THE MONOTONY OF SMARANDACHE FUNCTIONS OF FIRST KIND

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Smarandache functions of first kind are defined in [1] thus:

$$S_n : N^* \to N^*, \quad S_1(k) = 1 \text{ and } S_n(k) = \max_{1 \le j \le r} \{S_{p_j}(i_j k)\},$$

where $n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r}$ and S_{p_j} are functions defined in [4].

They Σ_1 - standardise $(N^*,+)$ in $(N^*,\le,+)$ in the sense that

$$\Sigma_1 : \quad \max\{S_n(a), S_n(b)\} \le S_n(a+b) \le S_n(a) + S_n(b)$$

for every $a,b \in N^*$ and Σ_2 - standardise $(N^*,+)$ in $(N^*,\le,\cdot)$ by

$$\Sigma_2 : \quad \max\{S_n(a), S_n(b)\} \le S_n(a+b) \le S_n(a) \cdot S_n(b), \text{ for every } a,b \in N^*$$

In [2] it is prooved that the functions S_n are increasing and the sequence $\{S_{p^i}\}_{i \in N^*}$ is also increasing. It is also proved that if p,q are prime numbers, then

$$p \cdot i < q \Rightarrow S_{p^i} < S_q \text{ and } i < q \Rightarrow S_i < S_q,$$

where $i \in N^*$.

It would be used in this paper the formula

$$S_p(k) = p(k - i_k), \text{ for same } i_k \text{ satisfying } 0 \le i_k \le \left[\frac{k-1}{p}\right], \text{ (see [3])} \qquad (1)$$

1. Proposition. *Let* p *be a prime number and* $k_1, k_2 \in N^*$. *If* $k_1 < k_2$ *then* $i_{k_1} \le i_{k_2}$, *where* i_{k_1}, i_{k_2} *are defined by* (1).

Proof. It is known that $S_p : \mathrm{N}^* \to \mathrm{N}^*$ and $S_p(k) = pk$ for $k \le p$. If $S_p(k) = mp^\alpha$ with $m, \alpha \in \mathrm{N}^*, (m,p) = 1$, there exist α consecutive numbers:

$n, n+1, \ldots, n+\alpha-1$ so that
$k \in \{n, n+1, \ldots, n+\alpha-1\}$ and
$S_p(n) = S_p(n+1) = \cdots = S(n+\alpha-1)$,

this means that S_p is stationed the $\alpha-1$ steps $(k \to k+1)$.

If $k_1 < k_2$ and $S_p(k_1) = S_p(k_2)$, because $S_p(k_1) = p(k_1 - ik_1)$, $S_p(k_2) = p(k_2 - ik_2)$ it results $i_{k_1} < i_{k_2}$.

If $k_1 < k_2$ and $S_p(k_1) < S_p(k_2)$, it is easy to see that we can write:

$$i_{k_1} = \beta_1 + \sum_{\substack{\alpha \\ mp^\alpha < S_p(k_1)}} (\alpha - 1), \quad \text{where} \quad \beta_1 = 0 \text{ for } S_p(k_1) \neq mp^\alpha, \quad \text{if } S_p(k_1) = mp^\alpha$$

then $\beta_1 \in \{0,1,2,\dots,\alpha-1\}$

and

$$i_{k_2} = \beta_2 + \sum_{\substack{\alpha \\ mp^\alpha < S_p(k_2)}} (\alpha - 1), \quad \text{where} \quad \beta_2 = 0 \text{ for } S_p(k_2) \neq mp^\alpha, \text{ if } S_p(k_2) = mp^\alpha \text{ then}$$

$\beta_2 \in \{0,1,2,\dots,\alpha-1\}$.

