The Convergence of Smarandache Harmonic Series

Sabin Tabirca* ***Tatiana Tabirca*****

* Buckinghamshire University College, Computer Science Division

** "Transilvania" University of Brasov, Computer Science Department

The aim of this article is to study the series $\sum_{n\geq 2}\frac{1}{S^m(n)}$ called Smarandache harmonic series. The article shows that the series $\sum_{n\geq 2}\frac{1}{S^2(n)}$ is divergent and studies from the numerical point of view the sequence $a_n = \sum_{i=2}^{n}\frac{1}{S^2(i)} - \ln(n)$.

1. Introduction

The studies concerning the series with Smarandache numbers have been done recently and represents an important research direction on Smarandache's notions. The question of convergence or divergence were resolved for several series and the sums of some series were proved to be irrational.

The most important study in this area has been done by Cojocaru [1997]. He proved the following results:

1. If $(x_n)_{n\geq 0}$ is an increasing sequence then the series $\sum_{n\geq 0}\frac{x_{n+1}-x_n}{S(x_n)}$ is divergent.

   As a direct consequence, the following series $\sum_{n\geq 2}\frac{1}{S(n)}$, $\sum_{n\geq 1}\frac{1}{S(2\cdot n+1)}$ and $\sum_{n\geq 1}\frac{1}{S(4\cdot n+1)}$ are divergent.

2. The series $\sum_{n\geq 2}\frac{n^\alpha}{S(2)\cdot S(3)\cdot\ldots\cdot S(n)}$ is convergent and the sum $\sum_{n\geq 2}\frac{1}{S(2)\cdot S(3)\cdot\ldots\cdot S(n)}$ is in then interval $\left(\frac{71}{100},\frac{101}{100}\right)$.

3. The series $\sum_{n\geq 0}\frac{1}{S(n)!}$ is converges to a number in the interval $\left(\frac{717}{1000},\frac{1253}{1000}\right)$.

4. The series $\sum_{n\geq 0}\frac{S(n)}{n!}$ converges to an irrational number.

Jozsef [1997] extended Cojocaru's result and proved that the series $\sum_{n\geq 0}(-1)^n\cdot\frac{S(n)}{n!}$ also converges to an irrational number.

2. Divergence of the Series $\sum_{n\geq 2}\frac{1}{S^2(n)}$

In this section, the divergence of the series $\sum_{n\geq 2}\frac{1}{S^2(n)}$ will be proved based on an inequality which we shall establish in Lemma 1.

Lemma 1.

$$\lim_{n\to\infty} n\cdot\left(\frac{\pi^2}{8}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}\right)=\frac{1}{4} \tag{1}$$

Proof

The proof is based on the well-known formula

$$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{(2i+1)^2}=\frac{\pi^2}{8} \tag{2}$$

and on a double inequality for the quantity $n\cdot\left(\frac{\pi^2}{8}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}\right)$.

Let m be a natural number such that $m>n$. We then have

$$\sum_{i=n}^{m}\frac{1}{(2i+1)\cdot(2i-1)}=\frac{1}{2}\cdot\sum_{i=n}^{m}\left(\frac{1}{2i-1}-\frac{1}{2i+1}\right)=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2m+1}\right) \tag{3}$$

$$\sum_{i=n}^{m}\frac{1}{(2i+1)\cdot(2i+3)}=\frac{1}{2}\cdot\sum_{i=n}^{m}\left(\frac{1}{2i+1}-\frac{1}{2i+3}\right)=\frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2m+3}\right) \tag{4}$$

The difference $\sum_{i=0}^{m}\frac{1}{(2i+1)^2}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}=\sum_{i=n+1}^{m}\frac{1}{(2i+1)^2}$ is studied using (3-4) to obtain the inequalities (5-6).

$$\sum_{i=n+1}^{m}\frac{1}{(2i+1)^2}<\sum_{i=n+1}^{m}\frac{1}{(2i+1)(2i-1)}=\frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2m+1}\right) \tag{5}$$

$$\sum_{i=n+1}^{m}\frac{1}{(2i+1)^2}>\sum_{i=n+1}^{m}\frac{1}{(2i+1)(2i+3)}=\frac{1}{2}\left(\frac{1}{2n+3}-\frac{1}{2m+3}\right) \tag{6}$$

Therefore, the inequality

$$\frac{1}{2}\left(\frac{1}{2n+3}-\frac{1}{2m+3}\right) < \sum_{i=0}^{m}\frac{1}{(2i+1)^2}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2} < \frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2m+1}\right) \tag{7}$$

holds for all $m>n$. If $m\to\infty$ then the inequality (7) becomes

$$\frac{1}{2(2n+3)} < \frac{\pi^2}{8}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2} < \frac{1}{2(2n+1)} \quad \text{and} \tag{8}$$

$$\frac{n}{2(2n+3)} < n\left(\frac{\pi^2}{8}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}\right) < \frac{n}{2(2n+1)}. \tag{9}$$

The inequality (9) gives the limit $\lim_{n\to\infty} n\cdot\left(\frac{\pi^2}{8}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}\right)=\frac{1}{4}$.

