THE FACTORIAL SIGNATURE OF NATURAL NUMBERS

by

Ion Bălăcenoiu

In this paper we define the factorial signature for natural numbers and with this we obtain several results.

1. DEFINITION The system $(j_1, j_2, \dots, j_r) \in N^{*r}$ is a system of factorial exponents if $\exists\, s \in N^*$ so that $s! = p_1^{j_1} \cdot p_2^{j_2} \cdot \dots \cdot p_{\pi(s)}^{j_{\pi(s)}}$, where $2 = p_1 < 3 = p_2 < \dots < p_{\pi(s)} \le s$, $\pi(s) = r$.

Obviously, for every natural number $s > 1$ there exists a system of factorial exponents $(j_1, j_2, \dots, j_{\pi(s)})$.

Because $s! = \prod_{i=1}^{\pi(s)} p_i^{e_{p_i}(s)}$, where $e_{p_i}(s)$ are Legendre's exponents, it is true that: $e_{p_1}(s) \ge e_{p_2}(s) \ge \dots e_{p_{\pi(s)}}(s) = 1$.

Therefore for every system of factorial exponents $(j_1, j_2, \dots, j_r)$ it results that $j_1 \ge j_2 \ge \dots \ge j_r = 1$.

It exists a finite number of system of factorial exponents with r components. Indeed, they correspond those natural numbers with the property: $p_r! \le s! < p_{r-1}!$

If $(j'_1, j'_2, \dots, j'_r)$ and $(j''_1, j''_2, \dots, j''_r)$ are systems of factorial exponent corresponding as n respectively m, then

$n < m \Rightarrow j'_1 \le j''_1,\ j'_2 \le j''_2, \dots, j'_{r-1} \le j''_{r-1}, j'_r = j''_r = 1.$

If $\pi(n) = \pi(n+1)$, then $n+1$ is a composite number and their systems of factorial exponents have the same number of components.

If $n+1$ is a prime number, then $\pi(n+1) = \pi(n) + 1$ and if $(j_1, j_2, \ldots, j_{\pi(n)} = 1)$ is the system of exponents of adequate factorial for n, then the system of exponents of adequate factorial for $n+1$ is:

$$(j_1, j_2, \ldots, j_{\pi(n)} = 1,\ j_{\pi(n+1)} = 1)$$

Two systems of factorial exponents with r components, adequate as two different natural numbers, have different components and equal components, too.

2. DEFINITION Let $n \in N$, $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots p_{i_t}^{\alpha_{i_t}}$, and let s be the smallest positive integer such that $s!$ is divisible by n. Then the factorial signature for n (denoted by s.f.(n!)) is: $s.f.(n) = \left\{p_{i_{k_1}}^{\alpha_{i_{k_1}}}, p_{i_{k_2}}^{\alpha_{i_{k_2}}}, \ldots, p_{i_{k_r}}^{\alpha_{i_{k_r}}}\right\}$ where $\left\{p_{i_{k_1}}^{\alpha_{i_{k_1}}}, p_{i_{k_2}}^{\alpha_{i_{k_2}}}, \ldots, p_{i_{k_r}}^{\alpha_{i_{k_r}}}\right\}$ is the largest subset for $\left\{p_{i_1}^{\alpha_{i_1}}, p_{i_2}^{\alpha_{i_2}}, \ldots, p_{i_t}^{\alpha_{i_t}}\right\}$ so that there are $\beta_{i_{k_j}} \geq \alpha_{i_{k_j}} \geq 1$, $j \in \overline{1, r}$ with $p_{i_{k_j}}^{\beta_{i_{k_j}}} \nmid (s-1)!$ and $p_{i_{k_j}}^{\beta_{i_{k_j}}} \mid s!$.

It is considered $s.f.(0) = \varnothing$, $s.f.(1) = \{1\}$.

Obviously: $e_{p_{i_{k_j}}}(s-1) < \beta_{i_{k_j}} \leq e_{p_{i_{k_j}}}(s)$, $j = \overline{1, r}$.