Now is obviously that $k_1 < k_2$ and $S_p(k_1) < S_p(k_2) \Rightarrow i_{k_1} \le i_{k_2}$. We note that, for $k_1 < k_2$, $i_{k_1} = i_{k_2}$ iff $S_p(k_1) < S_p(k_2)$ and $\{mp^\alpha \mid \alpha > 1 \text{ and } mp^\alpha \le S_p(k_1)\} = \{mp^\alpha \mid \alpha > 1 \text{ and } mp^\alpha < S_p(k_2)\}$

2. Proposition. *If p is a prime number and $p \ge 5$, then $S_p > S_{p-1}$ and $S_p > S_{p+1}$.*

Proof. Because $p-1 < p$ it results that $S_{p-1} < S_p$. Of course $p+1$ is even and so:

(i) if $p+1 = 2^i$, then $i > 2$ and because $2i < 2^i - 1 = p$ we have $S_{p+1} < S_p$.

(ii) if $p+1 \neq 2^i$, let $p+1 = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r}$, then $S_{p+1}(k) = \max\limits_{1 \le j \le r} \{S_{p_j^{i_j}}(k)\} = S_{p_m^{i_m}}(k) =$

$= S_{p_m}(i_m \cdot k)$.

Because $p_m \cdot i_m \le p_m^{i_m} \le \dfrac{p+1}{2} < p$ it results that $S_{p_m^{i_m}}(k) < S_p(k)$ for $k \in N^*$, so that $S_{p+1} < S_p$.

3. Proposition. *Let p,q be prime numbers and the sequences of functions*

$$\{S_{p^i}\}_{i \in N^*}, \quad \{S_{q^j}\}_{j \in N^*}$$

If $p < q$ and $i \le j$, then $S_{p^i} < S_{q^j}$.

Proof. Evidently, if $p < q$ and $i \le j$, then for every $k \in N^*$

$$S_{p^i}(k) \le S_{p^j}(k) < S_{q^j}(k)$$

so,

$$S_{p^i} < S_{q^j}$$

4. Definition. *Let p,q be prime numbers. We consider a function S_{q^j}, a sequence of functions $\{S_{p^i}\}_{i \in N^*}$, and we note:*

$$i_{(j)} = \max_i \left\{ i \mid S_{p^i} < S_{q^j} \right\}$$

$$i^{(j)} = \min_i \left\{ i \middle| S_{q^j} < S_{p^i} \right\},$$

then $\{k \in N | i_{(j)} < k < i^{(j)}\} = \Delta_{p^j(q^j)} = \Delta_{i(j)}$ *defines the interference zone of the function* S_{q^j} *with the sequence* $\left\{S_{p^i}\right\}_{i \in N^*}$.

5. ***Remarque.***

a) If $S_{q^j} < S_{p^i}$ for $i \in \mathbf{N}^*$, then now exists $i_{(j)}$ and $i^{(j)} = 1$, and we say that S_{q^j} is separately of the sequence of functions $\left\{S_{p^i}\right\}_{i \in N^*}$.

b) If there exist $k \in \mathbf{N}^*$ so that $S_{p^k} < S_{q^j} < S_{p^{k+1}}$, then $\Delta_{p^j(q^j)} = \varnothing$ and say that the function S_{q^j} does not interfere with the sequence of functions $\left\{S_{p^i}\right\}_{i \in N^*}$.

6. **Definition**. *The sequence* $\{x_n\}_{n \in N^*}$ *is generaly increasing if*

$$\forall n \in N^* \ \exists m_0 \in N^* \text{ so that } x_m \geq x_n \text{ for } m \geq m_0.$$

7. ***Remarque***. If the sequence $\{x_n\}_{n \in N^*}$ with $x_n \geq 0$ is generaly increasing and boundled, then every subsequence is generaly increasing and boundled.

8. **Proposition**. *The sequence* $\{S_n(k)\}_{n \in N^*}$, *where* $k \in N^*$, *is in generaly increasing and boundled.*

Proof. Because $S_n(k) = S_{n^k}(1)$, it results that $\{S_n(k)\}_{n \in N^*}$ is a subsequence of $\{S_m(1)\}_{m \in N^*}$.

