THE SEMILATTICE WITH CONSISTENT RETURN

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Let p be a prime number. In [5] is defined the function S_p as $S_p:N^* \to N^*, S_p(a)=k$, where k is the smallest positive integer so that p^a is a divizor for $k!$.

A Smarandache function of first kind is defined for each $n \in N^*$ in [1], as numerical function $S_n:N^* \to N^*$, so that:

i) if $n = u^i$, where $u=1$ or $u=p$, then $S_n(a)=k$, k being the smallest positive integer with the property that $k! = M \cdot u^{ia}$.

ii) if $n = p_1^{i_1} \cdot p_2^{i_2} \cdot \dots \cdot p_r^{i_r}$, then $$S_n(a) = \max_{1 \le j \le r} \left\{ S_{p_j}(i_j a) \right\}.$$

It is proved that:

Σ_1 $$\max\left\{S_n(a), S_n(b)\right\} \le S_n(a+b) \le S_n(a) + S_n(b)$$

Σ_2 $$S_n(a+b) \le S_n(a) \cdot S_n(b)$$

In [2] is proved that:

i) the function S_n is monotonously increasing,

ii) the sequence of functions $\left\{S_{p^i}\right\}_{i \in N^*}$ is monotonously increasing.

iii) for p, q - prime numbers such that: $p < q \Rightarrow S_p < S_q$ and $p \cdot i < q \Rightarrow S_{p^i} < S_q$, where $i \in N^*$

iv) if $n<p$, then $S_n < S_p$.

In [3] it is proved:

i) for $p \ge 5$, $S_p > \max\left\{S_{p-1}, S_{p+1}\right\}$

ii) for p, q - prime numbers, $i, j \in N^*$

$$p < q \text{ and } i \le j \Rightarrow S_{p^i} < S_{q^j}$$

iii) the sequence of functions $\{S_n\}_{n \in N^*}$ is generaly increasing boundled

iv) if $n = p_1^{i_1} \cdot p_2^{i_2} \cdot \dots \cdot p_r^{i_r}$, there are $k_1, k_2, \dots, k_m \in \{1, 2, \dots, r\}$ so that for each $t \in \overline{1,m}$ there is $q_t \in N^*$ so that

$$S_n(q_t) = S_{p_{k_t}^{i_{k_t}}}(q_t)$$

and for each $l \in N^*$ we have:

$$S_n(l) = \max_{1 \le t \le m} \left\{ S_{p_{k_t}^{i_{k_t}}}(l) \right\}.$$

We define the set $\left\{ p_{k_t}^{i_{k_t}} \middle| t \in \overline{1,m} \right\}$ as the set of active factors of n and the others factors as the pasive factors.

Let $N_{p_1 \cdot p_2 \cdots p_r} = \left\{ n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r} \middle| i_1, i_2, \ldots, i_r \in N^* \right\}$, where $p_1 < p_2 < \cdots < p_r$ are prime numbers. Then

$$N^{p_1 p_2 \ldots p_r} = \left\{ n \in N_{p_1 p_2 \ldots p_r} \middle| n \text{ has } p_1^{i_1}, p_2^{i_2}, \ldots, p_r^{i_r} \text{ as active factors} \right\}$$

is the S-active cone.

A Smarandache function of second kind is defined for each $k \in N^*$ in [1], as the function $S^k : N^* \to N^*$ where $S^k(n) = S_n(k)$.

It is proved that:

Σ_3 $$\max\left\{ S^k(a), S^k(b) \right\} \le S^k(a \cdot b) \le S^k(a) + S^k(b)$$

Σ_4 $$S^k(a \cdot b) \le S^k(a) \cdot S^k(b)$$

In [4] it is proved that:

i) for $k, n \in N^*$ the formula $S^k(n) \le n \cdot k$ is true

ii) all prime numbers $p \ge 5$ are maximal points for S^k and

$$S^k(p) = p\left[k - i_p(k)\right], \quad \text{where} \quad 0 \le i_p(k) \le \left[\frac{k-1}{p} \right]$$

iii) the function S^k has its relative minimum values for every $n = p!$, where p is a prime number and $p \ge \max\{3, k\}$

iv) the numbers kp for p prime number, $k \in N^*$ and $p > k$, are the fixed points of S^k

v) the function S^k have the following properties:

a) $S^k = 0 \quad (n^{1+\varepsilon})$, for $\varepsilon > 0$

b) $\limsup_{n \to \infty} \dfrac{S^k(n)}{n} = k$

c) S^k is, "generally speaking", incresing, thus:

$$\forall n \in N^*, \exists m_0 \in N \text{ so that } \forall m \geq m_0 \Rightarrow S^k(m) \geq S^k(n)$$