3. DEFINITION The type of the factorial signature for n is noted $T[s.f.(n)]$ and $T[s.f.(0)] = 0$, $T[s.f.(n)] = s$, for $n > 0$, where s is the smallest positive integer such that $n \mid s!$.

4. EXAMPLE

a) Let $n = 120 = 2^3 \times 3 \times 5$, therefore $p_1 = 2$, $p_2 = 3$, $p_3 = 5$; $\alpha_1 = 3$, $\alpha_2 = 1$, $\alpha_3 = 1$. Obviously the smallest positive integer s thus so that $n \mid s!$ is $s = 5$. Indeed, $s.f.(120) = \{5\}$ because $\{5\}$ is the largest subset of $\{2^3, 3, 5\}$ in the sense that (see definition 2) it exist $\beta_3 = \alpha_3 = 1$ so that $5^{\beta_3} \nmid 4!$ and $5^{\beta_3} \mid 5!$.

b) Let $n = p^\alpha$, then $s.f.(p^\alpha) = \{p^\alpha\}$ and $T[s.f.(p^\alpha)] = s$ iff $e_p(s-1) < \alpha \leq e_p(s)$.

5. PROPOSITION Let $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_t}^{\alpha_{i_t}}$, $p_{i_1} < p_{i_2} < \ldots < p_{i_t}$, with $s.f.(n) = \left\{p_{i_{k_1}}^{\alpha_{i_{k_1}}}, p_{i_{k_2}}^{\alpha_{i_{k_2}}}, \ldots, p_{i_{k_r}}^{\alpha_{i_{k_r}}}\right\}$, and $T[s.f.(n)] = s > 1$ then it exists at least an element $p_{i_{k_j}}^{\alpha_{i_{k_j}}}$, $j = \overline{1, r}$ so that

$e_{p_{i_{k_j}}}(s-1) < \alpha_{i_{k_j}} \leq e_{p_{i_{k_j}}}(s)$ and $T\left[s.f.\left(p_{i_{k_j}}^{\alpha_{i_{k_j}}}\right)\right] = s.$

Proof. Let $p_{i_{k_1}} < p_{i_{k_2}} < \ldots < p_{i_{k_r}}$.

Because $T[s.f.(n)] = s > 1$ it results that $n \mid s!$ and it exists $\beta_{i_{k_j}} \geq \alpha_{i_{k_j}} \geq 1$ so that $e_{p_{i_{k_j}}}(s-1) < \beta_{i_{k_j}} \leq e_{p_{i_{k_j}}}(s)$.

If does not exist $j \in \overline{1,r}$ so that $e_{p_{i_{k_j}}}(s-1) < \alpha_{i_{k_j}} \leq e_{p_{i_{k_j}}}(s)$, then $p_{i_{k_r}} < p_{\pi(s)}$ because $p_{i_{k_r}} = p_{\pi(s)}$ it implies that $\alpha_{i_{k_r}} = e_{p_{i_{k_r}}}(s) = e_{p_{\pi(s)}}(s) = 1 = \beta_{i_{k_r}}$ and $e_{p_{\pi(s)}}(s-1) = 0$.

Using $\alpha_{i_{k_j}} \leq \beta_{i_{k_j}} \leq e_{p_{i_{k_j}}}(s)$ it results that $\alpha_{i_{k_j}} \leq e_{p_{i_{k_j}}}(s-1)$, $j = \overline{1,r}$. Thus we have $T[s.f.(n)] \leq s - 1 < s$, which is not possible.

Therefore it exists $j \in \overline{1,r}$ so that $e_{p_{i_{k_j}}}(s-1) < \alpha_{i_{k_j}} \leq e_{p_{i_{k_j}}}(s)$ and in consequence $T\left[s.f.\left(p_{i_{k_j}}^{\alpha_{i_{k_j}}}\right)\right] = s.$

We can observe that $p_{i_{k_j}}^{\alpha_{i_{k_j}}}$ indicates the type $T[s.f.(n)]$.

