TWO FUNCTIONAL EQUATIONS

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Abstract: In this paper we solve two problems concerning the pseqdo Smarandache function.

Key words: pseudo Smarandache function, sum of distinct divisors; divisors function

For any positive integer n, let $Z(n)$, $b(n)$ and $d(n)$ denote the pseudo Smarandache function, the sum of distinct divisors and the divisors function of n respectively. In [1], Ashbacher proposed the following two problems.

Problem 1. Is there infinite many positive integers n of the equation

$$Z(n) = \delta(n) \tag{1}$$

with $n \neq 2^r$, where r is a nonnegative integer.

Supported by the National Natural Science Foundation of China (No.10271104), the Guangdong Provincial Natural Science Foundation (No.011781) and the Natural Science Foundation of the Education Department of Guangdong Province (No.0161).

Problem 2. How many positive integer solutions n are there to the equation

$$Z(n)=d(n). \tag{2}$$

In this paper we completely solve these problems as follows.

Theorem 1. The equation (1) has only the positive integer solutions $n=2^r$, where r is a nonnegative integer.

Theorem 2. The equation (2) has only the positive integer solutions $n=1$, 3 and 10.

Proof of Theorem 1. It is a well kown fact that $n=2^r$ is a solution of (1). Let n be a positive integer solution of (1) with $n\neq 2^r$. Then, by [3], we have

$$Z(n)<n. \tag{3}$$

Since $\delta(n)\geq n+1$, (1) is impossible by (3). The theorem is proved.

Proof of Theorem 2. By [1], a computer search up through $n=10000$ yielded (2) only the solutions $n=1$, 3 and 10. Let n be a positive integer solution with $n>10000$, and let

$$n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k} \tag{4}$$

be the factorization of n. By [2, Theorem 273], we have

$$d(n)=(\alpha_1+1)(\alpha_1+1)\cdots(\alpha_k+1). \tag{5}$$

On the other hand, let $t=Z(n)$. Since

$$\frac{1}{2}t(t+1)\equiv 0 \pmod{n}, \tag{6}$$

we have $t(t+1)\geq 2n$. It implies that

$$Z(n)=t\geq\frac{1}{2}\left(\sqrt{8n+1}\right)>1.414\sqrt{n}, \tag{7}$$

since n$>$10000. For any prime p and any positive integer α, let

$$f\left(p^{\alpha}\right)=\frac{p^{\alpha/2}}{\alpha+1}. \tag{8}$$

Then, by (2),(4),(5),(7) and (8), we get

$$1.414 f\left(p_1^{\alpha_1}\right) f\left(p_2^{\alpha_2}\right) \cdots f\left(p_k^{\alpha_k}\right) < 1. \tag{9}$$

Since

$$f\left(p^{\alpha}\right) \geq \begin{cases} 1, & \text{if } p=2 \text{ and } a>6 \text{ or } p=3 \text{ and } a>1. \\ \dfrac{\sqrt{5}}{2}, & \text{if } p>3, \end{cases} \tag{10}$$

we find from (4) that (9) is impossible if $n>10000$. Thus, the theorem is proved.

References

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