The notions of the SMARANDACHE GROUP and the SMARANDACHE BOOLEAN RING

D. Talukdar.
Deptt. of Mathematics
Nalbari College, Nalbari
Assam, India

The notions of the Snmarandache group and the Smarandache Boolean ring are introduced here with the help of group action and ring action i.e. module respectively. The centre of the Smarandache groupoid is determined. These are very important for the study of Algebraic structures.

1. The centre of the Smarandache groupoid :

Definition 1.1

An element a of the smarandache groupoid $(\mathbb{Z}p, \Delta)$ is said to be conjugate to b if there exists r in $\mathbb{Z}p$ such that $a = r \Delta b \Delta r$.

Definition 1.2

An element a of the smarandache groupoid $(\mathbb{Z}p, \Delta)$ is called a self conjugate element of $\mathbb{Z}p$ if $a = r \Delta a \Delta r$ for all $r \in \mathbb{Z}p$.

Definition 1.3

The set $\mathbb{Z}p^*$ of all self conjugate elements of $(\mathbb{Z}p, \Delta)$ is called the centre of $\mathbb{Z}p$ i.e.

$$\mathbb{Z}p^* = \{ a \in \mathbb{Z}p : a = r \Delta a \Delta r \ \forall\, r \in \mathbb{Z}p \}.$$

Definition 1.4

Let $(\mathbb{Z}_9, +_9)$ be a commutative group, then $\mathbb{Z}_9=\{ 0, 1, 2, 3, 4, 5, 6, 7, 8 \}$. If the elements of $\mathbb{Z}_9$ are written as 3-adic numbers, then

$$\mathbb{Z}_9 = \{ (00)_3, (01)_3, (02)_3, (10)_3, (11)_3, (12)_3, (20)_3, (21)_3, (22)_3 \}$$

and $(\mathbb{Z}_9, \Delta)$ is a smarandache groupoid of order 9. Conjugacy relations among the elements of $\mathbb{Z}_9$ are determined as follows :

