THE AVERAGE VALUE OF THE SMARANDACHE FUNCTION

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Given a positive integer n, let $P(n)$ denote the largest prime factor of n and $S(n)$ denote the smallest integer m such that n divides $m!$

The function $S(n)$ is known as the Smarandache function and has been intensively studied [1]. Its behavior is quite erratic [2] and thus all we can reasonably hope for is a statistical approximation of its growth, e.g., an average. It appears that the sample mean $E(S)$ satisfies [3]

$$E(S(N)) = \frac{1}{N} \cdot \sum_{n=1}^{N} S(n) = O\left(\frac{N}{\ln(N)}\right)$$

as N approaches infinity, but I don't know of a rigorous proof. A natural question is if some other sense of average might be more amenable to analysis.

Erdös [4,5] pointed out that $P(n) = S(n)$ for almost all n, meaning

$$\lim_{N\to\infty} \frac{\left|\{n \le N: P(n) < S(n)\}\right|}{N} = 0 \qquad \text{that is,} \qquad \left|\{n \le N: P(n) < S(n)\}\right| = o(N)$$

as N approaches infinity. Kastanas [5] proved this to be true, hence the following argument is valid. On one hand,

$$\lambda = \lim_{n\to\infty} E\left(\frac{\ln(P(n))}{\ln(n)}\right) \le \lim_{n\to\infty} E\left(\frac{\ln(S(n))}{\ln(n)}\right) = \lim_{N\to\infty} \frac{1}{N} \cdot \sum_{n=1}^{N} \frac{\ln(S(n))}{\ln(n)}$$

The above summation, on the other hand, breaks into two parts:

$$\lim_{N\to\infty} \frac{1}{N} \cdot \left(\sum_{P(n)=S(n)} \frac{\ln(P(n))}{\ln(n)} + \sum_{P(n)<S(n)} \frac{\ln(S(n))}{\ln(n)} \right)$$

The second part vanishes:

$$\lim_{N\to\infty}\frac{1}{N}\cdot\left(\sum_{P(n)<S(n)}\frac{\ln(S(n))}{\ln(n)}\right)\le\lim_{N\to\infty}\frac{1}{N}\cdot\left(\sum_{P(n)<S(n)}1\right)=\lim_{N\to\infty}\frac{o(N)}{N}=0$$

while the first part is bounded from above:

$$\lim_{N\to\infty}\frac{1}{N}\cdot\left(\sum_{P(n)=S(n)}\frac{\ln(P(n))}{\ln(n)}\right)\le\lim_{N\to\infty}\frac{1}{N}\cdot\sum_{n=1}^{N}\frac{\ln(P(n))}{\ln(n)}=\lim_{n\to\infty}E\left(\frac{\ln(P(n))}{\ln(n)}\right)=\lambda$$

We deduce that

$$\lim_{n\to\infty}E\left(\frac{\ln(S(n))}{\ln(n)}\right)=\lambda=0.6243299885\ldots$$

where λ is the famous Golomb-Dickman constant [6-9]. Therefore $\lambda\cdot n$ is the asymptotic average number of digits in the output of S at an n-digit input, that is, 62.43% of the original number of digits. As far as I know, this result about the Smarandache function has not been published before.

A closely related unsolved problem concerns estimating the variance of S.

References

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