6. DEFINITION The complement until a factorial (see [2]), is $b : N^* \to N^*$, $b(n) = k$, where k is the smallest positive integer so that $n\,b(n)$ is a factorial. Thus $n\,b(n) = m!$.

Obviously, if $n\,b(n) = m!$, then $m!$ is the smallest factorial divisible by n, therefore $n\,b(n) = [\eta(n)]!$ where η is Smarandache function see [1].

It is easy to see that $b(n!) = 1$ and $b(p) = (p-1)!$ p is a prime number.

Because $\eta(n!) = n$ it results $b(n) = \dfrac{[\eta(n)]!}{\eta(n!)}$.

7. PROPOSITION Let p be a prime number and $p > m$, then $b(m! \cdot p) = \dfrac{(p-1)!}{m!}$

Proof. Obviously, $p!$ is the smallest factorial divisible by $m!\,p$.

Therefore $b(m! \cdot p) = \dfrac{p!}{m!\,p} = \dfrac{(p-1)!}{m!}$

8. PROPOSITION

$T[s.f.(n)] = s$ iff $nb(n) = s!$

Proof. Obviously, $T[s.f.(n)] = s \Leftrightarrow s!$ is the smallest factorial divisible by $n \Leftrightarrow n\,b(n) = s!$.

9. DEFINITION We define the echivalent relation: $s.f.(n) \approx s.f.(m) \Leftrightarrow$ $\Leftrightarrow$ $nbn = mb(m)$.

We note $\hat{s!} = \{n \in N^* / nb(n) = s!\}$.

10. REMARK Obviously, if $s.f.(n) = s.f.(m)$ then $s.f.(n) \approx s.f.(m)$. If $s.f.(n) \approx s.f.(m)$ it does not result that $s.f.(n) = s.f.(m)$. If $s.f.(n) = s.f.(m)$ it does not result that $n = m$. If $n\,b(n) = s!$ it results that $s.f.(n) \approx s.f.(s!)$ because $s!\,b(s!) = s!$.

We also observe that $T[s.f.(n)] = s \Leftrightarrow n \in \hat{s!}$.

If p is a prime number, then $p \in \hat{p!}$ because $p\,b(p) = p!$. It is easy to see that $s.f.(p) = s.f.(p!) = \{p\}$.

Because $p! = p_1^{e_{p_1}(p)} \cdot p_2^{e_{p_2}(p)} \cdot \ldots p$, where $2 = p_1 < 3 = p_2 < \ldots < p$, it results $\hat{p!} = \{p, p_1p, p_2p, p_1^2p, p_1p_2p, \ldots, p_1^{e_{p_1}(p)} \cdot p_2^{e_{p_2}(p)} \cdot \ldots \cdot p\}$.

If $s = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_k}^{\alpha_{i_k}}$, then $s.f.(s!) = \{p_{i_1}^{e_{p_{i_1}}(s)}, p_{i_2}^{e_{p_{i_2}}(s)}, \ldots, p_{i_k}^{e_{p_{i_k}}(s)}\}$

11. PROPOSITION If $(n, m) = 1$ and $n, m \in \hat{s!}$ then $n \cdot m \in \hat{s!}$ and $s.f.(n \cdot m) = [s.f.(n)] \cup [s.f.(m)]$.

Proof. Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_t^{\alpha_t}$ and $m = q_1^{\gamma_1} \cdot q_2^{\gamma_2} \cdot \ldots \cdot q_h^{\gamma_h}$ be the cannonical decomposition of n and m. Obviously, because $(n, m) = 1$ it results $p_i \neq q_j$ for $i \in \overline{1, t}$, $j \in \overline{1, h}$.

Let $s.f.(n) = \{p_{i_1}^{\alpha_{i_1}}, p_{i_2}^{\alpha_{i_2}}, \ldots, p_{i_r}^{\alpha_{i_r}}\}$ and $s.f.(m) = \{q_{j_1}^{\gamma_{j_1}}, \ldots, q_{j_k}^{\gamma_{j_k}}\}$.