$0 \Delta 0 \Delta 0 = 0$	$0 \Delta 1 \Delta 0 = 1$	$0 \Delta 2 \Delta 0 = 2$	$0 \Delta 3 \Delta 0 = 3$	$0 \Delta 4 \Delta 0 = 4$
$1 \Delta 0 \Delta 1 = 0$	$1 \Delta 1 \Delta 1 = 1$	$1 \Delta 2 \Delta 1 = 0$	$1 \Delta 3 \Delta 1 = 3$	$1 \Delta 4 \Delta 1 = 4$
$2 \Delta 0 \Delta 2 = 0$	$2 \Delta 1 \Delta 2 = 1$	$2 \Delta 2 \Delta 2 = 2$	$2 \Delta 3 \Delta 2 = 3$	$2 \Delta 4 \Delta 2 = 4$
$3 \Delta 0 \Delta 3 = 0$	$3 \Delta 1 \Delta 3 = 1$	$3 \Delta 2 \Delta 3 = 2$	$3 \Delta 3 \Delta 3 = 3$	$3 \Delta 4 \Delta 3 = 4$
$4 \Delta 0 \Delta 4 = 0$	$4 \Delta 1 \Delta 4 = 1$	$4 \Delta 2 \Delta 4 = 0$	$4 \Delta 3 \Delta 4 = 3$	$4 \Delta 4 \Delta 4 = 4$
$5 \Delta 0 \Delta 5 = 0$	$5 \Delta 1 \Delta 5 = 1$	$5 \Delta 2 \Delta 5 = 2$	$5 \Delta 3 \Delta 5 = 3$	$5 \Delta 4 \Delta 5 = 4$
$6 \Delta 0 \Delta 6 = 0$	$6 \Delta 1 \Delta 6 = 1$	$6 \Delta 2 \Delta 6 = 2$	$6 \Delta 3 \Delta 6 = 3$	$6 \Delta 4 \Delta 6 = 4$
$7 \Delta 0 \Delta 7 = 0$	$7 \Delta 1 \Delta 7 = 1$	$7 \Delta 2 \Delta 7 = 0$	$7 \Delta 3 \Delta 7 = 3$	$7 \Delta 4 \Delta 7 = 4$
$8 \Delta 0 \Delta 8 = 0$	$8 \Delta 1 \Delta 8 = 1$	$8 \Delta 2 \Delta 8 = 2$	$8 \Delta 3 \Delta 8 = 3$	$8 \Delta 4 \Delta 8 = 4$
$0 \Delta 5 \Delta 0 = 5$	$0 \Delta 6 \Delta 0 = 6$	$0 \Delta 7 \Delta 0 = 7$	$0 \Delta 8 \Delta 0 = 8$	
$1 \Delta 5 \Delta 1 = 3$	$1 \Delta 6 \Delta 1 = 6$	$1 \Delta 7 \Delta 1 = 7$	$1 \Delta 8 \Delta 1 = 6$	
$2 \Delta 5 \Delta 2 = 5$	$2 \Delta 6 \Delta 2 = 6$	$2 \Delta 7 \Delta 2 = 7$	$2 \Delta 8 \Delta 2 = 8$	
$3 \Delta 5 \Delta 3 = 5$	$3 \Delta 6 \Delta 3 = 0$	$3 \Delta 7 \Delta 3 = 1$	$3 \Delta 8 \Delta 3 = 2$	
$4 \Delta 5 \Delta 4 = 3$	$4 \Delta 6 \Delta 4 = 0$	$4 \Delta 7 \Delta 4 = 1$	$4 \Delta 8 \Delta 4 = 0$	
$5 \Delta 5 \Delta 5 = 5$	$5 \Delta 6 \Delta 5 = 0$	$5 \Delta 7 \Delta 5 = 1$	$5 \Delta 8 \Delta 5 = 2$	
$6 \Delta 5 \Delta 6 = 5$	$6 \Delta 6 \Delta 6 = 6$	$6 \Delta 7 \Delta 6 = 7$	$6 \Delta 8 \Delta 6 = 8$	
$7 \Delta 5 \Delta 7 = 3$	$7 \Delta 6 \Delta 7 = 6$	$7 \Delta 7 \Delta 7 = 7$	$7 \Delta 8 \Delta 7 = 6$	
$8 \Delta 5 \Delta 8 = 5$	$8 \Delta 6 \Delta 8 = 6$	$8 \Delta 7 \Delta 8 = 7$	$8 \Delta 8 \Delta 8 = 8$	

Table - 1

Here $\mathbf{Z}_9^* = \{ 0, 1, 3, 4 \}$, the set of all self conjugate elements of $\mathbf{Z}_9$ is called the centre of $(\mathbf{Z}_9, \Delta)$. Again $(\mathbf{Z}_9^*, \Delta)$ is an abelian group.

D-form of the Smarandache groupoid $(\mathbf{Z}_9, \Delta)$ is given by $D_8 = \{ 0, 2, 6, 8 \}$. Again (D_8, Δ) is an abelian group. The group table (2) and group table (3) are given below.

Δ	0	1	3	4
0	0	1	3	4
1	1	0	4	3
3	3	4	0	1
4	4	3	1	0

Table - 2

Δ	0	2	6	8
0	0	2	6	8
2	2	0	8	6
6	6	8	0	2
8	8	6	2	0

Table - 3

From table (2) and table (3); it is clear that $(\mathbf{Z}_9^*, \Delta)$ and (D_8, Δ) are isomorphic to each other.

Definition 1.3

The groups $(\mathbb{Z}p^*, \Delta)$ and $(D_{\sup(\mathbb{Z}p)}, \Delta)$ of the Smarandache groupoid $(\mathbb{Z}p, \Delta)$ are isomorphic to each other.

