{1}{n} + \frac{31}{5} \tag{7}$$

where $\pi(x)$ denotes the number of prim numbers less than or equal to x. Thus, the complexity order for the average $E(S(n))$ is indeed $O\left(\frac{n}{\log n}\right)$.

2. THE COMPLEXITY ORDER FOR THE ERDOS FUNCTION

Some of the above results are used to find the complexity order of $E(P(n))$. Based on the well-known formula $\lim_{n\to\infty} \frac{\pi(n)}{\frac{n}{\ln n}} = 1$, Equation (7) gives

$$\frac{1}{2} \leq \liminf_{n\to\infty} \frac{E(S(n))}{\frac{n}{\ln n}} \leq \limsup_{n\to\infty} \frac{E(S(n))}{\frac{n}{\ln n}} \leq 1. \tag{8}$$

Theorem 1.

$$\liminf_{n\to\infty} \frac{E(S(n))}{\frac{n}{\ln n}} = \liminf_{n\to\infty} \frac{E(P(n))}{\frac{n}{\ln n}}, \ \limsup_{n\to\infty} \frac{E(S(n))}{\frac{n}{\ln n}} = \limsup_{n\to\infty} \frac{E(P(n))}{\frac{n}{\ln n}} \tag{9}$$

Proof Let us denote $A = \{i = \overline{1,n} \mid S(i) > P(i)\}$ the set of the numbers that do not satisfy the equation $S(i)=P(i)$. The cardinal of this set is $|A| = n \cdot e^{-\left(\sqrt{2}+a_n\right)\sqrt{\ln n \cdot \ln\ln n}}$, where $\lim_{n\to\infty} a_n = 0$.

The proof is started from the following equation

$$\left|E(S(n)) - E(P(n))\right| = \frac{1}{n} \cdot \left|\sum_{i=1}^{n} S(i) - \sum_{i=1}^{n} P(i)\right| = \frac{1}{n} \cdot \left[\sum_{i\in A} \left(S(i) - P(i)\right)\right] \leq \sum_{i\in A} \frac{S(i)}{n}. \tag{10}$$

Because we have $\left(\forall\, i = \overline{1,n}\right) S(i) \leq n$, Equation (10) gives

$$\left|E(S(n)) - E(P(n))\right| \leq |A| = n \cdot e^{-\left(\sqrt{2}+a_n\right)\sqrt{\ln n \cdot \ln\ln n}} \text{ and}$$

$$\left|\frac{E(S(n))}{\frac{n}{\ln n}} - \frac{E(P(n))}{\frac{n}{\ln n}}\right| \leq \ln n \cdot e^{-\left(\sqrt{2}+a_n\right)\sqrt{\ln n \cdot \ln\ln n}}. \tag{11}$$

Because $\lim_{n\to\infty} a_n = 0$, the equation $\lim_{n\to\infty} \ln n \cdot e^{-(\sqrt{2}+a_n)\sqrt{\ln n \cdot \ln\ln n}} = 0$ is found true, thus $\liminf_{n\to\infty} \frac{E(S(n))}{\frac{n}{\ln n}} = \liminf_{n\to\infty} \frac{E(P(n))}{\frac{n}{\ln n}}$, $\limsup_{n\to\infty} \frac{E(S(n))}{\frac{n}{\ln n}} = \limsup_{n\to\infty} \frac{E(P(n))}{\frac{n}{\ln n}}$ holds. ◆

Theorem 2 is obtained as a direct consequence of Theorem1.

Theorem 2

$$E(P(n)) = O(\frac{n}{\ln n})$$

Proof The equation $\frac{1}{2} \le \liminf_{n\to\infty} \frac{E(P(n))}{\frac{n}{\ln n}} \le \limsup_{n\to\infty} \frac{E(P(n))}{\frac{n}{\ln n}} \le 1$ is found true applying Theorem 1. From that, there is a natural number N_1 such that that

$$(\forall\, n \ge N_1)\left(\frac{1}{2} - \varepsilon\right) \cdot \frac{n}{\ln n} \le E(P(n)) \le (1+\varepsilon) \cdot \frac{n}{\ln n}. \tag{12}$$

Therefore, the equation $E(P(n)) = O(\frac{n}{\ln n})$ holds. ◆

The right question that comes from (12) is the following "**Is the equation** $E(P(n)) \le \frac{n}{\ln n}$ **true?**". This has been investigated for all the natural numbers less than 1000000 and it has been found true. Equation (7) can be adapt to the average *E(P(n))* but obviously the inequality that is found is not an answer to the question. Therefore, we may conjecture the following: **The equation** $E(P(n)) \le \frac{n}{\ln n}$ **holds for all** $n>1$**.**

References

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