THE THIRD AND FOURTH CONSTANTS OF SMARANDACHE

by

Ion Cojocaru and Sorin Cojocaru

In the present note we prove the divergence of some series involving the Smarandache function, using an unitary method, and then we prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)}$$

is convergent to a number $s \in (71/100, 101/100)$ and we study some applications of this series in the Number Theory (third constant of Smarandache).

The Smarandache Function $S : N^* \to N$ is defined [1] such that $S(n)$ is the smallest integer k with the property that $k!$ is divisible by n.

Proposition 1. If $(x_n)_{n \geq 1}$ is a strict increasing sequence of natural numbers, then the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}, \tag{1}$$

where S is the Smarandache function, is divergent.

Proof. We consider the function $f : [x_n, x_{n+1}] \to \mathbf{R}$, defined by $f(x) = \ln \ln x$. It fulfils the conditions of the Lagrange's theorem of finite increases. Therefore there is $c_n \in (x_n, x_{n+1})$ such that :

$$\ln\ln x_{n+1} - \ln\ln x_n = \frac{1}{c_n \ln c_n}(x_{n+1} - x_n). \tag{2}$$

Because $x_n < c_n < x_{n+1}$, we have :

$$\frac{x_{n+1} - x_n}{x_{n+1}\ln x_{n+1}} < \ln\ln x_{n+1} - \ln\ln x_n < \frac{x_{n+1} - x_n}{x_n \ln x_n}, \quad (\forall) n \in N, \tag{3}$$

if $x_n \neq 1$.

We know that for each $n \in N^* \setminus \{1\}$, $\frac{S(n)}{n} \leq 1$, i.e.

$$0 < \frac{S(n)}{n \ln n} \leq \frac{1}{\ln n}, \tag{4}$$

from where it results that $\lim_{n\to\infty} \frac{S(n)}{n \ln n} = 0$. Hence there exists $k > 0$ such that $\frac{S(n)}{n \ln n} < k$, i.e., $n \ln n > \frac{S(n)}{k}$ for any $n \in N^*$, so

$$\frac{1}{x_n \ln x_n} < \frac{k}{S(x_n)}. \tag{5}$$

Introducing (5) in (3) we obtain :

$$\ln\ln x_{n+1} - \ln\ln x_n < k\frac{x_{n+1} - x_n}{S(x_n)},\ (\forall)n \in N^* \setminus \{1\}. \tag{6}$$

Summing up after n it results :

$$\sum_{n=1}^{m} \frac{x_{n+1} - x_n}{S(x_n)} > \frac{1}{k}(\ln\ln x_{m+1} - \ln\ln x_1).$$

Because $\lim_{m\to\infty} x_m = \infty$ we have $\lim_{m\to\infty} \ln\ln x_m = \infty$, i.e., the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$$

is divergent. The Proposition 1 is proved.

Proposition 2. Series $\sum_{n=2}^{\infty} \frac{1}{S(n)}$, where S is the Smarandache function, is divergent.

Proof. We use Proposition 1 for $x_n = n$.

Remarks. 1) If x_n is the n - th prime number, then the series $\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$ is divergent.

2) If the sequence $(x_n)_{n\geq 1}$ forms an arithmetical progression of natural numbers, then the series $\sum_{n=1}^{\infty} \frac{1}{S(x_n)}$ is divergent.

3) The series $\sum_{n=1}^{\infty} \frac{1}{S(2n+1)}$, $\sum_{n=1}^{\infty} \frac{1}{S(4n+1)}$ etc., are all divergent.

In conclusion, Proposition 1 offers us an unitary method to prove that the series having one of the precedent aspects are divergent.

Proposition 3. The series :

$$\sum_{n=2}^{\infty} \frac{1}{S(2) \cdot S(3) \cdots S(n)},$$

where S is the Smarandache function, is convergent to a number $s \in (71/100, 101/100)$.

Proof. From the definition of the Smarandache function it results $S(n) \leq n!$, $(\forall) n \in N^* \setminus \{1\}$, so $\frac{1}{S(n)} \geq \frac{1}{n!}$.

Summing up, begining with n = 2 we obtain :

$$\sum_{n=2}^{\infty} \frac{1}{S(2) \cdot S(3) \cdots S(n)} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.$$

The product $S(2) \cdot S(3) \ldots S(n)$ is greater than the product of prime numbers from the set $\{1, 2, \ldots, n\}$, because $S(p) = p$, for p = prime number. Therefore :

$$\frac{1}{\prod_{i=2}^{n} S(i)} < \frac{1}{\prod_{i=1}^{k} p_i}, \tag{7}$$

where p_k is the biggest prime number smaller or equal to n.

