THE FUNCTIONS $\theta_s(x)$ AND $\bar{\theta}_s(x)$

by V. Seleacu and Şt. Zanfir

Department of Mathematics, University of Craiova,
Craiova (1100), Romania

In this paper we define the function $\theta_s: N \setminus \{0,1\} \to N$ and $\bar{\theta}_s: N \setminus \{0,1,2\} \to N$ as follows :

$$\theta_s(x) = \sum_{\substack{p/x \\ 0<p\leq x \\ p-\text{prime}}} S(p^x); \quad \bar{\theta}_s(x) = \sum_{\substack{p \nmid x \\ 0<p\leq x \\ p-\text{prime}}} S(p^x),$$

where $S(p^x)$is the Smarandache function defined in [3] (S(n) is the smollest integer m such that m! is divisible by n).

For the begining we give some properties of the θ function. Let us observe that, from the definition of θ_s it results:

$\theta_s(2) = S(2^2) = 4$,
$\theta_s(3) = S(3^3) = 6$,
$\theta_s(4) = S(2^4) = 6$,
$\theta_s(5) = S(5^5) = 25$,
$\theta_s(6) = S(2^6) + S(3^6) = 7 + 15 = 22$,
$\theta_s(7) = S(7^7) = 49$,
$\theta_s(8) = S(2^8) = 10$,
$\theta_s(9) = S(3^9) = 21$,
$\theta_s(10) = S(2^{10}) + S(5^{10}) = 12 + 45 = 57$,
$\theta_s(11) = S(11^{11}) = 121$,
$\theta_s(12) = S(2^{17}) + S(3^{12}) = 43$.

We note also that if p-prime than $\theta_s(p^p) = p^2$.

Proposition 1. *The series* $\sum_{x\geq 2} (\theta_s(x))^{-1}$ *is convergent.*

Proof.
$$\sum_{x\geq 2} (\theta_s(x))^{-1} = \frac{1}{S(2^2)} + \frac{1}{S(3^3)} + \frac{1}{S(5^5)} + \frac{1}{S(2^6)+S(3^6)} +$$

$$+\frac{1}{S(7^7)} + \frac{1}{S(2^8)} + \frac{1}{S(3^9)} + \frac{1}{S(2^{10})+S(5^{10})} + \frac{1}{S(11^{11})} + \cdots$$

$$\leq \sum_{\substack{i=2 \\ p_i/x}}^{\infty} \left(\frac{1}{p_i^2} + \frac{1}{(p_{V(x)}-1)V(x)} \right) = \sum_{i=2}^{\infty} \frac{1}{p_i^2} + \sum_{p_{V(x)}/x} \frac{1}{(p_{V(x)}-1)V(x)},$$

where V(x) denote the number of the primes less or equal with x and divide by x.

Of course the series $\sum_{i=2}^{\infty}\frac{1}{p_i^2}$ and $\sum_{p_{V(x)}/x}\frac{1}{(p_{V(x)}-1)V(x)}$ are convergent, so the propositionis proved.

Proposition 2. *Let the sequence* $T(n)=1-\lg\theta_s(n)+\sum_{i=2}^{n}\frac{1}{\theta_s(i)}$. *Then* $\lim_{n\to\infty}T(n)=-\infty$.

The proof is imediate because the series $\sum_{n=2}^{\infty}\frac{1}{\theta_s(n)}$ is convergent according by the proposition 1.

Proposition 3. *The equation* $\theta_s(x)=\theta_s(x+1)$ (0) *has no solution if* x *is a prime.*

Proof. If x is a prime number the equation become

$x^2=\theta_s(x+1)$, where

$\theta_s(x+1)=S(p_{i_1}^{x+1})+S(p_{i_2}^{x+1})+\cdots+S(p_{i_{V(x+1)}}^{x+1})$.

Using the inequality

$$(p-1)\alpha<S(p^\alpha)\le p\alpha \tag{1}$$

given in [4], we have

$$\theta_s(x+1)\le(x+1)(p_{i_1}+p_{i_2}+\cdots+p_{i_{V(x+1)}})$$

Let us presume that the equation (0) has solution. We have the following relation:

$$x^2\le(x+1)(p_{i_1}+p_{i_2}+\cdots+p_{i_{V(x+1)}}) \tag{2}$$

and we prove that

$$p_{i_1}+p_{i_2}+\cdots+p_{i_{V(x+1)}}\le x-1 \tag{3}$$

for $x\ge 9$.