Proof is obvious.

2. The Smarandache group :

To introduce the Smarandache group, we have to explain group action on a set. Let A be a group and B a set. An action of A on B is a map, $B \times A \to B$ written $(b, p) \to b \Delta P$ such that

i) for every $p, q \in A$ and $b \in B$, we have
$((b, p), q) = ((b \Delta p) \Delta q) = (b \Delta p) \Delta q = b \Delta (p \Delta q)$

and ii) for every $b \in B$, we have $(b, 0) = b \Delta 0 = b$
where 0 denotes the identity element of the group A.

If a group A has an action on B, we say that B is a A-set or A-space. Here in this paper we shall use B(A) in place of "B is a A - space".

Note :

If B is a proper subgroup of A, then we get a map $A \times B \to A$ defined by $(a, b) \to a \Delta b \in A$. This is a group action of B on A. Then we say that A is a B - set or B - space i.e. A(B) is a B - space. In this paper, by proper subgroup, we mean a group contained in A, different from the trivial groups.

Definition 2.1

The smarandache group is defined to be a group A such that A(B) is a B - space, where B is a proper subgroup of A

Examples 2.2

i) The D - form of $(\mathbb{Z}p, \Delta)$ defined by
$D_{\sup(\mathbb{Z}p)} = \{ r \in \mathbb{Z}p : r \Delta C(r) = \mathrm{Sup}(\mathbb{Z}p)\} = A$ is a Smarandache group. If B is a proper subgroup of A, then the action of B on A is the map, $A \times B \to A$ defined by $(a, q) = a \Delta q$ for all $a \in A$ and $q \in B$. This action is a B - action i.e. A(B) is a B - space.

ii) The centre of $(\mathbb{Z}p, \Delta)$ defined by
$\mathbb{Z}p^* = \{ a \in \mathbb{Z}p : a = r \Delta a \Delta r \quad \forall r \in \mathbb{Z}p\} = A$ is a Smarandache group. If B be a proper subgroup of A, then the action of B on A is the map, $A \times B \to A$ defined by $(a, p) = a \Delta p$ for all $a \in A$ and for all $p \in B$. This action is a B - space i.e. A(B) is a B - space.

iii) The Addition modulo m of two integers $r = (a_{n-1}a_{n-2} \text{------------------} a_1a_0)_m$ and $s = (b_{n-1}b_{n-2} \text{---------} b_1b_0)_m$ denoted by $/+_m\backslash$ and defined as

$$\begin{aligned} r /+_m\backslash s &= (a_{n-1}a_{n-2} \cdots \cdots a_1a_0)_m /+_m\backslash (b_{n-1}b_{n-2} \cdots \cdots b_1b_0)_m \\ &= (a_{n-1}/+_m\backslash b_{n-1}\ a_{n-2}/+_m\backslash b_{n-2} \cdots \cdots a_1/+_m\backslash b_1\ a_0/+_m\backslash b_0)_m \\ &= (c_{n-1}\ c_{n-2} \cdots \cdots c_1c_0)_m, \quad \text{where } c_i = a_i/+_m\backslash b_i \quad \text{for } i = 0, 1, 2, \ldots n-1. \end{aligned}$$

The group ($\mathbf{Z}p$, $/+_m\backslash$) is a Smarandache group. The group ($\mathbf{Z}p$, $/+_p\backslash$) contains a proper subgroup ($H = \{ 0, 1, 2, 3, \ldots \ldots \ldots p-1\}, /+_m\backslash$)

Then the action of H on $\mathbf{Z}p$ is the map $\mathbf{Z}p \times H \rightarrow \mathbf{Z}p$ defined by $(a, r) = a/+_m\backslash r$ for all $a \in \mathbf{Z}p$ and for all $r \in H$. This action is a H - space i.e. $\mathbf{Z}p(H)$ is a H - space.

