THE EQUATION $a^2(k+2, S(n)) = a^2(k+1, S(n)) + a^2(k, S(n))$

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Abstract. For any positive integer a, let $S(a)$ be the Smarandache function of a. For any positive integers r and b, let $a(r, b)$ be the first r digits of b. In this paper we prove that the title equation has no positive integer solutions (n, k).

Key words: Smarandache function, diophantine equation

Let **N** be the set of all positive integers. For any positive integer a, let $S(a)$ be the Smarandache function of a. For any positive integer

(1) $$b = \overline{t_s \cdots t_2 t_1}$$

with s digits, let

(2) $$\alpha(r,b) = \overline{t_s \cdots t_{s-r+1}}$$

be the first r digits of b. Recently, Bencze [1] proposed the following problem:

Problem Determine all solutions (n, k) of the equation

(3) $$\alpha^2(k+2, S(n)) = \alpha^2(k+1, S(n)) + \alpha^2(k, S(n)), n, k \in \mathbf{N}.$$

In this paper we completely solve the above-mentioned problem as follows.

Theorem The equation (3) has no solutions (n, k).

Proof. Let (n, k) be a solution of (3). It is a well known fact that $S(n)$ is a positive integer (see [2]). Let $b=S(n)$. We may assume that b

has s digits as (1). For any positive integer r, by the definition (2) of $\alpha(r, b)$, we have

$$\alpha(r+1,b)=\begin{cases}10\alpha(r,b)+t_{s-r+1}, & \text{if } r<s,\\ \alpha(r,b), & \text{if } r\geq s.\end{cases} \tag{4}$$

If $k>s-1$, then from (4) we get $\alpha(k+2, b)=\alpha(k+1, b)$. Hence, by (3), we obtain $\alpha(k, b)=0$, a contradiction.

If $k<s-1$, then from (4) we get

$$\alpha(k+2,b)\geq 10\cdot\alpha(k+1,b). \tag{5}$$

Hence, by (3) and (5), we get

$$99\cdot\alpha^2(k+1,b)\leq\alpha^2(k,b). \tag{6}$$

However, we see from (4) that $\alpha(k+1, b)\geqslant\alpha(k, b)$. Therefore, (6) is impossible. Thus, the equation (3) has no solutions (n, k).

References

[1] M. Bencze, Open questions for the Smarandache function, Smarandache Notions J. 12(2001), 201-203.

[2] F. Smarandache, A function in number theory, Ann. Univ. Timisoara XVIII, 1980.

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