1. **DEFINITION.** *Let* $\mathcal{M} = \{S_m(n) | n, m \in N^*\}$, *let* $A, B \in \mathcal{P}(N^*) \setminus \varnothing$ *and* $a = \min A$, $b = \min B, a^* = \max A, b^* = \max B$. *The set I is the set of the functions:*

$$I_A^B : N^* \to \mathcal{M}, \text{ with } I_A^B(n) = \begin{cases} S_a(b), n < \max\{a,b\} \\ S_{a_k}(b_k), \max\{a,b\} \leq n \leq \max\{a^k, b^k\} \\ \quad \text{where} \\ \quad a_k = \max_i \{a_i \in A | a_i \leq n\} \\ \quad b_k = \max_j \{b_j \in B | b_j \leq n\} \\ S_{a^*}(b^*), n > \max\{a^*, b^*\} \end{cases}$$

2. **EXAMPLES.**

a) $I_{\{3,8,10\}}^{\{6,10,12\}} : N^* \to \mathcal{M}$ and:

n	1	2	3	4	5	6	7	8	9	10	11	12	$n \geq 13$
$I_{\{3,8,10\}}^{\{6,10,12\}}$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_8(6)$	$S_8(6)$	$S_{10}(10)$	$S_{10}(10)$	$S_{10}(12)$	$S_{10}(12)$

b) Let $A = \{1,3,5,\ldots,2k+1,\ldots\}$
$B = \{2,4,6,\ldots,2k,\ldots\}$

$I_A^B : N^* \to \mathcal{M}$ and:

n	1	2	3	4	5	6	...	2k	2k+1	...
I_A^B	$S_2(1)$	$S_2(1)$	$S_2(3)$	$S_4(3)$	$S_4(5)$	$S_6(5)$	...	$S_{2k}(2k-1)$	$S_{2k}(2k+1)$	...

c) Let $A = \{5,9,10\}$ and $I_A^A, I_{N^*}^{N^*} : N^* \to \mathcal{M}$ with

n	1	2	3	4	5	6	7	8	9	10	$n\geq 11$
I_A^A	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_9(9)$	$S_{10}(10)$	$S_{10}(10)$
$I_{N^*}^{N^*}$	$S_1(1)$	$S_2(2)$	$S_3(3)$	$S_4(4)$	$S_5(5)$	$S_6(6)$	$S_7(7)$	$S_8(8)$	$S_9(9)$	$S_{10}(10)$	$S_n(n)$

It is easy to see that I_A^A is not the reduction of $I_{N^*}^{N^*}$ and $I_A^A(N^*)\subset I_{N^*}^{N^*}(N^*)$.

3. **REMARK.**

The functions whitch belongs to the set I have the folowing properties :

1) if $A_1\subset A_2$ and $n\in A_1$, then $I_{A_1}^B(n)=I_{A_2}^B(n)$

1') if $B_1\subset B_2$ and $n\in B_1$, then $I_A^{B_1}(n)=I_A^{B_2}(n)$

2) $I_{N^*}^{N^*}(n)=S_n(n)=S^n(n)$, the function $I_{N^*}^{N^*}$ is called the I - diagonal function and $I_{N^*}^{N^*}(N^*)$ is called the diagonal of $\mathscr{M}$.

3) for each $m\in N^*$ $I_{\{m\}}^{N^*}=S_m$ for $I_{\{m\}}^{N^*}(n)=S_m(n),\forall n\in N^*$.

3') for each $m\in N^*$ $I_{N^*}^{\{m\}}=S^m$ for $I_{N^*}^{\{m\}}(n)=S_n(m)=S^m(n),\forall n\in N^*$,

4) if $n\in A\cap B$, then $I_A^B(n)=I_{\{n\}}^{\{n\}}(n)=S_n(n)$.

4. **DEFINITION.** *For each pair $m,n\in N^*$, $S_m(n)$ and $S^m(n)$ are called the simetrical numbers relative to the diagonal of $\mathscr{M}$.*

S_m and S^m are called the simmetrical functions relative to the I-diagonal function $I_{N^*}^{N^*}$. As a rule, I_A^B and I_B^A are called the simmetrical functions relative to the I-diagonal function $I_{N^*}^{N^*}$.