iv) The set $\mathbf{Z}_{16} = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 \}$ can be written as

$$\mathbf{Z}_{16} = \{(00)_4, (01)_4, (02)_4, (03)_4, (10)_4, (11)_4, (12)_4, (13)_4, (20)_4, (21)_4, (22)_4, (23)_4, (30)_4, (31)_4, (32)_4, (33)_4 \}$$

is a smarandache group under the operation $/+_4\backslash$ and its group table is as follows:

$/+_4\backslash$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	0	5	6	7	4	9	10	11	8	13	14	15	12
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	0	1	2	7	4	5	6	11	8	9	10	15	12	13	14
4	4	5	6	7	8	9	10	11	12	13	14	15	0	1	2	3
5	5	6	7	4	9	10	11	8	13	14	15	12	1	2	3	0
6	6	7	4	5	10	11	8	9	14	15	12	13	2	3	0	1
7	7	4	5	6	11	8	9	10	15	12	13	14	3	0	1	2
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	10	11	8	13	14	15	12	1	2	3	0	5	6	7	4
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	8	9	10	15	12	13	14	3	0	1	2	7	4	5	6
12	12	13	14	15	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	15	12	1	2	3	0	5	6	7	4	9	10	11	8
14	14	15	12	13	2	3	0	1	6	7	4	5	10	11	8	9
15	15	12	13	14	3	0	1	2	7	4	5	6	11	8	9	10

Table - 4

From above group table, it is clear that ($H = \{ 0, 1, 2, 3, 8, 9, 10, 11 \}, /+_4\backslash$) is a subgroup of ($\mathbf{Z}_{16}$, $/+_4\backslash$). Then the action of H on $\mathbf{Z}_{16}$ is the map $\mathbf{Z}_{16} \times H \rightarrow \mathbf{Z}_{16}$ defined by $(a, r) = a/+_4\backslash r$ for all $a \in \mathbf{Z}_{16}$ and for all $r \in H$. This action is a H - space i.e. $\mathbf{Z}_{16}(H)$ is a H - space.

Here ($K = \{ 0, 1, 2, 3 \}, /+_4\backslash$) be a subgroup of ($H$, $/+_4\backslash$). Then the action of K on H is the map, $H \times K \rightarrow H$ defined by $(b, p) = b/+_4\backslash p$ for all $b \in H$ and for all $p \in K$. This action is a K - space i.e. H(K) is a K - space. Hence H is a Smarandache group contained in the Smarandache group $\mathbf{Z}_{16}$. So it is called the Smarandache sub-group.

3. The Smarandache sub-group :

Definition 3.1

The smarandache sub-group is defined to be a smarandache group B which is a proper subset of smarandache group A (with respect to the same induced operation).

4. Smarandache quotient group :

Let (H, Δ) be smarandache subgroup of the Smarandache group (B, Δ), then the quotient group $^{B}/_{H} = V$ is defined as smarandache quotient group such that V(K) is a K - space, where K is a proper subgroup of V i.e. the group action of K on V is a map $V \times K \to V$, defined by

$$((H \Delta a), H \Delta p) = (H \Delta a) \Delta (H \Delta p) \quad \text{for all} \quad H \Delta a \in V \quad \text{and} \quad H \Delta p \in K$$

5. Smarandache Boolean ring :

Definition 5.1

The intersection of two integers $r = (a_{n-1}a_{n-2} \quad \dots \quad \dots \quad a_1 a_0)_m$ and $s = (b_{n-1}b_{n-2} \quad \dots \quad \dots \quad b_1 b_0)_m$ denoted by $r \cap s$ and defined as

$$r \cap s = (a_{n-1} \cap b_{n-1}\, a_{n-2} \cap b_{n-2} \quad \dots \quad \dots \quad a_1 \cap b_1\, a_0 \cap b_0)_m$$
$$= (c_{n-1}c_{n-2} \quad \dots \quad \dots \quad c_1 c_0)_m$$

where $c_i = a_i \cap b_i = \min(a_i, b_i)$ for $i = 0, 1, 2, \quad \dots \quad \dots \quad, n-1$

If the equivalence classes of are expressed as m - adic numbers, then with binary operation $\cap$ is a groupoid, which contains some non trivial groups. This groupoid is smarandache groupoid. Here $(\mathbf{Z}p^*, \Delta, \cap)$ and $(\mathbf{Z}_{\sup(Zp)}, \Delta, \cap)$ are Boolean ring.

