The sequence of prime numbers

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This article lets out a law of recurrence in order to obtain the sequence of prime numbers $\{p_k\}_{k\geq 1}$ expressing p_{k+1} as a function of $p_1, p_2, \cdots, p_k$.

Suppose we can find a function $G_k(n)$ with the following property:

$$G_k(n) = \begin{cases} -1 & if \quad n < p_{k+1} \\ 0 & if \quad n = p_{k+1} \\ something & if \quad n > p_{k+1} \end{cases}$$

This is a variation of the Smarandache Prime Function [2].

Then we can write down a recurrence formula for p_k as follows.

Consider the product:

$$\prod_{s=p_k+1}^{m} G_k(s)$$

If $p_k < m < p_{k+1}$ one has

$$\prod_{s=p_k+1}^{m} G_k(s) = \prod_{s=p_k+1}^{m} (-1) = (-1)^{m-p_k}$$

If $m \geq p_{k+1}$

$$\prod_{s=p_k+1}^{m} G_k(s) = 0$$

since $G_k(p_{k+1}) = 0$

Hence

$$\sum_{m=p_k+1}^{2p_k} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s) =$$

$$= \sum_{m=p_k+1}^{p_{k+1}-1} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s) + \sum_{m=p_{k+1}}^{2p_k} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s)$$

(The second addition is zero since all the products we have the factor $G_k(p_{k+1}) = 0$)

$$= \sum_{m=p_k+1}^{p_{k+1}-1} (-1)^{m-p_k}(-1)^{m-p_k}$$

$$= p_{k+1} - 1 - (p_k + 1) + 1 = p_{k+1} - p_k - 1$$

so

$$p_{k+1} = p_k + 1 + \sum_{m=p_k+1}^{2p_k} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s)$$

which is a recurrence relation for p_k.

We now show how to find such a function $G_k(n)$ whose definition depends only on the first k primes and not on an explicit knowledge of p_{k+1}.

And to do so we define[1]:

$$T_k(n) = \sum_{i_1=0}^{\log_{p_1} n} \sum_{i_2=0}^{\log_{p_2} n} \cdots \sum_{i_k=0}^{\log_{p_k} n} \left(\begin{array}{c} n \\ \prod_{s=1}^{k} p_s^{i_s} \end{array} \right)$$

Let's see the value which $T_k(n)$ takes for all $n \geq 2$ integer. We distinguish two cases:

Case 1: $n < p_{k+1}$

The expression $p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$ with $i_1 = 0, 1, 2 \cdots \log_{p_1} n$ $i_2 = 0, 1, 2 \cdots \log_{p_2} n$ ... $i_k = 0, 1, 2 \cdots \log_{p_k} n$ all the values occur $1, 2, 3, \cdots, n$ each one of them only once and moreover some more values, strictly greater than n.

We can look at is. If $1 \leq m \leq n$ one obtains that $m < p_{k+1}$ for which $1 \leq m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \leq n$. From where one deduces that $1 \leq p_s^{\alpha_s} \leq n$ and for it $0 \leq \alpha_s \leq \log_{p_s} n$ for all $s = 1, \cdots, k$

Therefore, for $i_s = \alpha_s \quad s = 1, 2, \cdots, k$ we have the value m. This value only appears once, the prime number descomposition of m is unique .

In fact the sums of $T_k(n)$ can be achieved up to the highest power of p_k contained in n instead of $\log_{p_k} n$.

Therefore one has that

$$T_k(n) = \sum_{i_1=0}^{\log_{p_1} n} \sum_{i_2=0}^{\log_{p_2} n} \cdots \sum_{i_k=0}^{\log_{p_k} n} \left(\begin{array}{c} n \\ \prod_{s=1}^{k} p_s^{i_s} \end{array} \right) = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n - 1$$

[1]Given that $i_s \quad s = 1, 2, \cdots, k$ only takes integer values one appreciates that the sums of $T_k(n)$ are until $E(\log_{p_s} n)$ where $E(x)$ is the greatest integer less than or equal to x.

since, in the case $p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$ would be greater than n one has that:

$$\binom{n}{\prod_{s=1}^{k} p_s^{i_s}} = 0$$

Case 2: $n = p_{k+1}$

The expression $p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$ with $i_1 = 0, 1, 2 \cdots \log_{p_1} n$ $i_2 = 0, 1, 2 \cdots \log_{p_2} n$ ... $i_k = 0, 1, 2 \cdots \log_{p_k} n$ the values occur $1, 2, 3, \cdots, p_{k+1} - 1$ each one of them only once and moreover some more values, strictly greater than p_{k+1}.One demonstrates in a form similar to case 1. It doesn't take the value p_{k+1} since it is coprime with $p_1, p_2, \cdots, p_k$.

Therefore,

$$T_k(n) = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} = 2^n - 2$$

In case 3: $n > p_{k+1}$ it is not necessary to consider it.

Therefore, one has:

$$T_k(n) = \begin{cases} 2^n - 1 & if \quad n < p_{k+1} \\ 2^n - 2 & if \quad n = p_{k+1} \\ something & if \quad n > p_{k+1} \end{cases}$$

and as a result:

$$G_k(n) = 2^n - 2 - T_k(n)$$

This is the summarized relation of recurrence:

Let's take $p_1 = 2$ and for $k \geq 1$ we define:

$$T_k(n) = \sum_{i_1=0}^{\log_{p_1} n} \sum_{i_2=0}^{\log_{p_2} n} \cdots \sum_{i_k=0}^{\log_{p_k} n} \binom{n}{\prod_{s=1}^{k} p_s^{i_s}}$$

$$G_k(n) = 2^n - 2 - T_k(n)$$

$$p_{k+1} = p_k + 1 + \sum_{m=p_k+1}^{2p_k} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s)$$

References:

(1) The Smarandache Notions Journal. Volume 11. Number 1-2-3. Page 59.

(2) E. Burton, "Smarandache Prime and Coprime Functions",
http://www.gallup.unm.edu/~smarandache/primfnct.txt

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