The Density of Generalized Smarandache Palindromes

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An integer is said to be a palindrome if it reads the same forwards and backwards. For example, 12321 is a palindromic number. It is easy to prove that the density of the palindromes is zero in the set of positive integers.

A *Generalized Smarandache Palindrome* (GSP) is any integer of the form

$a_1a_2a_3 \ldots a_na_n \ldots a_3a_2a_1$ or $a_1a_2a_3 \ldots a_{n-1}a_na_{n-1} \ldots a_3a_2a_1$

where all $a_1, a_2, a_3, \ldots a_n$ are integers having one or more digits [1], [2]. For example,

10101010 and 101010

are GSPs because they can be split into the forms

(10)(10)(10)(10) and (10)(10)(10)

and the segments are pairwise identical across the middle of the number.

As a point of clarification, we remove the possibility of the trivial case of enclosing the entire number

12345 written as (12345)

which would make every number a GSP. This possibility is eliminated by requiring that each number be split into at least two segments if it is not a regular palindrome.

Also, the number 100610

is considered to be a GSP, as the splitting

(10)(06)(10)

leads to an interior string that is a separate segment, which is a palindrome by default.

Obviously, since each regular palindrome is also a GSP and there are GSPs that are not regular palindromes, there are more GSPs than there are regular palindromes. Therefore, the density of GSPs is greater than or equal to zero and we consider the following question.

What is the density of GSPs in the positive integers?

The first step in the process is very easy to prove.

Theorem: The density of GSPs in the positive integers is greater than 0.1.

Proof: Consider a positive integer having an arbitrary number of digits.

$$a_n a_{n-1} \ldots a_2 a_1 a_0$$

and all numbers of the form

$$(k)a_{n-1} \ldots a_1(k)$$

are GSPs, and there are nine different choices for k. For each of these choices, one tenth of the values of the trailing digit would match it. Therefore, the density of GSPs is at least one tenth.

The simple proof of the previous theorem illustrates the basic idea that if the initial and terminal segments of the number are equal, then the number is a generalized palindrome and the values of the interior digits are irrelevant. This leads us to our general theorem.

Theorem: The density of GSPs in the positive integers is approximately 0.11.

Proof: Consider a positive integer having an arbitrary number of digits.

$$a_n a_{n-1} \ldots a_2 a_1 a_0$$

If the first and last digits are equal and nonzero, then the number is a generalized palindrome. As was demonstrated in the previous theorem, the likelihood of this is 0.10.

If $a_n = a_1$ and $a_{n-1} = a_0$, then the number is a GSP. Since the GSPs where $a_n = a_0$ have already been counted in the previous step, the conditions are

$$a_n = a_1 \text{ and } a_{n-1} = a_0 \quad \text{and } a_n \neq a_0$$

The situation is equivalent to choosing a nonzero digit for a_n, and decimal digits for a_{n-1} and a_0 that satisfy these conditions. This probability of this is easy to compute and is 0.009.

If $a_n = a_2$, $a_{n-1} = a_1$ and $a_{n-2} = a_0$, then the number is a GSP. To determine the probability here, we need to choose six digits, where a_n is nonzero and the digits do not also satisfy the conditions of the two previous cases. This is also easily computed, and the value is 0.0009.

The case where $a_n = a_3$, $a_{n-1} = a_2$, $a_{n-2} = a_1$ and $a_{n-3} = a_0$ is the next one, and the probability of satisfying this case after failing in the three previous cases is 0.0000891.

The sum of these probabilities is 0.10 + 0.009 + 0.0009 + 0.0000891, which is 0.1099891.

This process could be continued for initial and terminal segments longer and longer, but the probabilities would not be enough to make the sum 0.11.

References

1. G. Gregory, Generalized Smarandache Palindromes, http://www.gallup.unm.edu/~smarandache/GSP.htm.
2. F. Smarandache, Generalized Palindromes, Arizona State University Special Collections, Tempe.