Definition 5.2

The smarandache Boolean ring is defined to be a Boolean ring A such that the Abelian group (A, Δ) has both left and right B - module, where B is a non trivial sub-ring of A.

From above, we mean an Abelian group (A, Δ) together with a map, $B \times A \to A$, written $(b, p) = b \cap p \in A$ such that for every $b, c \in B$ and $p, q \in A$, we have

i) $b \cap (p \Delta q) = (b \cap p) \Delta (b \cap q)$

ii) $(b \Delta c) \cap p = (b \cap p) \Delta (c \cap p)$ iii) $(b \cap c) \cap p = b \cap (c \cap p)$

Again from the map, $A \times B \to A$, written $(p, b) = p \cap b \in A$ such that for every $p, q \in A$ and $b, c \in B$, we get

i) $(p \Delta q) \cap b = (p \cap b) \Delta (q \cap b)$

ii) $p \cap (b \Delta c) = (p \cap b) \Delta (p \cap c)$ iii) $p \cap (b \cap c) = (p \cap b) \cap c$

Definition 5.3

The smarandache Boolean sub-ring is defined to be a smrandache Boolean ring B which is a proper subset of a Smarandache Boolean ring A. (with respect to the same induced operation).

Definition 5.4

The smarandache Boolean ideal is defined to be an ideal B of Smarandache Boolean ring A such that the Abelian group (C, Δ) has both left and right B - module, where C is a proper subset of B. From above we mean an Abelian group (C, Δ) together with a map, $C \times B \to C$ written $(c, p) = C \cap P \in C$ such that this mapping satisfies all the postulates of both left and right B - module.

Examples 5.5

Let $(\mathbf{Z}_{256}, +_{256})$ be an Abelian group, then $\mathbf{Z}_{256} = \{ 0, 1, 2, \dots \dots \dots 255 \}$. If the elements $\mathbf{Z}_{256}$ of are written as 4 - adic numbers, then

$\mathbf{Z}_{256} = \{ (0000)_4, (0001)_4, (0002)_4, (0003)_4, (0010)_4, \dots \dots \dots \dots, (3333)_4 \}$and $(\mathbf{Z}_{256}, \Delta)$ is a smarandache groupoid of order 256. The centre of $(\mathbf{Z}_{256}, \Delta)$ is

$\mathbf{Z}^{*}_{256} = \{ 0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, 81, 84, 85 \}$. Here $(\mathbf{Z}^{*}_{256}, \Delta, \cap)$ is a Smarandache Boolean ring and its composition tables are given below :

Δ	0	1	4	5	16	17	20	21	64	65	68	69	80	81	84	85
0	0	1	4	5	16	17	20	21	64	65	68	69	80	81	84	85
1	1	0	5	4	17	16	21	20	65	64	69	68	81	80	85	84
4	4	5	0	1	20	21	16	17	68	69	64	65	84	85	80	81
5	5	4	1	0	21	20	17	16	69	68	65	64	85	84	81	80
16	16	17	20	21	0	1	4	5	80	81	84	85	64	65	68	69
17	17	16	21	20	1	0	5	4	81	80	85	84	65	64	69	68
20	20	21	16	17	4	5	0	1	84	85	80	81	68	69	64	65
21	21	20	17	16	5	4	1	0	85	84	81	80	69	68	65	64
64	64	65	68	69	80	81	84	85	0	1	4	5	16	17	20	21
65	65	64	69	68	81	80	85	84	1	0	5	4	17	16	21	20
68	68	69	64	65	84	85	80	81	4	5	0	1	20	21	16	17
69	69	68	65	64	85	84	81	80	5	4	1	0	21	20	17	16
80	80	81	84	85	64	65	68	69	16	17	20	21	0	1	4	5
81	81	80	85	84	65	64	69	68	17	16	21	20	1	0	5	4
84	84	85	80	81	68	69	64	65	20	21	16	17	4	5	0	1
85	85	84	81	80	69	68	65	64	21	20	17	16	5	4	1	0