Because $n, m \in \hat{s!}$ it results that s is the smallest positive integer so that $n \mid s!$, $m \mid s!$ and it exists $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_r}$ and $\delta_{j_1}, \delta_{j_2}, \ldots, \delta_{j_k}$ respectively so that $\beta_{i_u} \geq \alpha_{i_u} \geq 1$, $u \in \overline{1, r}$ and $\delta_{j_v} \geq \gamma_{j_v} \geq 1$, $v \in \overline{1, k}$ and

$$p_{i_u}^{\beta_{i_u}} \nmid (s-1)!, \quad p_{i_u}^{\beta_{i_u}} \mid s!$$

$$q_{j_v}^{\delta_{j_v}} \nmid (s-1)!, \quad q_{j_v}^{\delta_{j_v}} \mid s!$$

In $(n, m) = 1$ and $n \mid s!$, $m \mid s!$ it results that $n\,m \mid s!$. Because s is the smallest natural number such as $n \mid s!$ and $n\,m \mid s!$ it results that s is the smallest natural number such that $s!$ is divisible by $n \cdot m$, therefore $T[s.f.(nm)] = s$, so that $nm \in \hat{s!}$.

Obviously $nm = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_t^{\alpha_t} \cdot q_1^{\gamma_1} \cdot q_2^{\gamma_2} \ldots q_h^{\gamma_h}$ and

$$s.f.(n \cdot m) = \{p_{i_1}^{\alpha_{i_1}}, p_{i_2}^{\alpha_{i_2}}, \ldots, p_{i_r}^{\alpha_{i_r}}, q_{j_1}^{\gamma_{j_1}}, \ldots, q_{j_k}^{\gamma_{j_k}}\} = [s.f.(n)] \cup [s.f.(m)]$$

12. REMARK The proposition 11 can be also formulated in this way: if $n\,b(n) = s!$, $m\,b(m) = s!$ and $(n, m) = 1$, then $nmb(nm) = s!$.

It results $b(n \cdot m) = \frac{b(n)b(m)}{s!}$ if $(n, m) = 1$ and $n, m \in \hat{s!}$.

13. PROPOSITION

If $(n, m) = 1$ and $s.f.(n) \approx s.f.(m)$, then $b(n \cdot m) = (b(n), b(m))$.

Proof. Let $T[s.f.(m)] = s$, because $(n, m) = 1$ and $s.f.(n) \approx s.f.(m)$ then it results:

$$nb(n) = mb(m) = nmb(nm) = s!,$$

therefore $b(n \cdot m) = \frac{s!}{nm}$. Let us consider $d = (b(n), b(m))$, $b(n) = d \cdot a$ and $b(m) = d \cdot b$, where $(a, b) = 1$. Then $nb(m) = mb(m)$ implies that $na = mb$.

Because $(a, b) = 1$ it results $a \mid m$ and $b \mid n$, then we can write $n = hb$, then $hba = mb$, so that $m = ah$. Since $1 = (n, m) = (hb, ha) = h(a, b) = h$ it results $n = b$, $m = a$.

Then $(b(n), b(m)) = d = \frac{s!}{na} = \frac{s!}{nm} = b(n \cdot m)$.

14. PROPOSITION

Let $n = q_1^{\gamma_1} \cdot q_2^{\gamma_2} \cdot \ldots \cdot q_t^{\gamma_t}$ and $s.f.(n) = \{q_{j_1}^{\gamma_{j_1}}, \ldots, q_{j_r}^{\gamma_{j_r}}\}$. If $n \in \hat{s!}$ and $s.f.(s!) = \left\{p_{i_1}^{e_{p_{i_1}}(s)}, p_{i_2}^{e_{p_{i_2}}(s)}, \ldots, p_{i_k}^{e_{p_{i_k}}(s)}\right\}$, then $\{q_{j_1}, q_{j_2}, \ldots, q_{j_r}\} \subset \{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}$.