♣

In Lemma 2 we will prove an inequality for Smarandache's function.

Lemma 2.

$$S(2^k\cdot n)\le n \ (\forall n>2k>1). \tag{10}$$

Proof

Because $n>2k$ the product $n! = 1\cdot 2\cdot\ldots\cdot n$ contains the factors 2, 4, ..., $2k$. Therefore, the divisibility $n! = 1\cdot 2\cdot\ldots\cdot n = 2^k\cdot n\cdot m \vdots 2^k\cdot n$ holds resulting in the inequality $S(2^k\cdot n)\le n$.

♣

In the following, we analyse the summation $a_n=\sum_{i=1}^{2^n}\frac{1}{S^2(2^n+i)}$, where $n>0$.

Let us define the sets

$$A_n=\left\{i=\overline{1,2^n}: S(2^n+i)\le\frac{2^n+i}{(2^n,i)}\right\} \text{ and } B_n=\left\{i=\overline{1,2^n}: S(2^n+i)>\frac{2^n+i}{(2^n,i)}\right\} \tag{11}$$

which is a partition of the set $\left\{i=\overline{1,2^n}\right\}$.

Lemma 3.

If $i=2^k\cdot j$ satisfies the following conditions:

- $k \le n - \log_2(n) - 1$ (12)
- j is a odd number so that $j < 2^{n-k}$ (13)

then $i = 2^k \cdot j \in A_n$.

Proof

If k satisfies $k \le n - \log_2(n) - 1$ then $n - k \ge \log_2(n) + 1$ and the inequality

$$2^{n-k} + 1 \ge 2^{\log_2(n)+1} + 1 = 2n + 1 > 2k \tag{14}$$

holds.

Applying Lemma 2 and (14), the following inequality

$$S(2^n + i) = S(2^k(2^{n-k} + j)) \le 2^{n-k} + j = \frac{2^n + 2^k \cdot j}{(2^n, 2^k \cdot j)}$$

is found to be true. Therefore, the relationship $i = 2^k \cdot j \in A_n$ holds.

♣

Let $C_n = \left\{2^k \cdot j = \overline{1,2^n} \mid k \le n - \log_2(n) - 1,\ j\ odd,\ j < 2^{n-k}\right\}$ be the set of numbers which satisfies the conditions of Lemma 3. Thus, the inclusion $C_n \subseteq A_n$ holds.

Theorem 1 shows an inequality satisfied by the sequence a_n.

Theorem 1.

$$(\forall \varepsilon > 0)(\exists N_\varepsilon > 0)(\forall n > N_\varepsilon) a_n = \sum_{i=1}^{2^n} \frac{1}{S^2(2^n + i)} > (\frac{1}{4} - \varepsilon)\frac{1}{n-1}. \tag{15}$$

Proof

Let $\varepsilon > 0$ be a positive number.

The summation $a_n = \sum_{i=1}^{2^n} \frac{1}{S^2(2^n + i)}$ is split into two parts as follows

$$a_n = \sum_{i=1}^{2^n} \frac{1}{S^2(2^n + i)} = \sum_{i \in A_n} \frac{1}{S^2(2^n + i)} + \sum_{i \in B_n} \frac{1}{S^2(2^n + i)} > \sum_{i \in A_n} \frac{1}{S^2(2^n + i)}. \tag{16}$$

Because $C_n \subseteq A_n$, the inequality $\sum_{i \in A_n} \frac{1}{S^2(2^n + i)} > \sum_{i \in C_n} \frac{1}{S^2(2^n + i)}$ holds.

Consequently,

$$a_n > \sum_{i \in C_n} \frac{1}{S^2(2^n + i)} \tag{17}$$

is true.

