The function $\Pi_S(x)$

by

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In this paper are studied some properties of the numerical function $\Pi_S : N^* \to N$, $\Pi_S(x) = \{ m \in (0, x] / S(m) = \text{prime number} \}$, where $S(m)$ is the Smarandache function, defined in [1].

Numerical example :

$\Pi_S(1) = 0$, $\Pi_S(2) = 1$, $\Pi_S(3) = 2$, $\Pi_S(4) = 2$, $\Pi_S(5) = 3$, $\Pi_S(6) = 4$, $\Pi_S(7) = 5$, $\Pi_S(8) = 5$, $\Pi_S(9) = 5$, $\Pi_S(10) = 6$, $\Pi_S(11) = 7$, $\Pi_S(12) = 7$, $\Pi_S(13) = 8$, $\Pi_S(14) = 9$, $\Pi_S(15) = 10$, $\Pi_S(16) = 10$, $\Pi_S(17) = 11$, $\Pi_S(18) = 11$, $\Pi_S(19) = 12$, $\Pi_S(20) = 13$.

Proposion 1.

According to the definition we have :

a) $\Pi_S(x) \le \Pi_S(x+1)$,

b) $\Pi_S(x) = \Pi_S(x-1) + 1$, if x is a prime,

c) $\Pi_S(x) \le \varphi(x)$, if x is a prime,

where $\varphi(x)$ is the Euler's totient function.

Proposition 2.

The equation $\Pi_S(x) = \left[\frac{x}{2}\right]$, in the hypothesis $x \ne 1$ and $\Pi_S(x+1) = \Pi_S(x)$ has no solution in the following situation :

a) x is a prime,

b) x is a composite number, odd

c) x + 1 is the square of a positiv integer and x is odd.

Proof.

Using the reduction ad absurdum method we suppose that the equation $\Pi_S(x) = [\frac{x}{2}]$ has solution. Then $\Pi_S(x+1) = [\frac{x+1}{2}]$. Using the hypothesis we have :

$[\frac{x}{2}] = [\frac{x+1}{2}]$, false.

Because $x + 1$ is a perfect square we deduce that x is a composite number and because x is an uneven we obtain (b).

Proposition 3.

$\forall a \geq 2$ and $k \geq 2$ $S(a^k)$ is not a prime.

Proof.

If we suppose that $S(a^k) = p$ is a prime, then $p! = a^k p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n} p$ and $(a^k, p) = 1$. We deduce that $a^k / (p-1)! \Rightarrow$

$S(a^k) \leq p - 1 < p$, false.

Proposition 4.

$\forall x \in N^*$, we have :

$$\left[\frac{x}{2}\right] \leq \Pi_S(x) \leq x - [\sqrt{x}]$$

Proof.

We used the mathematical induction. In the particular case $x \in \{1, 2, 3, 4\}$ our inequality is verified by direct calculus.

We suppose that the inequality is verified for $x \in N^*$ and we proved it for $x + 1$.

We have the following cases :

1) $x + 1$ the prime number, with the subcases :

a) x is not a square of some integer. Then $\Pi_S(x+1) = \Pi_S(x) + 1$.

We suppose that $\Pi_S(x) \leq x - [\sqrt{x}]$

Let prove that $\Pi_S(x+1) \leq x + 1 - [\sqrt{x+1}]$.

It results that $\Pi_S(x+1) \leq x + 1 - [\sqrt{x+1}] \Leftrightarrow \Pi_S(x) \leq x - [\sqrt{x+1}]$.

It's enough to prove that $x - [\sqrt{x}] \leq x - [\sqrt{x+1}]$. This relation is true because from our hypothesis it results that $[\sqrt{x}] = [\sqrt{x+1}]$.

For the left side of the inequality we have $\Pi_S(x) \geq \left[\frac{x}{2}\right]$, true, and let prove that $\Pi_S(x+1) > \left[\frac{x+1}{2}\right]$.

Because $\Pi_S(x+1) = \Pi_S(x) + 1$ we have to prove that $\Pi_S(x) + 1 \geq \left[\frac{x+1}{2}\right]$

Therefore $\Pi_S(x) \geq \left[\frac{x+1}{2}\right] - 1$, that is a true relation.

b) x perfect square.

We suppose that $\Pi_S(x) \leq x - [\sqrt{x}]$ is true. Then :

$\Pi_S(x) \le x + 1 - [\sqrt{x+1}] \Rightarrow \Pi_S(x) + 1 \le x + 1 - [\sqrt{x+1}] \Leftrightarrow \Pi_S(x) \le x - [\sqrt{x+1}]$.
That is a true relation because $[\sqrt{x}] = [\sqrt{x+1}]$. For the left inequality the demonstration is analogous with (a)

2) x prime

a) $x - 1$ is not a perfect square.

We suppose that $\Pi_S(x) \le x - [\sqrt{x}]$ is true.

Let prove that $\Pi_S(x+1) \le x + 1 - [\sqrt{x+1}]$.

In this case we have the following two situations :

(i) If $\Pi_S(x+1) = \Pi_S(x) + 1$, then we must prove that :

$\Pi_S(x) + 1 \le x + 1 - [\sqrt{x+1}]$.

Supposing that $\Pi_S(x) \ge \left[\frac{x}{2}\right]$ is true, let show that $\Pi_S(x+1) \ge \left[\frac{x+1}{2}\right]$ or $\Pi_S(x) + 1 \ge \left[\frac{x+1}{2}\right]$, therefore $\Pi_S(x) \ge \left[\frac{x+1}{2}\right] - 1$ and that results from the hypothesis.

