THE SOLUTION OF SOME DIOPHANTINE EQUATIONS RELATED TO SMARANDACHE FUNCTION

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In the present note wesolve two diophantine eqations concerning the Smarandache function.

First, we try to solve the diophantine eqation :

$$S(x^y) = y^x \qquad (1)$$

It is porposed as an open problem by F. Smarandache in the work [1], pp. 38 (the problem 2087).

Because $S(1) = 0$, the couple $(1,0)$ is a solution of eqation (1). If $x = 1$ and $y \geq 1$, the eqation there are no $(1,y)$ solutions. So, we can assume that $x \geq 2$. It is obvious that the couple $(2,2)$ is a solution for the eqation (1).

If we fix $y = 2$ we obtain the equation $S(x^2) = 2^x$. It is easy to verify that this eqation has no solution for $x \in \{3,4\}$, and for $x \geq 5$ we have $2^x > x^2 \geq S(x^2)$, so $2^x > S(x^2)$. Consequently for every $x \in N^* \setminus \{2\}$, the couple $(x,2)$ isn't a solution for the eqation (1).

We obtein the equation $S(2^y) = y^2$, $y \geq 3$, fixing $x = 2$. It is know that for p = prime number we have the ineqality:

$$S(p^r) \leq p \bullet r \qquad (2)$$

Using the inequality (2) we obtein the inequality $S(2^y) \leq 2 \bullet y$. Because $y \geq 3$ implies $y^2 > 2y$, it results $y^2 > S(2^y)$ and we can assume that $x \geq 3$ and $y \geq 3$.

We consider the function $f: [3,\infty] \to R_+^*$ defined by $f(x) = \frac{y^x}{x^y}$, where $y \geq 3$ is fixed.

This function is derivable on the considered interval, and $f'(x) = \frac{y^x x^{y-1}(x \ln y - y)}{x^{2y}}$. In the point $x_o = \frac{y}{\ln y}$ it is equal to zero, and $f(x_o) = f(\frac{y}{\ln y}) = y^{\frac{1}{\ln y}} (\ln y)^y$.

Because $y \geq 3$ it results that $\ln y > 1$ and $y^{\frac{1}{\ln y}} > 1$, so $f(x_o) > 1$. For $x > x_o$, the function f is strict incresing, so $f(x) > f(x_o) > 1$, that leads to $y^x > x^y \geq S(x^y)$, respectively $y^x > S(x^y)$. For $x < x_o$, the function f is strict decreasing, so $f(x) > f(x_o) > 1$ that lands to $y^x > S(x^y)$. There fore, the only solution of the eqaution (1) are the couples $(1,0)$ and $(2,2)$.

SOLVING THE DIOPHANTINE EQUATION

$$x^y - y^x = S(x) \qquad (3)$$

It is obvious that the couples (1,1) is a solution of the eqaution (3).

Because $x^y-y^x = S(x)$ it results $x \neq y$ (otherwise we have $S(x)=0$, i.e., $x = 1 = y$). We prove that the equation (3) has an unique solution.

Case I : $x > y$. Therefore it exists $a \in N^*$ so $x = y + a$, $(y + a)^y - y^{y+a} = S(y+a)$ or $(1 + \frac{a}{y})^y - y^a = \frac{S(y+a)}{y^y}$. But $(1+\frac{a}{y})^y < e^a$. It results $e^a - y^a > \frac{S(y+a)}{y^y}$, false inequality for $y > e$ ($e^a - y^a < 0$ for $y > e$). So we have $y = 1$ or $y = 2$. If $y = 1$ we have $x-1 = S(x)$. In this situation it is obvious that x is a compound number. If $x = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$ is the factorization of x into prims wich $p_i \neq p_j$, $a_i \neq 0$, $i,j = \overline{1,n}$, then we have $S(x) = \max_{1 \le i \le n} S(p_i^{a_i}) = S(p_e^{a_e})$, $1 \le e \le r$. But, because $S(x) = S(p_e^{a_e}) < p_e a_e < x - 1$ it results that $S(x) < x - 1$, that is fals.

If $y = 2$, we have $x^2 - 2^x = S(x)$. For $x \ge 4$ we obtein $x^2 - 2^x < 0$, and for $x \in \{2,3\}$ there is no solution.

Case II: $x < y$. Therefore it exists $b > 0$ such that $y = x + b$. Then we have $x^{x+b}-(x+b)x=S(x)$, so $x^b - (1 + \frac{b}{x})^x = \frac{S(x)}{x^x} \le \frac{x}{x^x} \le 1$.

But, because $(1 + \frac{b}{x})^x < e^b$ we obtain $x^b - e^b < 1$, which is a false inequality for $x \ge 4$. If $x = 2$, then $2^y - y^2 = 2$, an equation which fas no solution because $2^y - y^2 \ge 7$ for $y \ge 5$.

If $x = 3$, then $3^y - y^3 = 3$, an equation which has no solutions for $y \in \{1,2,3\}$, because, if $y \ge 4$ it results $3^y - y^3 \ge 17$.

Therefore the equation (3) admits an unique solution (1,1).

REFERENCES

[1] **F. Smarandache** : *An infinity of unsolved problems concerning a Function in the Number Theory* (Presented at the 14th American Romanian Academy Anual cOnvention, hold in Los Angeles, California, hosted by the University of Southern California, from April 20 to April 22, 1989).

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