Table - 5

$\cap$	0	1	4	5	16	17	20	21	64	65	68	69	80	81	84	85
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
4	0	0	4	4	0	0	4	4	0	0	4	4	0	0	4	4
5	0	1	4	5	0	1	4	5	0	1	4	5	0	1	4	5
16	0	0	0	0	16	16	16	16	0	0	0	0	16	16	16	16
17	0	1	0	1	16	17	16	17	0	1	0	1	16	17	16	17
20	0	0	4	4	16	16	20	20	0	0	4	4	16	16	20	20
21	0	1	4	5	16	17	20	21	0	1	4	5	16	17	20	21
64	0	0	0	0	0	0	0	0	64	64	64	64	64	64	64	64
65	0	1	0	1	0	1	0	1	64	65	64	65	64	65	64	65
68	0	0	4	4	0	0	4	4	64	64	68	68	64	64	68	68
69	0	1	4	5	0	1	4	5	64	65	68	69	64	65	68	69
80	0	0	0	0	16	16	16	16	64	64	64	64	80	80	80	80
81	0	1	0	1	16	17	16	17	64	65	64	65	80	81	80	81
84	0	0	4	4	16	16	20	20	64	64	68	68	80	80	84	84
85	0	1	4	5	16	17	20	21	64	65	68	69	80	81	84	85

Table - 6

Some smarandache Boolean sub-rings of $(\mathbb{Z}^{*}_{256}, \Delta, \cap)$ are given below :

i) $H_1 = \{ 0, 1, 4, 5, 16, 17, 20, 21 \}$
$H_2 = \{ 0, 1, 4, 5, 64, 65, 68, 69 \}$
$H_3 = \{ 0, 1, 4, 5, 80, 81, 84, 85 \}$
$H_4 = \{ 0, 5, 16, 21, 64, 69, 80, 85 \}$
$H_5 = \{ 0, 1, 16, 17, 64, 65, 80, 81 \}$
$H_6 = \{ 0, 1, 4, 5 \}$
$H_7 = \{ 0, 1, 16, 17 \}$
$H_8 = \{ 0, 1, 64, 65 \}$
$H_9 = \{ 0, 1, 80, 81 \}$ etc.

Here smarandache Boolean subrings H_1, H_2, H_6, H_7, H_8 are ideals of $(\mathbb{Z}^{*}_{256}, \Delta)$.

References :

1. David M. Burton - Elementary number theory 2nd edition, University book stall New Delhi (1994).

2. Mc Coy, N.H - Introduction to Modern Algebra Boston Allyu and Bacon INC (1965)

3. Talukdar, D & Das N.R.- Measuring associativity in a groupoid of natural numbers The Mathematical Gazette Vol. 80. No.- 488 (1996), 401 - 404

4. Talukdar, D - Some Aspects of inexact groupoids J. Assam Science Society 37(2) (1996), 83 - 91

5. Talukdar, D - A Klein 2^n - group, a generalization of Klein 4 group GUMA Bulletin Vol. 1 (1994), 69 - 79

6. Hall, M - The theory of groups Macmillan Co. 1959.

7. Talukdar, D - Klein 2^n – group Action on a set of Fuzzy Subsets. The Journal of Fuzzy Mathematics. Vol. 6, 1998.