Proof. Because $s.f.(s!) = \left\{p_{i_1}^{e_{p_{i_1}}(s)}, p_{i_2}^{e_{p_{i_2}}(s)}, \ldots, p_{i_k}^{e_{p_{i_k}}(s)}\right\}$ it results $p_{i_h}^{e_{p_{i_h}}(s)} \nmid (s-1)!$ and $p_{i_h}^{e_{p_{i_h}}(s)} \mid s!$ for $h = \overline{1, k}$, therefore $p_{i_h} \mid s$, thus we have $s = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_k}^{\alpha_{i_k}}$, where $1 \le \alpha_{i_h} \le e_{p_{i_h}}(s)$ for $h = \overline{1, k}$.

Because $n \in \hat{s!}$ it exists $\beta_{i_m} \ge \gamma_{i_m} \ge 1$, for $m = \overline{1, r}$ so that $q_{j_m}^{\beta_{j_m}} \mid (s-1)!$ and $q_{j_m}^{\beta_{j_m}} \mid s!$, thus $q_{j_m} \mid s$. Therefore $\{q_{j_1}, q_{j_2}, \ldots, q_{j_r}\} \subset \{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}$.

15. DEFINITION Let $n, m \in \hat{s!}$ and $s.f.(n) = \{p_{i_1}^{\alpha_{i_1}}, p_{i_2}^{\alpha_{i_2}}, \ldots, p_{i_r}^{\alpha_{i_r}}\}$, $s.f.(m) = \{q_{j_1}^{\gamma_{j_1}}, q_{j_2}^{\gamma_{j_2}}, \ldots, q_{j_k}^{\gamma_{j_k}}\}$ then $s.f.(n) \subsetneqq s.f.(m)$ iff

$\{p_{i_1}, p_{i_2}, \dots, p_{i_r}\} \subset \{q_{j_1}, q_{j_2}, \dots, q_{j_k}\}$ and for every $p_{i_s} = q_{j_l}$ it implies $\alpha_{i_s} \le \gamma_{j_l}$.

16. REMARK Obviously "$\subseteq$" is a partial order relation in the set of factorial signatures of numbers which belongs to $\hat{s!}$. For any $n \in \hat{s!}$ it results $s.f.(n) \subseteq s.f.(s!)$, so that $s.f(s!)$ is the maximal element. If $s = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_u}^{\alpha_{i_u}}$ then the minimal elements in the factorial signatures of numbers which belongs to $\hat{s!}$ are:

$$s.f.\left\{p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1}\right\} = \left\{p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1}\right\}, \quad h \in \overline{1, u}$$

because $p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1} \in \hat{s!}$ and for any $x \in \hat{s!}$ so that $s.f.(x) \subseteq s.f.\left(p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1}\right)$ it results $s.f.(x) = \left\{p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1}\right\}$

17. PROPOSITION For any $m \in N$, $\eta^{-1}(m) = \hat{m!}$, where η is Smarandache function.

Proof. Let $n \in \hat{m!}$, then $nb(n) = [\eta(n)]! = m!$, therefore $\eta(n) = m$, or $n \in \eta^{-1}(m)$. Conversely, if $n \in \eta^{-1}(m)$ it results $\eta(n) = m$, that $nb(n) = [\eta(n)]!$ and therefore $n \in \hat{m!}$.

18. DEFINITIONS In $\hat{s!}$ it is considered the equivalent relation: $n \approx m \Leftrightarrow s.f.(n) = s.f.(m)$. The echivalent class for n is $\tilde{n} = \{m \in \hat{s!} \mid s.f.(n) = s.f.(m)\}$. The set of equivalent classes in $\hat{s!}$ it noted with $\hat{\hat{s!}}$. In $\hat{\hat{s!}}$ it is considered partial order relation $\tilde{n} \lesssim \tilde{m} \Leftrightarrow s.f.(n) \subseteq s.f.(m)$.