There are the inequalities :

$$S = \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} = \frac{1}{S(2)} + \frac{1}{S(2)S(3)} + \frac{1}{S(2)S(3)S(4)} + \cdots +$$

$$+ \frac{1}{S(2)S(3)\cdots S(k)} + \cdots < \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 5} + \frac{4}{2 \cdot 3 \cdot 5 \cdot 7} +$$

$$+ \frac{2}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} + \cdots + \frac{p_{k+1} - p_k}{p_1 p_2 \cdots p_k} + \cdots. \tag{8}$$

Using the inequality $p_1 p_2 \cdots p_k > p_{k+1}^3$, $(\forall) k \geq 5$ [2], we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{1}{p_6^2} + \frac{1}{p_7^2} + \cdots + \frac{1}{p_{k+1}^2} + \cdots. \tag{9}$$

We note $P = \frac{1}{p_6^2} + \frac{1}{p_7^2} + \cdots$ and observe that $P < \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \cdots$.

It results :

$$P < \frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{12^2}\right),$$

where

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{(EULER).}$$

Introducing in (9) we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{\pi^2}{6} - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \cdots - \frac{1}{12^2}.$$

Estimating with an approximation of an order not more than $\frac{1}{10^2}$, we find :

$$0,71 < \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} < 1,01. \tag{10}$$

The Proposition 3 is proved.

Remark. Giving up at the right increase from the first terms in the inequality (8) we can obtain a better right framing :

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\cdots S(n)} < 0,97. \tag{11}$$

Proposition 4. Let α be a fixed real number, $\alpha \geq 1$. Then the series $\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)}$ is convergent (fourth constant of Smarandache).

Proof. Be $(p_k)_{k\geq 1}$ the sequence of prime numbers. We can write :

$$\frac{2^\alpha}{S(2)} = \frac{2^\alpha}{2} = 2^{\alpha-1}$$

$$\frac{3^\alpha}{S(2)S(3)} = \frac{3^\alpha}{p_1p_2}$$

$$\frac{4^\alpha}{S(2)S(3)S(4)} < \frac{4^\alpha}{p_1p_2} < \frac{p_3^\alpha}{p_1p_2}$$

$$\frac{5^\alpha}{S(2)S(3)S(4)S(5)} < \frac{5^\alpha}{p_1p_2p_3} < \frac{p_4^\alpha}{p_1p_2p_3}$$

$$\frac{6^\alpha}{S(2)S(3)S(4)S(5)S(6)} < \frac{6^\alpha}{p_1p_2p_3} < \frac{p_4^\alpha}{p_1p_2p_3}$$

....

$$\frac{n^\alpha}{S(2)S(3)\cdots S(n)} < \frac{n^\alpha}{p_1p_2\cdots p_k} < \frac{p_{k+1}^\alpha}{p_1p_2\cdots p_k},$$

where $p_i \le n$, $i \in \{1, ..., k\}$, $p_{k+1} > n$.

Therefore

$$\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{\infty} \frac{(p_{k+1} - p_k)\cdot p_{k+1}^\alpha}{p_1p_2\cdots p_k} <$$

$$< 2^{\alpha-1} + \sum_{k=1}^{\infty} \frac{p_{k+1}^{\alpha+1}}{p_1p_2\cdots p_k}.$$

Then it exists $k_0 \in \mathbf{N}$ such that for any $k \ge k_0$ we have :

$$p_1p_2\cdots p_k > p_{k+1}^{\alpha+3}.$$

Therefore

$$\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{k_0-1} \frac{p_{k+1}^{\alpha+1}}{p_1p_2\cdots p_k} + \sum_{k\ge k_0} \frac{1}{p_{k+1}^2}$$

Because the series $\sum_{k \geq k_0} \frac{1}{p_{k+1}^2}$ is convergent it results that the given series is convergent too.

Consequence 1. It exists $n_0 \in N$ so that for each $n \geq n_0$ we have $S(2)S(3) \ldots S(n) > n^{\alpha}$.

Proof. Because $\lim_{n \to \infty} \frac{n^{\alpha}}{S(2)S(3)\cdots S(n)} = 0$, there is $n_0 \in N$ so that

$$\frac{n^{\alpha}}{S(2)S(3)\cdots S(n)} < 1 \text{ for each } n \geq n_0.$$

Consequence 2. It exists $n_0 \in N$ so that :

$$S(2) + S(3) + \cdots + S(n) > (n-1) \cdot n^{\frac{\alpha}{n-1}} \text{ for each } n \geq n_0.$$

Proof. We apply the inequality of averages to the numbers S(2), S(3), ..., S(n) :

$$S(2) + S(3) + \cdots + S(n) > (n-1) \sqrt[n-1]{S(2)S(3)\cdots S(n)} > (n-1) n^{\frac{\alpha}{n-1}}, \ \forall n \geq n_0.$$

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DEPARTMENT OF MATHEMATICS

UNIVERSITY OF CRAIOVA, CRAIOVA 1100, ROMANIA