THE EQUALITY $\beta^2(k+2,S(n))=\beta^2(k+1,S(n))+\beta^2(k,S(n))$

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Abstract. For any positive integer a, let $S(a)$ be the Smarandache function of a. For any positive integer r and b, let $\beta(r, b)$ be the last r digits of b. In this paper we determine all positive integer pairs (n, k) for which the title equality holds.

Key words: Smarandache function, digit, equality

For any positive integer a, let $S(a)$ be the Smarandache function of a. For any positive integer

(1) $$b=\overline{t_s\cdots t_2t_1}$$

with s digits, let

(2) $$\beta(r,b)=\overline{t_r\cdots t_1}$$

be the last r digits of b. Recently, Bencze [1] proposed the following problem:

Problem Determine all positive integer pairs (n, k) for which

(3) $$\beta^2(k+2,S(n))=\beta^2(k+1,S(n))+\beta^2(k,S(n)).$$

In this paper we completely solve the above-mentioned problem as follows.

Theorem A positive integer pair (n, k) satisfies (3) if and only if n satisfy

(4) $$S(n)=10^{k+2}c+10^{k}d,$$

where c is a nonnegative integer, d is a positive integer with $1 \leqslant d \leqslant 9$.

By the definition of the Smarandache function (see [2]), we have $S(m!)=m$ for any positive integer m. Therefore, by the above theorem, we obtain the following corollary immediately.

Corollary For any fixed positive integer k, there exists infinitely many positive integers

(5) $$n=\left(10^{k+2}c+10^{k}d\right)!, c \geq 0, d=1,2,\cdots,9,$$

Satisfying (3).

The proof of Theorem Let (n, k) be a positive integer pair satisfying (3), and let $b=S(n)$. Then b is a positive integer. We may assume that b has s digits as (1). For any positive integer r, by the definition (2) of $\beta(r, b)$, we have

(6) $$0 \leq \beta(r,b) < 10^{r}$$

and

(7) $$\beta(r+1,b)=\beta(r,b)+10^{r}t_{r+1}.$$

If $t_{k+2} \neq 0$, then from (6) and (7) we get

(8) $$\beta(k+2,b) \geq \beta(k+1,b)+10^{k+1} > \beta(k+1,b)+\beta(k,b).$$

It implies that

(9) $$\beta^{2}(k+2,b) > \beta^{2}(k+1,b)+\beta^{2}(k,b),$$

which contradicts (3).

If $t_{k+2}=0$, then from (7) we get

(10) $$\beta(k+2,b)=\beta(k+1,b).$$

Substitute (10) into (3), we get $\beta(k, b)=0$. It implies that $t_1=\cdots=t_k=0$

by (2). Thus, $b=S(n)$ satisfies (4). The theorem is proved.

References

[1] M. Bencze, Open questions for the Smarandache function, Smarandache Notions J. 12(2001), 201-203.

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