(ii) If $\Pi_S(x+1) = \Pi_S(x)$. We have to prove that $\Pi_S(x) \le x+1 - [\sqrt{x+1}]$

Of course this inequality is true. For the left side of the inequality we have to prove that $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]$. If we admit $\left[\frac{x}{2}\right] \le \Pi_S(x) < \left[\frac{x+1}{2}\right]$ we obtain that $\Pi_S(x) = \left[\frac{x}{2}\right], x \ne 1$.

According to the *Proposition 2*. this inequality can't be true.

Therefore we have $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]$.

Let observe that $x + 1$ is not a perfect square, if $x > 3$ is a prime number. For $x = 3$ the inequality is verified by calculus.

3) x is an even composed number. Then :

a) If $x + 1$ is a prime.

We know that $\Pi_S(x+1) = \Pi_S(x) + 1$. Then supposing $\Pi_S(x) \le x - [\sqrt{x}]$.
We have to prove that $\Pi_S(x+1) \le x + 1 - [\sqrt{x+1}]$ or $\Pi_S(x) = x - \left[\sqrt{x+1}\right]$.

This is true, because $[\sqrt{x}] = \left[\sqrt{x+1}\right]$.

For the left inequality we have to show $\Pi_S(x+1) \ge \left[\frac{x+1}{2}\right]$,
or $\Pi_S(x) + 1 \ge \left[\frac{x+1}{2}\right]$. But $\Pi_S(x) \ge \left[\frac{x+1}{2}\right] - 1$, is true.

b) If $x + 1$ is an odd composite number, then

(i) If $\Pi_S(x+1) = \Pi_S(x) + 1$, the demonstration is the same as at (a).

(ii) If $\Pi_S(x+1) = \Pi_S(x)$, we have to prove that $\Pi_S(x) \le x + 1 - \left[\sqrt{x+1}\right]$

Obvious.

The left inequality is obvious.

c) $x + 1$ perfect square.

Using *Proposition 3* we have only the case $\Pi_S(x+1) = \Pi_S(x)$. Then if we consider to be true the relation $\Pi_S(x) \le x - [\sqrt{x}]$.

Let prove that $\Pi_S(x+1) < x+1-\left[\sqrt{x+1}\right]$.

But $\Pi_S(x) \le x+1-\left[\sqrt{x+1}\right]$ is true.

For the left inequality we suppose that $\Pi_S(x) \ge \left[\frac{x}{2}\right]$ is true. We have to prove that $\Pi_S(x+1) \ge \left[\frac{x+1}{2}\right]$.

Because $\Pi_S(x+1) = \Pi_S(x)$ it results $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]$.

So, we must have $\left[\frac{x}{2}\right] \ge \left[\frac{x+1}{2}\right]$. This is true, because $x+1$ is an odd number.

4) x is an odd composed number.

a) If $x+1$ is even composed number the proof is the same as in (2a).

For the right inequality we have :

(i) If $\Pi_S(x+1) = \Pi_S(x)+1$ and we suppose that $\Pi_S(x) \le x-[\sqrt{x}]$, let to prove that $\Pi_S(x+1) \le x+1-\left[\sqrt{x+1}\right]$.

This relation lead us to $\Pi_S(x) \le x-\left[\sqrt{x+1}\right]$. This is true because $[\sqrt{x}] = \left[\sqrt{x+1}\right]$.

(ii) If $\Pi_S(x+1) = \Pi_S(x)$ the proof is obvious.

b) If $x+1$ is a perfect square.

In this case according to the *Proposition 3* we have only the situation $\Pi_S(x+1) = \Pi_S(x)$. The right sided inequality is obvious and the left side inequality has the same proof as for (2a).

5) If x is a perfect square.

a) If x is a prime and the only situation is that $\Pi_S(x+1) = \Pi_S(x)+1$. The demonstration is obvious.

b) If $x+1$ is a composite number.

For the right inequality we have :

(i) If $\Pi_S(x+1) = \Pi_S(x+1)$, the proof is analogous as in the preceding case.

(ii) If $\Pi_S(x+1) = \Pi_S(x)$ the proof is obvious.

For the left inequality :

If $x+1$ is an odd composite number the relation is obvious.

If $x+1$ is an even composite number then :

if $\Pi_S(x+1) = \Pi_S(x)+1$, the proof is analogous with (a).

if $\Pi_S(x+1) = \Pi_S(x)$ then x can be just an odd perfect square.

We suppose that $\Pi_S(x) \ge \left[\frac{x}{2}\right]$ is true.

To show that $\Pi_S(x) \ge \left[\frac{x+1}{2}\right]$, if we suppose, again, that $\Pi_S(x) < \left[\frac{x+1}{2}\right]$

it results

$\left[\frac{x}{2}\right] \le \Pi_S(x) < \left[\frac{x+1}{2}\right]$, and we have $\Pi_S = \left[\frac{x}{2}\right]$.

Proposition 5.

$$\lim_{n\to\infty} [\ \Pi_S(2n) - \Pi_S(n)\ ] = \infty.$$

Proof.

According to the *Proposition 4* we have :

$\Pi_S(n) \le n - \left[\sqrt{n+1}\right] < n < \Pi_S(2n) \Rightarrow$

$\Pi_S(2n) - \Pi_S(n) > \left[\sqrt{n+1}\right]$ and $\lim_{n\to\infty} \left[\sqrt{n+1}\right] = \infty$.

Referencies

1) F. Smarandache. A function in the Number Theory, An. University of Timisoara, Ser. St. Mat. vol. XVII, fasc. 1 (1980)

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