5. **DEFINITION.** *Let us consider the following rule $\top : I\times I\to I$, $I_A^B\top I_C^D=I_{A\cup C}^{B\cup D}$. It is easy to see that $\top$ is idempotent, commutative and associative, so that:*

i) $I_A^B\top I_A^B=I_A^B$

ii) $I_A^B\top I_C^D=I_C^D\top I_A^B$

iii) $\left(I_A^B\top I_C^D\right)\top I_E^F=I_A^B\top\left(I_C^D\top I_E^F\right)$, where $A,B,C,D,E,F\in\mathscr{P}(N^*)\setminus\varnothing$

6. **DEFINITION.** *Let us consider the following relative partial order relation ρ, where:*

$$\rho\subset I\times I,$$
$$I_A^B\,\rho\, I_C^D \Leftrightarrow A\subset C \text{ and } B\subset D.$$

It is easy to see that $(I,\top,\rho)$ is a semilattice.

7. **DEFINITION.** *The elements* $u, v \in I$ *are* ρ *- preceded if there is* $w \in I$ *so that:*

$$w \, \rho \, u \text{ and } w \, \rho \, v.$$

8. **DEFINITION.** *The elements* $u, v \in I$*, are* ρ *- strictly preceded by* w *if:*

i) $w \, \rho \, u$ *and* $w \, \rho \, v$.

ii) $\forall x \in I \setminus \{w\}$ *so that* $x \, \rho \, u$ *and* $x \, \rho \, v \Rightarrow x \, \rho \, w$.

9. **DEFINITION.** *Let us defined:*

$$I^* = \{(u,v) \in I \times I | u, v \text{ are } \rho\text{-preceded}\}$$
$$I^\# = \{(u,v) \in I \times I | u, v \text{ are } \rho\text{-strictly preceded}\}.$$

It is evidently that $(u,v) \in I^* \Leftrightarrow (v,u) \in I^*$ and $(u,v) \in I^\# \Leftrightarrow (v,u) \in I^\#$.

10. **DEFINITION.** *Let us consider* $\mathrm{T}^\# = U \times U$, $U \subset I$ and let us consider the following rule: $\perp : I^\# \to W$, $W \subset I$, $I_A^B \perp I_C^D = I_{A \wedge C}^{B \wedge D}$ and the ordering partial relation $r \subset U \times U$ so that $I_A^B \, r \, I_C^D \Leftrightarrow I_C^D \, \rho \, I_A^B$.

The structure $(I^\#, \perp, r)$ is called the return of semilattice (I, T, ρ).

11. **DEFINITION.** *The following set*

$$\mathcal{B} = \{I_A^B \in I | A \cap B \neq \varnothing\}$$

is called the base of return $(I^\#, \perp, r)$.

12. **REMARK.** The base of return has the following properties:

i) if $I_A^B \in \mathcal{B} \Rightarrow I_B^A \in \mathcal{B}$

ii) for $\varnothing \neq X \subset N^*, I_X^X \in \mathcal{B}$

iii) for $I_A^B \in \mathcal{B}$ is true the following equivalence $\varnothing \neq X \subset C_{N^*}(A \wedge B) \Leftrightarrow$ non existence of $I_X^X \perp I_A^B$.

13. **PROPOSITION.** *For* $I_A^B \in \mathcal{B}$ *there exists* $n \in N^*$ *so that* $I_A^B(n) = I_{N^*}^{N^*}(n)$.

Proof. Because $A \cap B \neq \varnothing$ it results that there exists $n \in A \wedge B$ so that:

$$I_A^B(n) = S_n(n) = I_{N^*}^{N^*}(n).$$

It results that for $I_A^B \in \mathcal{B}$ then I_A^B has at least a point of contact with I-diagonal function.