The sequence $\{S_m(1)\}_{m \in N^*}$ is generaly increasing and boundled because:

$$\forall m \in N^* \ \exists t_0 = m! \text{ so that } \forall t \geq t_0 \ S_t(1) \geq S_{t_0}(1) = m \geq S_m(1).$$

From the remarque 7 it results that the sequence $\{S_n(k)\}_{n \in N^*}$ is generaly increasing boundled.

9. **Proposition**. *The sequence of functions* $\{S_n\}_{n \in N^*}$ *is generaly increasing boundled.*

Proof. Obviously, the zone of interference of the function S_m with $\{S_n\}_{n \in N^*}$ is the set

$$\Delta_{n(m)} = \{k \in N^* | n_{(m)} < k < n^{(m)}\} \text{ where}$$

$$n_{(m)} = \max\{n \in N^* | S_n < S_m\}$$

$$n^{(m)} = \min\{n \in N^* | S_m < S_n\}.$$

The interference zone $\Delta_{n(m)}$ is nonemty because $S_m \in \Delta_{n(m)}$ and finite for $S_1 \le S_m \le S_p$, where p is one prime number greater than m.

Because $\{S_n(1)\}$ is generaly increasing it results:

$$\forall m \in N^* \ \exists t_0 \in N^* \text{ so that } S_t(1) \ge S_m(1) \text{ for } \forall t \ge t_0.$$

For $r_0 = t_0 + n^{(m)}$ we have

$$S_r \ge S_m \ge S_m(1) \text{ for } \forall r \ge r_0,$$

so that $\{S_n\}_{n \in N^*}$ is generaly increasing boundled.

10. ***Remarque.***

a) For $n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r}$ are posible the following cases:

1) $\exists\, k \in \{1,2,\ldots,r\}$ so that

$$S_{p_j^{i_j}} \le S_{p_k^{i_k}} \text{ for } j \in \{1,2,\ldots,r\},$$

then $S_n = S_{p_k^{i_k}}$ and $p_k^{i_k}$ is named the dominant factor for n.

2) $\exists\, k_1, k_2, \ldots, k_m \in \{1,2,\ldots,r\}$ so that :

$\forall t \in \overline{1,m} \quad \exists q_t \in N^*$ so that $S_n(q_t) = S_{p_{k_t}^{i_{k_t}}}(q_t)$ and

$$\forall\, l \in N^* \quad S_n(l) = \max_{1 \le t \le m} \left\{ S_{p_{k_t}^{i_{k_t}}}(l) \right\}.$$

We shall name $\{p_{k_t}^{i_{k_t}} \,|\, t \in \overline{1,m}\}$ the active factors, the others wold be name passive factors for n.

b) We consider

$$N_{p_1 p_2} = \{n = p_1^{i_1} \cdot p_2^{i_2} \,|\, i_1, i_2 \in N^*\}, \text{ where } p_1 < p_2 \text{ are prime numbers.}$$

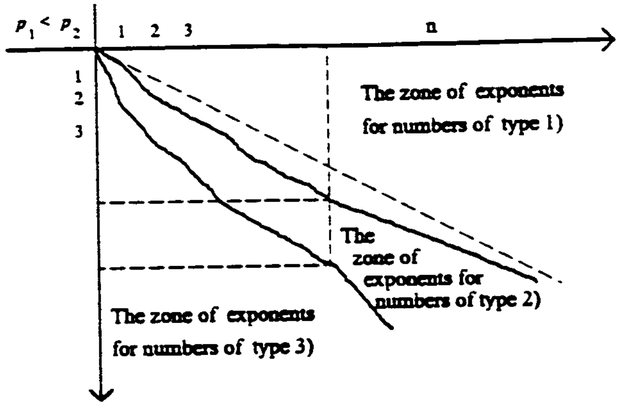
For $n \in N_{p_1 p_2}$ appear the following situations:

1) $i_1 \in (0, i_1^{(i_2)}]$, this means that $p_1^{i_1}$ is a pasive factor and $p_2^{i_2}$ is an active factor.