19. REMARK Each class $\tilde{n} \in \hat{\hat{s!}}$ is a set of elements which belongs to $\hat{s!}$, and it is total ordered in the sense of the relation $\le$ It is also finite, therefore it has a minimum and a maximum. If $s.f.(n) = \{p_{i_1}^{\alpha_{i_1}}, p_{i_2}^{\alpha_{i_2}}, \dots, p_{i_r}^{\alpha_{i_r}}\}$, then in the class $\tilde{n}$ the smallest number is $\bar{n} = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_r}^{\alpha_{i_r}}$ and the other numbers of $\tilde{n}$ are $A \cdot \bar{n}$, with $A = p_{h_1}^{\varepsilon_1} \cdot p_{h_2}^{\varepsilon_2} \cdot \ldots \cdot p_{h_k}^{\varepsilon_k}$, with $p_{h_j} \nmid s$ and $p_{h_j} \mid s!$ and $0 \le \varepsilon_j \le e_{p_{h_j}}(s)$, $j = \overline{1, k}$, where $\left\{p_{h_1}^{e_{p_{h_1}}(s)}, \dots, p_{h_k}^{e_{p_{h_k}}(s)}\right\} = \left\{p_1^{e_{p_1}(s)}, \dots, p_{\pi(s)}^{1}\right\} - s.f.(s!)$.

The largest number of $\tilde{n}$ is:

$$\bar{n}\, p_{h_1}^{e_{p_{h_1}}(s)} \cdot p_{h_2}^{e_{p_{h_2}}(s)} \cdots \cdot p_{h_k}^{e_{p_{h_k}}(s)} .$$

Minimal elements of $\hat{\hat{s!}}$, in the sense of the partial order relation $\lesssim$, are the classes which respectively have the elements: $p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1}$, $h=\overline{1,r}$. The maximal class of $\hat{\hat{s!}}$ has $s!$ as element.

If $\tilde{n} \lesssim \tilde{m}$ and $\tilde{n} \neq \tilde{m}$, the absolute value of the difference between two different numbers in the class $\tilde{m}$ is larger than the smallest between absolute values of differences between two different numbers of the class $\tilde{n}$.

If $\tilde{n} \lesssim \tilde{m}$ and $\tilde{n} \neq \tilde{m}$, the absolute value of the difference between a number of $\tilde{n}$ and a number of $\tilde{m}$ is larger or equal than the smallest number of $\tilde{n}$ and therefore it is larger or equal than the smallest number of the minimal class comparable (in the sense of the partial order relation $\lesssim$) with $\tilde{n}$.

20. EXAMPLE Let $s = 12 = 2^2 \cdot 3$, then $s! = 12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, $s.f.(12!) = \{2^{10}, 3^5\}$.

Let us consider the set of natural numbers with the factorial signature of type 12, so that $\hat{12!} = \{n \in N / nb(n) = 12!\} = \{n \in N / s.f.(n) = s.f.(12!)\}$.

Obviously $\eta^{-1}(12) = \hat{12!}$.

The minimal elements of $\hat{12!}$, in the sense of the partial order relation $\lesssim$, are: $\widetilde{2^{10-2-1}} = \widetilde{2^9}$ and $\widetilde{3^{5-1-1}} = \widetilde{3^5}$.

Factorial signatures of numbers of $\hat{12!}$ are ordered in the following way:

$$\{2^9\} \subseteq \left\{ \begin{array}{l} \{2^{10}\} \subseteq \{2^{10},3\} \subseteq \{2^{10},3^2\} \subseteq \{2^{10},3^3\} \subseteq \{2^{10},3^4\} \subseteq \\ \{2^9,3\} \subseteq \{2^9,3^2\} \subseteq \{2^9,3^3\} \subseteq \{2^9,3^4\} \subseteq \Big\} \{2^9,3^5\} \subseteq \end{array} \right\} \{2^{10},3^5$$

$$\{3^5\} \subseteq \{3^5,2\} \subseteq \{3^5,2^2\} \subseteq \ldots \subseteq \{3^5,2^8\} \subseteq$$