Let $\eta=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots\cdot p_r^{\alpha_r}$, $p_i\ne p_j$, $i\ne j$, the decomposition of n into primes. We define the function $f(n)=1+\alpha_1p_1+\cdots+\alpha_rp_r$ and we show that $f(n)\le n-2$ for $n\ge 9$. If $1\le n<9$ the precedent inequality is verified by calculus). For $n\ge 9$, we prove the inequality by induction:

$f(9) = 7, \ f(10) = 8, \ f(12) = 8 < 10$, true.

Now let us suppose that $f(n) \le n-2, \ \forall\, n \ge 12$, and we show that $f(n+1) \le n-1$.
In this case we have three different situations:
I) $n+1 = h = k_1 \cdot k_2$, where k_1, k_2 are composed members. Using the true relation, $f(h) = f(k_1 \cdot k_2) = f(k_1) + f(k_2) - 1$, we have

$$f(h) = f(n+1) = f(k_1) + f(k_2) - 1 \le k_1 - 2 + k_2 - 2 - 1 = k_1 \cdot k_2 - 8 - 5 \le$$
$$\le h - 2 = n + 1 - 2 = n - 1 \Rightarrow f(n+1) \le n - 1.$$

II) $n+1 = h = k_1 \cdot k_2$, where, k_1 – prime, k_2 – compounded,

$$f(h) = f(k_1) + f(k_2) - 1 \le k_1 + 1 + k_2 - 2 - 1 \le k_1 \cdot k_2 - 2 = n - 1.$$

III) $n+1 = h = k_1 \cdot k_2$, where k_1, k_2 – prime,

$$f(h) = 1 + k_1 + k_2 = k_1 k_2 + 2 - (k_1 - 1)(k_2 - 1) \le h + 2 - 4 = h - 2 = n - 1.$$

Conclusion: $f(n) \le n - 2, \ \forall\, n \ge 9$.
Then $f(n) \le n + 2 \Rightarrow 1 + \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r \le n - 2 \Rightarrow$

$$\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r \le n - 3.$$

We obtain

$$p_1 + p_2 + \cdots + p_r \le \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r \le n - 3 \le n - 2$$

Using (3) in (2) we have

$$x^2 \le (x+1)(x-1) \Rightarrow x^2 < x^2 - 1, \text{ imposible.}$$

Proposition 4. *The equation $\theta_s(x) = \theta_s(x+1)$ has no solution for $(x+1)$ - prime.*

Proof. We have $\theta_s(x+1) = (x+1)^2$.
We suppose that the equation has solution and with the inequalityes (1) is must that

$$(x+1)^2 \le x(p_{i_1}^x + p_{i_2}^x + \cdots + p_{i_{v(x)}}^x) \le x^2, \text{ imposible.}$$

We give some particular value for $\bar{\theta}_s(x) = \sum_{\substack{p^x | x \\ p-\text{prime}}} S(p^x)$;

$\bar{\theta}_s(3) = S(2^3) = 4$

$\bar{\theta}_s(4) = S(3^4) = 9$

$\bar{\theta}_s(5) = S(2^5) + S(3^5) = 20$

$\bar{\theta}_s(6) = S(5^6) = 30$

$\bar{\theta}_s(7) = S(2^7) + S(3^7) + S(5^7) =$
$=8+18+36=62$

$\bar{\theta}_s(8) = S(3^8) + S(5^8) + S(7^8) = 18 + 35 + 49 = 102$

$\bar{\theta}_s(9) = S(2^9) + S(5^9) + S(7^9) = 12 + 40 + 56 = 108$

$\bar{\theta}_s(10) = S(3^{10}) + S(7^{10}) = 24 + 63 = 87$

$\bar{\theta}_s(11) = S(2^{11}) + S(3^{11}) + S(5^{11}) + S(7^{11}) = 16 + 27 +$
$+50+73=163$