If $i = 2^h \cdot j \in C_n \subseteq A_n$ then $S(2^n + i) \le \dfrac{2^n + i}{(2^n, i)} = 2^{n-k} + j$ holds. This inequality is applied in (17) resulting in

$$a_n > \sum_{2^k \cdot j \in C_n} \frac{1}{\left(2^{n-k} + j\right)^2} = \sum_{k \le n - \log_2(n) - 1} \; \sum_{j\ odd, j < 2^{n-k}} \frac{1}{\left(2^{n-k} + j\right)^2} \tag{18}$$

The right side of (18) is equivalent to the following summations

$$k = 0 \Rightarrow \frac{1}{(2^n + 1)^2} + \frac{1}{(2^n + 3)^2} + \ldots + \frac{1}{(2^{n+1} - 1)^2} +$$

$$k = 1 \Rightarrow + \frac{1}{(2^{n-1} + 1)^2} + \frac{1}{(2^{n-1} + 3)^2} + \ldots + \frac{1}{(2^n - 1)^2} +$$

...

$$k = n - ([\log_2(n)] + 1) \Rightarrow \frac{1}{(2^{[\log_2(n)]+1} + 1)^2} + \frac{1}{(2^{[\log_2(n)]+1} + 3)^2} + \ldots + \frac{1}{(2^{[\log_2(n)]+2} - 1)^2} +$$

therefore, the sum is equal to $\displaystyle\sum_{i = 2^{[\log_2(n)]}}^{2^n - 1} \frac{1}{(2i + 1)^2}$.

The inequality (13) becomes

$$a_n > \sum_{i = 2^{[\log_2(n)]}}^{2^n - 1} \frac{1}{(2i + 1)^2} = \sum_{i=1}^{2^n - 1} \frac{1}{(2i + 1)^2} - \sum_{i=1}^{2^{[\log_2(n)]} - 1} \frac{1}{(2i + 1)^2}. \tag{19}$$

Based on Lemma1, a natural number N_ε can be found so that the inequalities (20-21) hold simultaneous true for all $n > N_\varepsilon$.

$$\frac{\pi^2}{8} - \left(\frac{1}{4} - \varepsilon\right) \cdot \frac{1}{2^n - 1} > \sum_{i=1}^{2^n - 1} \frac{1}{(2i + 1)^2} > \frac{\pi^2}{8} - \left(\frac{1}{4} + \varepsilon\right) \cdot \frac{1}{2^n - 1} \tag{20}$$

$$\frac{\pi^2}{8} - \left(\frac{1}{4} - \varepsilon\right) \cdot \frac{1}{2^{[\log_2(n)]} - 1} > \sum_{i=1}^{2^{[\log_2(n)]} - 1} \frac{1}{(2i + 1)^2} > \frac{\pi^2}{8} - \left(\frac{1}{4} + \varepsilon\right) \cdot \frac{1}{2^{[\log_2(n)]} - 1} \tag{21}$$

Using (20-21), the inequality (19) is transformed as follows

$$a_n > \sum_{i=1}^{2^n - 1} \frac{1}{(2i + 1)^2} - \sum_{i=1}^{2^{[\log_2(n)]} - 1} \frac{1}{(2i + 1)^2} > \frac{\pi^2}{8} - \left(\frac{1}{4} + \varepsilon\right) \frac{1}{2^n - 1} - \frac{\pi^2}{8} + \left(\frac{1}{4} - \varepsilon\right) \frac{1}{2^{[\log_2(n)]} - 1} \Rightarrow$$

$$a_n > \left(\frac{1}{4} - \varepsilon\right) \frac{1}{2^{[\log_2(n)]} - 1} - \left(\frac{1}{4} + \varepsilon\right) \frac{1}{2^n - 1} > \left(\frac{1}{4} - \varepsilon\right) \frac{1}{2^{[\log_2(n)]} - 1} \Rightarrow$$

$$a_n > \left(\frac{1}{4} - \varepsilon\right) \frac{1}{2^{\log_2(n)} - 1} = \left(\frac{1}{4} - \varepsilon\right) \frac{1}{n-1} \qquad (22).$$

The inequality (22) is true for all $n > N_\varepsilon$.

♣

The divergence of the series $\sum_{n\geq 2} \frac{1}{S^2(n)}$ is proved based on the inequality (22).

Theorem 2.

The series $\sum_{n\geq 2} \frac{1}{S^2(n)}$ is divergent.

Proof

Theorem 1 is applied starting from the obvious equation $\sum_{n\geq 2} \frac{1}{S^2(n)} = \sum_{n\geq 1} a_n$.

Let $\varepsilon>0$ be a positive number. There exists a number $N_\varepsilon>0$ so that the inequality $a_n > \left(\frac{1}{4} - \varepsilon\right) \frac{1}{n-1}$ holds for all $n > N_\varepsilon$. The divergence of the series is given by $\sum_{n\geq 2} \frac{1}{S^2(n)} = \sum_{n\geq 1} a_n \geq \sum_{n\geq N_\varepsilon} a_n \geq \left(\frac{1}{4} - \varepsilon\right) \cdot \sum_{n\geq N_\varepsilon} \frac{1}{n-1} = \infty$.

♣

Consequence 1.