Classes of numbers of $\hat{12!}$ are presented in next table:

n →	s.f.(n)	s.f.(n)	← n	s.f.(n)	← n
		$\{2^9\}$	2^9 $2^9 \cdot 5$ $2^9 \cdot 7$ $2^9 \cdot 11$ $2^9 \cdot 5^2$ $2^9 \cdot 5 \cdot 7$ $2^9 \cdot 5 \cdot 11$ $2^9 \cdot 7 \cdot 11$ $2^9 \cdot 5 \cdot 7 \cdot 11$ $2^9 \cdot 5^2 \cdot 7 \cdot 11$	$\{3^5\}$	3^5 $3^5 \cdot 5$ $3^5 \cdot 7$ $3^5 \cdot 11$ $3^5 \cdot 5^2$ $3^5 \cdot 5 \cdot 7$ $3^5 \cdot 5 \cdot 11$ $3^5 \cdot 7 \cdot 11$ $3^5 \cdot 5 \cdot 7 \cdot 11$ $3^5 \cdot 5^2 \cdot 7 \cdot 11$
2^{10} $2^{10} \cdot 5$ $2^{10} \cdot 7$ $2^{10} \cdot 11$ $2^{10} \cdot 5^2$ $2^{10} \cdot 5 \cdot 7$ $2^{10} \cdot 5 \cdot 11$ $2^{10} \cdot 7 \cdot 11$ $2^{10} \cdot 5 \cdot 7 \cdot 11$ $2^{10} \cdot 5^2 \cdot 7 \cdot 11$	$\{2^{10}\}$	$\{2^9, 3\}$	$2^9 \cdot 3$ $2^9 \cdot 3 \cdot 5$ $2^9 \cdot 3 \cdot 7$ $2^9 \cdot 3 \cdot 11$ $2^9 \cdot 3 \cdot 5^2$ $2^9 \cdot 3 \cdot 5 \cdot 7$ $2^9 \cdot 3 \cdot 5 \cdot 11$ $2^9 \cdot 3 \cdot 7 \cdot 11$ $2^9 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ $2^9 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11$	$\{3^5, 2\}$	$3^5 \cdot 2$ $3^5 \cdot 2 \cdot 5$ $3^5 \cdot 2 \cdot 7$ $3^5 \cdot 2 \cdot 11$ $3^5 \cdot 2 \cdot 5^2$ $3^5 \cdot 2 \cdot 5 \cdot 7$ $3^5 \cdot 2 \cdot 5 \cdot 11$ $3^5 \cdot 2 \cdot 7 \cdot 11$ $3^5 \cdot 2 \cdot 5 \cdot 7 \cdot 11$ $3^5 \cdot 2 \cdot 5^2 \cdot 7 \cdot 11$
$2^{10} \cdot 3$ ---- $2^{10} \cdot 3 \cdot 5^2 \cdot 7 \cdot 11$	$\{2^{10}, 3\}$	$\{2^9, 3^2\}$	$2^9 \cdot 3^2$ ----- $2^9 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$	$\{3^5, 2^2\}$	$3^5 \cdot 2^2$ ----- $3^5 \cdot 2^2 \cdot 5^2 \cdot 7 \cdot 11$
------------	------	-------	-------------	-------	------------
$2^{10} \cdot 3^3$ ---- $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11$	$\{2^{10}, 3^3\}$	$\{2^9, 3^4\}$	$2^9 \cdot 3^4$ ----- $2^9 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	$\{3^5, 2^8\}$	$3^5 \cdot 2^8$ ----- $3^5 \cdot 2^8 \cdot 5^2 \cdot 7 \cdot 11$
$2^{10} \cdot 3^4$ ---- $2^{10} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	$\{2^{10}, 3^4\}$	$\{2^9, 3^5\}$	$2^9 \cdot 3^5$ ----- $2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$		
$2^{10} \cdot 3^5$ ---- $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	$\{2^{10}, 3^5\}$				

REFERENCES

[1] Bălăcenoiu I., Smarandache Numerical Functions. Smarandache Function Journal, Vol. 4-5, No. 1, 1994, 6-13.

[2] "The Florentin Smarandache papers" special collection, Arizona, State University, Hayden Library, Temple, Box 871006, AZ 85287-1006, USA.

CURRENT ADDRESS

Brazda lui Novac, Bloc G_5, Ap. 161

1100, Craiova

Romania