$\bar{\theta}_s(12) = S(2^{12}) + S(7^{12}) + S(11^{12}) = 50+77+$
$+121=248$

$\bar{\theta}_s(13) = S(2^{13}) + S(3^{13}) + S(5^{13}) + S(7^{13}) + S(11^{13}) =$
$=16+27+60+84+132=319.$

Proposition 5. *The series* $\sum_{x\geq 3}(\bar{\theta}_s(x))^{-1}$ *is convergent.*

Proof.

$$\sum_{x\geq 3}(\bar{\theta}_s(x))^{-1} = \frac{1}{S(2^3)} + \frac{1}{S(3^4)} + \frac{1}{S(2^5)+S(3^5)} + \frac{1}{S(2^8)+S(5^8)+S(7^8)} + \frac{1}{S(2^9)+S(5^9)+S(7^9)}$$

$$+\cdots \leq \frac{1}{S(2^3)} + \frac{1}{S(3^4)} + \frac{1}{S(5^6)} + \frac{1}{S(2^7)} + \cdots \leq \sum_{\substack{x\geq 3 \\ p_i \nmid x \\ p_i - \text{prime}}} \frac{1}{S(p_i^x)} \leq \sum_{\substack{x\geq 3 \\ p_i \nmid x \\ p_i - \text{prime}}} \frac{1}{p_i^2} \leq \sum_{x\geq 3}\frac{1}{x^2}.$$

Because the series $\sum_{x\geq 3}\frac{1}{x^2}$ is convergent, we have that our series is convergent.

Proposition 6. *If* $T(n) = 1 - \lg \bar{\theta}_s(n) + \sum_{i=3}^{n}\frac{1}{\bar{\theta}_s(i)}$ *then* $\lim_{n\to\infty} T(n) = -\infty$.

Proposition 7. *The equation* $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$ *has no solution if* x+1=p-*prime.*

Proof. If x+1 is prime be wouldn't divide with any of prime numbers then him

$$\bar{\theta}_s(x+1) = \sum_{\substack{p \nmid x+1 \\ 0<p\leq x+1}} S(p^{x+1}) = S(p_{i_1}^{x+1}) + S(p_{i_2}^{x+1}) + \cdots + S(p_{i_{\pi(x)}}^{x+1}).$$

The number x is divisible with at least two prime numbers then him. In the case $\bar{\theta}_s(x) = \sum_{\substack{p \nmid x \\ 0<p\leq x}} S(p^x)$ will have at least two terms $S(p_{i_k}^x)$ less then they are in $\bar{\theta}_s(x+1)$.

Moreover $S(p_i^x) \leq S(p_i^{x+1})$ and it results that $\bar{\theta}_s(x) < \bar{\theta}_s(x+1)$.

Proposition 8. *The equation* $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$ *has no solution if* x=p-prime, $x \geq 9$.

Proof. using the function $F_s(x) = \sum_{\substack{0<p\leq x \\ p-\text{prime}}} S(p^x)$ defined in [2] we have

$$F_s(x) = \theta_s(x) + \bar{\theta}_s(x)$$

$$F_s(x+1) = \theta_s(x+1) + \bar{\theta}_s(x+1).$$

If our equation have solution $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$ then

$$F_s(x) - F_s(x+1) = \theta_s(x) - \theta_s(x+1)$$

or

$$F_s(x) - F_s(x+1) = x^2 - \theta_s(x+1).$$

Is known [2] that $F_s(x) - F_s(x+1) < 0$. We have $x^2 - \theta_s(x+1) < 0 \Rightarrow x^2 < \theta_s(x+1)$.
Using (3) we have

$\theta_s(x+1) \leq (x+1)(x-1) = x^2 - 1$, therefore $x^2 < x^2 - 1$, imposible.

For x<9 is verified by calculus that the equation $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$ has not solution.

Proposed problem

1. $\theta_s(x) = \theta_s(x+1)$, x, x+1 are composed numbers.
2. $\bar{\theta}_s(x) = \bar{\theta}_s(x+1)$, x, x+1, are composed numbers.

Calculate

3. $\lim_{n\to\infty} \dfrac{\theta_s(n)}{n^\alpha}$, $\alpha \in \mathbf{R}$.

4. $\lim_{n\to\infty} \dfrac{\bar{\theta}_s(n)}{n^\alpha}$, $\alpha \in \mathbf{R}$.

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