The Integral Values of $\log_{k^n} S(n^k)$

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Abstract: Let k, n be distinct positive integers. In this paper, we prove that $\log_{k^n} S(n^k)$ is never a positive integer.

Key words: Smarandache function, logarithm, integral value.

For any positive integer a, let S(a) denote the Smarandache function of a. In [2, Problem 22], Muller posed the following problem:

Problem: Is it possible to find two distinct positive integers k and n such that $\log_{k^n} S(n^k)$ is a positive integer?

In this paper, we completely solve the above problem as follows:

Theorem: For any distinct positive integers k and n, $\log_{k^n} S(n^k)$ is never a positive integer.

Proof: If $\log_{k^n} S(n^k)$ is a positive integer, then we have $k > 1$, $n > 1$ and

(1) $\log_{k^n} S(n^k) = m$,

where m is a positive integer. By (1), we get

(2) $S(n^k) = k^{nm}$.

By (1), we have

(3) $S(n^k) = S(n^{k-1} * n) \leq S(n^{k-1}) + S(n) \leq \ldots kS(n)$.

Therefore, by (2) and (3), we get

(4) $$k^{nm} \leq kS(n) \leq kn.$$

If $k > n > 1$, then from (4) we obtain

(5) $$k^2 \leq k^n \leq k^{nm} \leq kn \leq k(k-1) < k^2$$

a contradiction. If $n > k > 1$, then we have

(6) $$2^n \leq k^n \leq k^{nm} \leq kn \leq (n-1)n.$$

It is impossible, since $n \geq 3$. Thus, the theorem is proved.

References

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