2) $i_1 \in (i_{1(i_2)}, i_1^{(i_2)})$ this means that $p_1^{i_1}$ and $p_2^{i_2}$ are active factors.

3) $i_1 \in [i_1^{(i_2)}, \infty)$ this means that $p_1^{i_1}$ is a active factor and $p_2^{i_2}$ is a pasive factor.

For $p_1 < p_2$ the repartion of exponents is represently in following scheme:



For numbers of type 2) $i_1 \in (i_{1(i_2)}, i_1^{(i_2)})$ and $i_2 \in (i_{2(i_1)}, i_2^{(i_1)})$

c) I consider that

$$N_{p_1 p_2 p_3} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} \mid i_1, i_2, i_3 \in N^*\},$$

where $p_1 < p_2 < p_3$ are prime numbers.

Exist the following situations:

1) $n \in N^{p_j}, j = 1,2,3$ this means that $p_j^{i_j}$ is active factor.

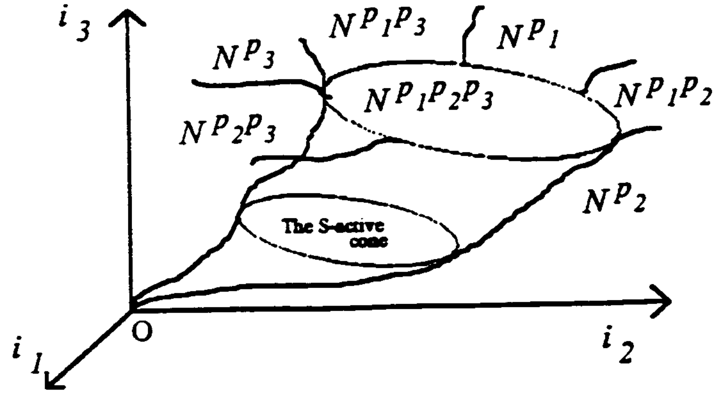
2) $n \in N^{p_j p_k}, j \neq k;\ j,k \in \{1,2,3\}$, this means that $p_j^{i_j}, p_k^{i_k}$ are active factors.

3) $n \in N^{p_1 p_2 p_3}$, this means that $p_1^{i_1}, p_2^{i_2}, p_3^{i_3}$ are active factors. $N^{p_1 p_2 p_3}$ is named the S-active cone for $N_{p_1 p_2 p_3}$.

Obviously

$$N^{p_1 p_2 p_3} = \{n = p_1^{i_1} p_2^{i_2} p_3^{i_3} \mid i_1, i_2, i_3 \in N^* \text{ and } i_k \in (i_{k(i_j)}, i_k^{(i_j)}) \text{ where } j \neq k; j,k \in \{1,2,3\}\}.$$

The repartision of exponents is represented in the following scheme:



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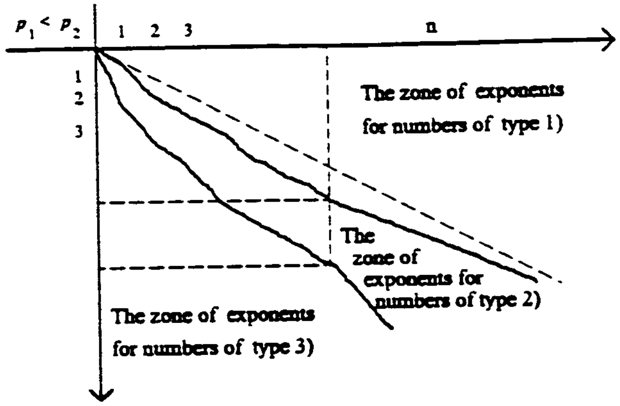

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c) I consider that

$$N_{p_1 p_2 p_3} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} \mid i_1, i_2, i_3 \in N^*\},$$

where $p_1 < p_2 < p_3$ are prime numbers.

