THE SMARANDACHE FRIENDLY NATURAL NUMBER PAIRS

Maohua Le

Abstract. In this paper we completely determinate all the Smarandache friendly natural number pairs.

Key words: Smarandache friendly natural number pair, Pell equation, positive integer solution

Let **Z**, **N** be the sets of all integers and positive integers respectively. Let a, b be two positive integers with $a<b$. Then the pair (a, b) is called a Smarandache friendly natural number pair if

(1) $$a+(a+1)+\cdots\cdots+b=ab.$$

For example, (1, 1), (3, 6), (15, 35), (85, 204) are Smarandache friendly natural number pairs. In [2], Murthy showed that there exist infinitely many such pairs. In this paper we shall completely determinate all Smarandache friendly natural number pairs.

Let

(2) $$\alpha=1+\sqrt{2},\quad \beta=1-\sqrt{2}.$$

For any positive integer n, let

(3) $$P(n)=\frac{1}{2}\left(\alpha^n+\beta^n\right),\ Q(n)=\frac{1}{2\sqrt{2}}\left(\alpha^n-\beta^n\right)$$

Notice that $1+\sqrt{2}$ and $\left(1+\sqrt{2}\right)^2=3+2\sqrt{2}$ are the fundamental solutions of Pell equations

(4) $$x^2-2y^2=-1, x, y \in \mathbf{N},$$

and

(5) $$x^2-2y^2=1, x, y \in \mathbf{N},$$

respectively. By [1, Chapter 8], we obtain the following two lemmas immediately.

Lemma 1. All solutions (x, y) of (4) are given by

(6) $$x=P(2m+1), y=Q(2m+1), \quad m \in \mathbf{Z}, m \geqslant 0.$$

Lemma 2. All solutions (x, y) of (5) are given by

(7) $$x=P(2m), y=Q(2m), m \in \mathbf{N}.$$

We now prove a general result as follows.

Theorem. If (a, b) is a Smarandache friendly natural number pair, then either

(8) $$a=(P(2m+1)+2Q(2m+1))Q(2m+1),$$
$$b=(P(2m+1)+2Q(2m+1))(P(2m+1)+Q(2m+1)), m \in Z, m \geq 0$$

or

(9) $$a=(P(2m)+Q(2m))P(2m),$$
$$b=(P(2m)+Q(2m))(P(2m)+2Q(2m)), m \in N.$$

Proof. Let (a, b) be a Smarandache friendly natural number pair. Since

(10) $$a+(a+1)+\ldots+b=(1+2+\ldots+b)-(1+2+\ldots+(a-1))$$
$$=\frac{1}{2}b(b+1)-\frac{1}{2}a(a-1)=\frac{1}{2}(b+a)(b-a+1),$$

we get from (1) that

(11) $$(b+a)(b-a+1)=2ab.$$

Let $d=\gcd(a, b)$. Then we have

(12) $$a=da_1, \quad b=db_1,$$

where a_1, b_1 are positive integers satisfying

(13) $$a_1 < b_1, \gcd(a_1, b_1)=1.$$

Substitute (12) into (11), we get

(14) $$(b_1+a_1)(d(b_1-a_1)+1)=2da_1b_1.$$

Since $\gcd(a_1, b_1)=1$ by (13), we get $\gcd(a_1b_1, a_1+b_1)=1$. Similarly, we have $\gcd(d, d(b_1-a_1)+1)=1$. Hence, we get from (14) that

(15) $$d \mid b_1+a_1, \quad a_1b_1 \mid d(b_1-a_1)+1.$$

Therefore, by (14) and (15), we obtain either

(16) $$b_1+a_1=d, \quad d(b_1-a_1)+1=2a_1b_1$$

or

(17) $$b_1+a_1=2d, \quad d(b_1-a_1)+1=a_1b_1$$

If (16) holds, then we have

(18) $$d(b_1-a_1)+1=(b_1+a_1)(b_1-a_1)+1=b_1^2-a_1^2+1=2a_1b_1.$$

whence we get

(19) $$(b_1-a_1)^2-2a_1^2=-1.$$

It implies that $(x, y)=(b_1-a_1, a_1)$ is a solution of (4). Thus, by Lemma 1, we get (8) by (16).

If (17) holds, then we have

$$(20) \qquad d(b_1 - a_1) + 1 = \frac{1}{2}(b_1 + a_1)(b_1 - a_1) + 1 = \frac{1}{2}(b_1^2 - a_1^2) + 1 = a_1 b_1.$$

Since gcd (a_1, b_1)=1 by (13), we see from (17) that both a_1 and b_1 are odd. If implies that $(b_1-a_1)/2$ is a positive integer. By (20), we get

$$(21) \qquad a_1^2 - 2\left(\frac{b_1 - a_1}{2}\right)^2 = 1.$$

We find from (21) that $(x, y)=(a_1, (b_1-a_1)/2)$ is a solution of (5). Thus, by Lemma 2, we obtain (9) by (17). The theorem is proved.

References

[1] Mordell, L. J., Diophantine equations, London: Academic Press, 1968.

[2] Murthy, A., Smarandache friendly numbers and a few more sequences, Smarandache Notions J., 2001, 12: 264-267.

Department of Mathematics
Zhanjiang Normal College
Zhanjing, Guangdong
P. R. CHINA