14. **REMARK.** From the 1. it results:

$$I_{\{n\}}^{B}(n)=S_n(b_n), \text{ where } b_n=\begin{cases} b,\ n<b=\min B \\ b_k,\ b\le n\le b^*=\max B \\ \qquad \text{where} \\ \quad b_k=\max\{x\in B|x\le n\} \\ b^*\ ,n>b^* \end{cases}$$

and

$$I_{A}^{\{m\}}(m)=S^m(a_m), \text{ where } a_m=\begin{cases} a,\ m<a=\min A \\ a_k,\ a\le m\le a^*=\max A \\ \qquad \text{where} \\ \quad a_k=\max\{x\in A|x\le m\} \\ a^*\ ,m>a^* \end{cases}$$

15. **PROPOSITION.** *There are true the following equivalences:*

$$\left(I_A^B,I_C^D\right)\in I^{\#}\Leftrightarrow I_A^C,I_B^D\in\mathcal{B}\Leftrightarrow\exists n,m\in N^* \text{ so that:}$$

$I_A^B(n)=I_{\{n\}}^B(n)=S_n(b_n)$, $I_C^D(n)=I_{\{n\}}^D(n)=S_n(d_n)$, $I_A^B(m)=I_A^{\{m\}}(m)=S^m(a_m)$, and $I_C^D(m)=I_C^{\{m\}}(m)=S^m(c_m)$ where a_m,b_n,c_m,d_n are defined in the sense of 14.

If $n\le m$, then $n\le a_m$, $c_m\le m$.

Proof. Evidently,
$\left(I_A^B,I_C^D\right)\in I^{\#}\Leftrightarrow A\cap C\ne\varnothing$ and $B\cap D\ne\varnothing\Leftrightarrow I_A^C,I_B^D\in\mathcal{B}$.

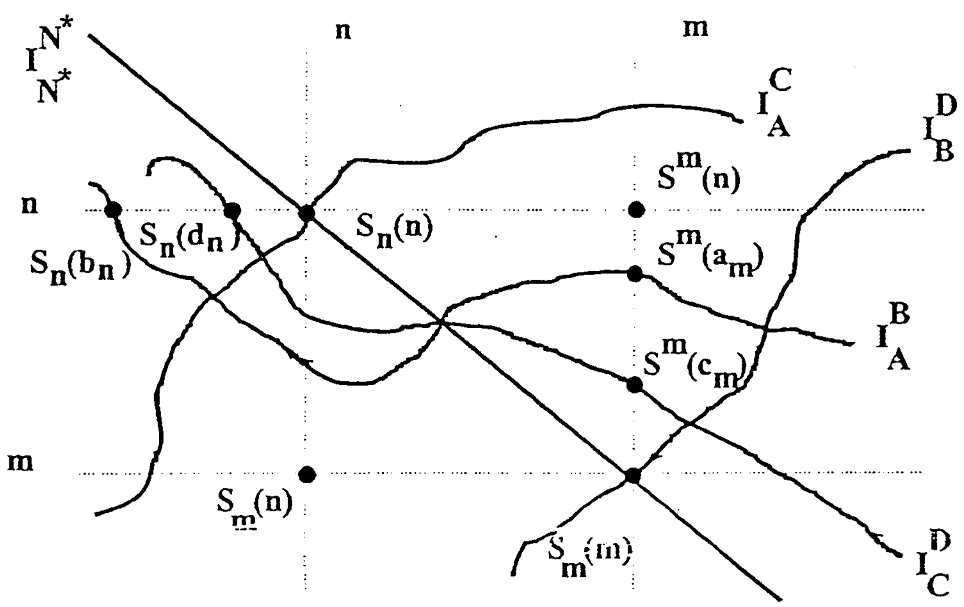
Because $A\cap C\ne\varnothing$ and $B\cap D\ne\varnothing$ it exists $n\in A\cap C$ and $m\in B\cap D$. Then:
$I_A^B(n)=I_{\{n\}}^B(n)=S_n(b_n)$, $I_C^D(n)=I_{\{n\}}^D(n)=S_n(d_n)$
$I_A^B(m)=I_A^{\{m\}}(m)=S^m(a_m)$, $I_C^D(m)=I_C^{\{m\}}(m)=S^m(c_m)$.

Conversely, if there exist $n\in N^*$ so that $I_A^B(n)=S_n(b_n)$ and $I_C^D(n)=S_n(d_n)$, then because $I_A^B(n)=S_n(b_n)$ it results $n=a_k=\max\limits_i\{a_i\in A|a_i\le n\}$, so that $n\in A$. Because $I_C^D(n)=S_n(d_n)$ it results $n\in C$.

Therefore $A\cap C\ne\varnothing$, thus, finally, $I_A^C\in\mathcal{B}$. It is also proved $I_B^D\in\mathcal{B}$ in the some way.