If $m\leq 2$ then the series $\sum_{n\geq 2} \frac{1}{S^m(n)}$ is divergent.

Proof

The statement follows directly from divergence of the series $\sum_{n\geq 2} \frac{1}{S^2(n)}$ and the inequality $\sum_{n\geq 2} \frac{1}{S^2(n)} \leq \sum_{n\geq 2} \frac{1}{S^m(n)}$.

♣

3. About the Sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$

In this section the sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$ is evaluated and some remarks concerning the sequence values are made[1].

n	a_n	n	a_n	n	a_n
500	-3.14	100000	11.19	500000	31.15
1000	-2.97	200000	17.95	1000000	47.74
1500	-2.75	300000	23.09	1500000	56.80
2000	-2.55	400000	27.38	2000000	66.05
2500	-2.35	500000	31.15	2500000	74.14
3000	-2.14	600000	34.53	3000000	81.45
3500	-1.95	700000	37.63	3500000	88.13
4000	-1.79	800000	40.51	4000000	94.34
4500	-1.60	900000	43.20	4500000	100.15
5000	-1.44	1000000	45.74	5000000	105.63

Table 1. The values for the sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$

Because $\sum_{n\geq 2} \frac{1}{S^2(n)}$ is divergent, it is natural to find the convergence order for the series.

Firstly, we evaluate the sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$ and its values are presented in Table 1. Analysing the results from Table 1, the following remarks are obvious:

1. The sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$ can be considered *pseudo*-monotone.

[1] Numerical results presented in the tables have been calculated by Henry Ibstedt. The algorithm and its implementation will be included in *Computer Analysis of Number Sequences, H.Ibstedt, American Research Press (to appear summer 1998).*

2. The sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$ satisfies the inequality $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n) > 0 \; \forall n: 50000 < n < 5000000$. If the inequality holds for all n>50000 then it is evident that $\sum_{n \geq 2} \frac{1}{S^2(n)}$ diverges.

3. Because (the values of) the sequence a_n is *pseudo*-increasing we conjecture that $\lim_{n \to \infty} \left(\sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n) \right) = \infty$.

Secondly, the sequence $b_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n) - \ln(\ln(n))$ is evaluated in Table 2.

n	b_n	n	b_n	n	b_n
500	-3.14	100000	4.83	500000	1.83
1000	0.17	200000	3.08	1000000	1.26
1500	0.21	300000	2.43	1500000	1.02
2000	0.2	400000	2.07	2000000	0.87
2500	0.21	500000	1.83	2500000	0.77
3000	0.2	600000	1.65	3000000	0.7
3500	0.18	700000	1.52	3500000	0.65
4000	0.17	800000	1.4	4000000	0.61
4500	0.18	900000	1.33	4500000	0.57
5000	0.16	1000000	1.26	5000000	0.53

Table 2. The values for the sequence $b_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n) - \ln(\ln(n))$.

This sequence is more unpredictable than the sequence a_n. The only thing, which can be remarked is the decreasing behaviour. We have not been able to predict if this sequence is convergent yet.

4. Conclusions

A proof more simple than the proof presented in this article can be obtained using a convergence test similar to the *condensation* test [Nicolescu *et.al.* 1974]. According to this test, if $(a_n)_{n>0}$ is a decreasing sequence of positive numbers then the series $\sum_{n>0} a_n$ is convergent if and only if the series $\sum_{n>0} 2^n \cdot a_{2^n}$ is convergent. The sequence $\left(\frac{1}{S^m(n)}\right)_{n>1}$ satisfies that $\sum_{n>0} 2^n \cdot \frac{1}{S^m(2^n)}$ is divergent. In spite of that, we cannot conclude that the series $\sum_{n>1} \frac{1}{S^m(n)}$ is divergent because the sequence $\left(\frac{1}{S^m(n)}\right)_{n>1}$ is not decreasing.

Acknowledgements

I would like to thank Henry Ibstedt for his support according me to improve the quality of this article and for the numerical computation that he provided me.

References

Cojocaru, I and Cojocaru, S (1997) On a Function in Number Theory, *Smarandache Notions Journal*, **8**, 164-169.

Ibstedt, H. (1997) *Surfing on the Ocean of Numbers - a few Smarandache Notions and Similar Topics*, Erhus University Press.

Jozsef, S. (1997) On the Irrationality of Certain Alternative Smarandache Series, *Smarandache Notions Journal*, **8**, 143-144.

Niculescu, I., Dinculeanu, A. and Marcus, S. (1974) *Analiza Matematica*. Editura Tehnica, Bucuresti.

Smarandache, F (1980) A Function in Number Theory, *Analele Univ. Timisoara*, **XVIII**.