Exist the following situations:

1) $n \in N^{p_j}, j = 1,2,3$ this means that $p_j^{i_j}$ is active factor.

2) $n \in N^{p_j p_k}, j \neq k;\ j,k \in \{1,2,3\}$, this means that $p_j^{i_j}, p_k^{i_k}$ are active factors.

3) $n \in N^{p_1 p_2 p_3}$, this means that $p_1^{i_1}, p_2^{i_2}, p_3^{i_3}$ are active factors. $N^{p_1 p_2 p_3}$ is named the S-active cone for $N_{p_1 p_2 p_3}$.

Obviously

$$N^{p_1 p_2 p_3} = \{n = p_1^{i_1} p_2^{i_2} p_3^{i_3} \mid i_1, i_2, i_3 \in N^* \text{ and } i_k \in (i_{k(i_j)}, i_k^{(i_j)}) \text{ where } j \neq k; j,k \in \{1,2,3\}\}.$$

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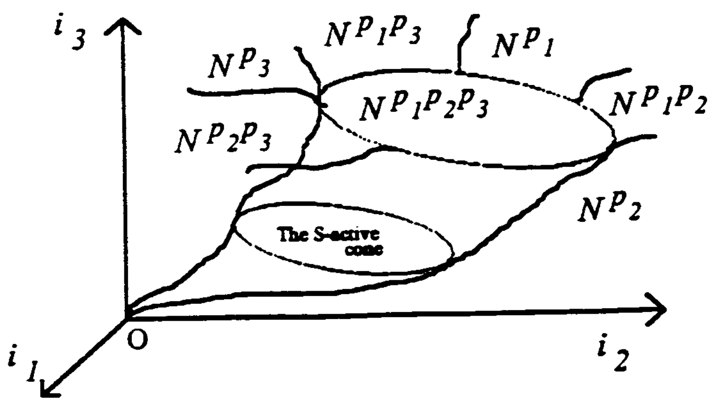

d) Generaly, I consider $N_{p_1 p_2 \cdots p_r} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot \cdots \cdot p_r^{i_r} \mid i_1, i_2, \ldots, i_r \in N^*\}$, where $p_1 < p_2 < \cdots < p_r$ are prime numbers.

On $N_{p_1 p_2 \cdots p_r}$ exist the following relation of equivalence:

$$n \rho m \Leftrightarrow n \text{ and } m \text{ have the same active factors.}$$

This have the following clases:

- $N^{p_{j_1}}$, where $j_1 \in \{1,2,\ldots,r\}$.

$n \in N^{p_{j_1}} \Leftrightarrow n$ hase only $p_{j_1}^{i_{j_1}}$ active factor

- $N^{p_{j_1} p_{j_2}}$, where $j_1 \neq j_2$ and $j_1, j_2 \in \{1,2,\ldots,r\}$.

$n \in N^{p_{j_1} p_{j_2}} \Leftrightarrow n$ has only $p_{j_1}^{i_{j_1}}$, $p_{j_2}^{i_{j_2}}$ active factors.

. . . .

$N^{p_1 p_2 \cdots p_r}$ wich is named S-active cone.

$N^{p_1 p_2 \cdots p_r} = \{n \in N_{p_1 p_2 \cdots p_r} \mid n \text{ has } p_1^{i_1}, p_2^{i_2}, \ldots, p_r^{i_r} \text{ active factors}\}$.

Obviously, if $n \in N^{p_1 p_2 \cdots p_r}$, then $i_k \in (i_{k(i_j)}, i_k^{(i_j)})$ with $k \neq j$ and $k, j \in \{1,2,\ldots,r\}$.

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