If $n\le m$, because $n\in A\cap C$ it results that $n\in\{x\in A|x\le m\}$ and $n\in\{y\in C|y\le m\}$, therefore $n\le a_m\le m$ and $n\le c_m\le m$.

This is presented in the following scheme:



16. **DEFINITION**. *The return* $(L^{\#},\perp,r)$ *of semillatice* $(L,\top,\rho)$ *is:*

a) null, ,if $L^{\#}=\{(u,u)|u\in L\}=\Delta_L$.
b) weak, if $\text{card}L^{\#}<\text{card}(L\times L\setminus L^{\#})$
c) consistent, if $\text{card}L^{\#}=\text{card}(L\times L-L^{\#})$
d) vigour, if $\text{card}L^{\#}>\text{card}(L\times L-L^{\#})$
e) total, if $L^{\#}=L\times L$.

17. **PROPOSITION**. *The return* $(I^{\#},\perp,r)$ *of the semilattice* $(I,\top,\rho)$ *is consistent.*

Proof. Evidently, $\text{card}(\mathcal{P}(N^*)\setminus\varnothing)=\aleph$, $\text{card}\,I=\text{card}\left[(\mathcal{P}(N^*)-\varnothing)\times(\mathcal{P}(N^*)-\varnothing)\right]=\aleph$ and $\text{card}(I\times I)=\aleph$.

Let us consider $\mathcal{F}=\{(A,C)|A,C\in\mathcal{P}(N^*)-\varnothing,A\cap C=\varnothing\}$ and $\overline{\mathcal{F}}=\{(A,C)|A,C\in\mathcal{P}(N^*)-\varnothing,$ $A\cap C\neq\varnothing\}$.

$\text{card}\mathcal{F}=\text{card}\overline{\mathcal{F}}=\aleph$. Indeed, if $A\cap C=\varnothing$ it results that $C_N\cdot A\cup C_N\cdot C=N^*$; because for every $X\in P(N^*)-\varnothing$ $\exists Y=N^*\setminus X$ so that $X\cup Y=N^*$ then it results $\text{card}\mathcal{F}=\text{card}\mathcal{P}(N^*)=\aleph$. Because for each (A,C), $A,C\in\mathcal{P}(N^*)-\varnothing,A\cap C=\varnothing$, it exist at least two $(A_1,C_1),(A_2,C_2)$ with $A_1\cap C_1\neq\varnothing,A_2\cap C_2\neq\varnothing$ it results $\text{card}\overline{\mathcal{F}}\geq\text{card}\mathcal{F}=\aleph$.

Since $\text{card}\overline{\mathscr{F}} \leq \text{card}\left[(\mathscr{P}(N^*)-\varnothing)\times(\mathscr{P}(N^*)-\varnothing)\right]=\aleph$ finally $\text{card}\overline{\mathscr{F}}=\aleph$. Because $\text{card}I^\# = \text{card}(\overline{\mathscr{F}}\times\overline{\mathscr{F}})=\aleph$ and $\text{card}(I\times I)-I^\# = \text{card}(\mathscr{F}\times\mathscr{F})=\aleph$ it results that $(I^\#,\perp,r)$ is a return consistent.

18. **REMARK.** Generaly, it is interesting the folowing problems:

i) what relations, operations, structures can be defined on

$$M=\left\{S_m(n)\middle| n,m\in N^*\right\}?$$

ii) what relations, operations, structures can be defined on

$$\mathscr{H}=\{f\,|\,f:N^*\to\mathscr{M}\}?$$

REFERENCES

[1] I. Bălăcenoiu, *Smarandache Numerical Functions*, S.F.J. vol.4, 1994, p.6-13.

[2] I. Bălăcenoiu, V. Seleacu, *Some properties od Smarandache Functions of the type I*, S.F.J., vol.6, 1995, p.16-20.

[3] I. Bălăcenoiu, *The Monotony of Smarandache Functions of First Kind*, S.N.J, vol.7, 1996, p.39-44.

[4] I. Bălăcenoiu, C. Dumitrescu, *Smarandache Functions of the second kind*, S.F.J., vol.6, 1995, p.55-58.

[5] F. Smarandache, *A function in the Numbers Theory*, An.Univ.Timişoara, seria st.mat., vol. XVIII, fasc.1, p.79-88, 1980.